ORDER OF MAGNITUDE OF FOURIER TRANSFORMS

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The remark (by N. Wiener) that a pair f,g of Fourier transforms cannot both be very small at infinity has been made precise in various ways. For example, Hardy (see [3], p. 175) proved that if $f(x)$ and $g(x)$ are both $O(|x|^n \exp[-x^2/2])$ then both are of the form $P_n(x)\exp[-x^2/2]$, where the degree of the polynomial P_n does not exceed n. This implies in particular that if in addition one of the functions is $o(\exp[-x^2/2])$, both vanish identically. This corollary has been generalized by Levinson [2] to cover much more general rates of growth and also one-sided approach to ∞. Here I deal with another case in the spirit of Hardy's theorem.

THEOREM. *If* $f(x)$ *and* $g(x)$ *are Fourier transforms and both are* $O(\exp[\tau|x|^\rho - x^2/2])$ *as* $|x| \to \infty$, *where* $0 < \rho < 2$, *then both are of the form* $\phi(x)\exp[-x^2/2]$, *where* $\phi(x)$ *is an entire function, at most of order* ρ *and finite type.*

This actually contains Hardy's theorem, in an ostensibly stronger form. In fact, under Hardy's hypothesis we have $\phi(x) = O(|x|^n)$ as $|x| \to \infty$ and $\phi(x)$ of order zero (since our hypotheses are satisfied with any positive ρ). But an entire function of order less than 1/2 cannot be $O(|x|^n)$ on a half-line without reducing to a polynomial. Hence we can replace the hypotheses of Hardy's theorem by the hypotheses of ours, with the additional requirement that one of f,g is $O(|x|^n \exp[-x^2/2])$, as $x \to +\infty$ (or $-\infty$), and obtain Hardy's conclusion.

It is natural to ask whether in the hypothesis $|x|^\rho$ can be replaced by a more general function, with $\phi(x)$ having a prescribed proximate order in the conclusion. It will be clear from the proof that something can be done along these lines, but only by subjecting the order function to two sets of special hypotheses demanded by the Fourier theory on one hand and the theory of entire functions on the other.

In proving the theorem we may assume that f and g are both even or both odd. We observe that

$$f(z) = \int_{-\infty}^{\infty} e^{izt} g(t)\, dt$$

represents $f(z)$ for all complex z, so that $f(z)$ is an entire function and

$$|f(x+iy)| \le A + A \int_0^\infty \exp[|y|t + \tau t^\rho - t^2/2]dt,$$

where A stands for various constants. Let $\sigma > \tau$; then

$$|f(x+iy)| \le A + A \max \exp[|y|t + \sigma t^\rho - t^2/2] \int_0^\infty \exp[(\tau - \sigma)t^\rho]dt$$

$$\le A + A \exp[|y|^2/2 + \sigma|y|^\rho + o(|y|^\rho)], \tag{1}$$

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as a straightforward calculation shows. According as $f(z)$ is even or odd, $F(z) = f(z^{1/2})$ or $F(z) = z^{-1/2} f(z^{1/2})$ is an entire function. If $h(\theta)$ is the indicator of $F(z)$ (for order 1),

$$h(\theta) = \limsup_{r \to \infty} r^{-1} \log |F(re^{i\theta})|,$$

our hypothesis states in particular that $h(0) \leq -1/2$, while (1) shows that $h(\theta) \leq 1/2$ for all θ. We now appeal to the following lemma ([1], p. 76).

LEMMA. *If* $F(z)$ *is an entire function of exponential type* τ *and* $h(0) = -\tau$, *then* $F(z)$ *is of the form* $e^{-\tau z}\Phi(z)$, *where* $\Phi(z)$ *is an entire function of exponential type* 0.

This shows that $F(z) = e^{-z/2}\Phi(z)$. Then (according as $f(z)$ is even or odd), $f(z)$ or $f(z)/z = \phi(z)\exp[-z^2/2]$, with $\phi(z) = \Phi(z^2)$, an entire function of growth not exceeding order 2, type 0. We then have $\phi(z) = O(\exp[\tau|z|^\rho])$ on the real axis and $\phi(z) = O(\exp[\sigma|z|^\rho])$, $\sigma > \tau$, on the imaginary axis. Consider, for $0 \leq \theta \leq \pi/2$, the function $\psi(z) = \phi(z)\exp[(\alpha + i\beta)z^\rho]$, whose modulus is

$$|\phi(z)| \exp[\,|z|^\rho(\alpha \cos \rho\theta - \beta \sin \rho\theta)\,].$$

If we take $\alpha = -\tau$, and $\alpha \cos \pi\rho/2 - \beta \sin \pi\rho/2 = -\sigma$, which we can do since $\rho < 2$, we have $\psi(z)$ satisfying the hypotheses of the Phragmén-Lindelöf theorem for the critical angle and so bounded in the first quadrant. Consequently $\phi(z)$ is at most of order ρ, finite type, in the first quadrant; and similarly in the other quadrants. (In fact, the type of $\phi(z)$, if it is of order ρ, does not exceed $\tau \sec \pi\rho/4$.) This establishes the conclusion for f, and the theorem is symmetrical in f and g.

REFERENCES

1. R. P. Boas, Jr., *Entire functions*, Academic Press, New York, 1954.

2. N. Levinson, *Restrictions imposed by certain functions on their Fourier transforms*, Duke Math. J. 6 (1940), 722-731.

3. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Oxford University Press, 1937.

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