A THEOREM ABOUT MAPPINGS OF A TOPOLOGICAL GROUP INTO THE CIRCLE

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1. INTRODUCTION. If S is the unit circle topological group with complex multiplication as its operation, and if G is a compact topological space, then the collection S^G of mappings of G into S with the compact-open topology is a commutative, metric topological group. For f_1 and f_2 in S^G, the product $f_1 \cdot f_2$ is the mapping defined by the relation

$$[f_1 \cdot f_2](x) = f_1(x)f_2(x), \textit{for all } x \in G,$$

and the inverse f^{-1} of any $f \in S^G$ is defined by the relation

$$[f^{-1}](x) = [f(x)]^{-1}, \textit{for all } x \in G.$$

The identity is the mapping f_0 defined by $f_0(x) \equiv 1$. A metric ρ for S is that defined by $\rho(z_1, z_2) = |z_1 - z_2|$, and the metric ρ^* for S^G may be defined by

$$\rho^*(f, g) = \sup_{x \in G} \rho[f(x), g(x)].$$

Following Eilenberg [1], a mapping $f \in S^G$ is said to be equivalent to 1 on a subset H of G ($f \sim 1$ on H) provided there exists a mapping $\phi : H \to R$, the real line, such that $f(x) = \exp[i\,\phi(x)]$ for all $x \in H$. Two mappings f and g in S^G are said to be equivalent on $H \subset G$ ($f \sim g$ on H) provided $f \cdot g^{-1} \sim 1$ on H. It is shown in [1] that $\sim$ is an equivalence relation, that if G is a separable metric space, then $f \sim g$ if and only if f is homotopic to g, and that $P(G) = \{f \mid f \in S^G, f \sim 1\}$ is algebraically a subgroup of S^G. In [1] and here also, if $|z_1 - z_2| < 2$, $[z_1, z_2]$ denotes the signed angle less than π through which the radius to z_1 must be rotated in order to coincide with the radius to z_2.

It will be shown in this paper that $P(G)$ is an open subgroup of S^G when G is compact, and that the factor group $S^G/P(G)$ is isomorphic to the character group of G when G is a compact, connected, commutative topological group satisfying the second axiom of countability. A corollary to this result is the fact that every mapping of such a topological group G into S is homotopic to an interior mapping. This corollary is analogous to a result of G. T. Whyburn [2].

2. With reference to the remark which follows, observe that the function $[z_1, z_2]$, for z_1 and z_2 in S and $|z_1 - z_2| < 2$, is continuous and that $\exp(i[z_1, z_2]) = z_2/z_1$.

(2.1) *If* G *is a compact topological space,* $P(G)$ *is an open subgroup of* S^G.

Proof. Suppose, for some $f \in P(G)$, that g is a mapping such that $\rho^*(f, g) < 2$. Then, for each $x \in G$, $[f(x), g(x)] < \pi$. Since $f \sim 1$ on G, there exists a continuous $\phi : G \to R$ such that $f(x) = \exp[i\phi(x)]$, for all $x \in G$. Let $\psi : G \to R$ be the function defined by

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$$\psi(x) = \phi(x) + [f(x), g(x)], \textit{ for all } x \in G.$$

Then ψ is continuous and

$$\begin{aligned}\exp[i\psi(x)] &= \exp[i\phi(x)]\exp(i[f(x), g(x)]) \\ &= f(x)\,[g(x)/f(x)] = g(x), \textit{ for all } x \in G.\end{aligned}$$

Therefore, $g \sim 1$ and $P(G)$ is open.

Thus $B(G)$ is a discrete topological group when G is compact.

It will be assumed henceforth that G is a compact topological group satisfying the second axiom of countability. The following three theorems are listed for reference. The first is an easy consequence of theorems in [3].

(2.2) *If* G *is connected and commutative, and if* U *is a neighborhood of the identity in* G, *then there exists a subgroup* H *of* G *such that* $H \subset U$ *and* G/H *is isomorphic with the direct product of a finite number of copies of* S.

(2.3) *If* $f \in S^S$, *then there exists an integer* p *such that* $f \sim z^p$ *on* S [1].

(2.4) *If* X *and* Y *are continua and* f *is a mapping of* $X \times Y$ *into* S *such that, for some* $(x_0, y_0) \in X \times Y$, $f \sim 1$ *on* $X \times (y_0)$ *and* $f \sim 1$ *on* $(x_0) \times Y$, *then* $f \sim 1$ *on* $X \times Y$ [1].

Considering the character group G^* of G as the group of all homomorphisms of G into S, define a function $F: G^* \to B(G)$ as follows: for $h \in G^*$, let $F(h)$ be the element of $B(G)$ containing h. This function is clearly single-valued, algebraically a homomorphism, and continuous.

(2.5) LEMMA. *If* G *is connected and commutative, then* F *is an isomorphism into.*

Proof. Suppose $h_1 \in G^*$ and $h_2 \in G^*$ are such that $h_1 \sim h_2$. Then $h_1/h_2 = g$ is a homomorphism and, since $g \sim 1$, there exists a continuous $\phi: G \to R$ such that $g(x) = \exp[i\,\phi(x)]$, for all $x \in G$. Since $g(e) = \exp[i\,\phi(e)] = 1$, where e is the identity of G, $\phi(e) = 2\pi k$, for some integer k. If $k \neq 0$, define ϕ' to be $\phi - 2\pi k$. Then $\phi'(e) = 0$ and $\exp[i\phi'(x)] = g(x)$, for each $x \in G$. Hence it may be assumed that $\phi(e) = 0$.

Now $\exp[i\phi(xy)] = g(xy) = g(x)g(y) = \exp[i(\phi(x) + \phi(y))]$, for all x and y in G; hence $\phi(xy) - [\phi(x) + \phi(y)] = 2\pi\eta(x,y)$, where $\eta: G \times G \to I$, the integers. Since η is continuous, $\eta(G \times G)$ is connected and, since $\eta(0,0) = 0$, $\eta(x,y) \equiv 0$. Therefore ϕ is a homomorphism and $\phi(G)$ is a compact, connected subgroup of R; hence $\phi(G) = \{0\}$, and $g(x) \equiv 1$. Thus $h_1 = h_2$, and F is one-to-one.

Consequently F^{-1} is single-valued and is continuous, since $B(G)$ is discrete.

(2.6) LEMMA. *If* $G = S \times S \times \cdots \times S$ (*a finite number* k *of copies of* S), *then* F *is an isomorphism onto.*

Proof. Suppose first that $k = 2$. Corresponding to $f \in S^{S\times S}$, let $f_1 = f \mid S \times (1)$ and $f_2 = f \mid (1) \times S$. By (2.3), there exist homomorphisms $h_1 \sim f_1$ and $h_2 \sim f_2$. Define $h: S \times S \to S$ by the equation $h(x,y) = h_1(x)h_2(y)$. It is easily verified that h is a continuous homomorphism; i.e., $h \in (S \times S)^*$. On $S \times (1)$, $f/h = f_1/h_1 \sim 1$ and, on $(1) \times S$, $f/h = f_2/h_2 \sim 1$. By (2.4), $f/h \sim 1$ on $S \times S$, so $f \sim h$. That is, F is onto.

On the assumption that F is onto when $G = S \times S \times \cdots \times S$ ($k - 1$ times), the completion of the proof by induction is similar to the preceding step and is omitted.

(2.7) THEOREM. *If* G *is a compact, connected, commutative topological group satisfying the second axiom of countability, then* $B(G) \approx G^*$.

Proof. It must be shown that F is onto; i.e., for $f \in S^G$, an $h \in G^*$ must be found such that $h \sim f$ on G. With a metric for G with respect to which translations are isometries, there exists a $\delta > 0$ such that, if $D \subset G$ is of diameter less than δ, then $f(D)$ is of diameter less than 1. Let U be the $\delta/2$-neighborhood of $e \in G$ and (invoking (2.2)) let H be a subgroup of G such that $H \subset U$ and $G/H \approx S \times S \times \cdots \times S$ (k times). Let $\alpha: G \to G/H$ be the natural mapping. Define $g: G/H \to S$ as follows: for a point $[xH]$ of G/H, let $g([xH])$ be the midpoint of the smallest arc of S containing $f(xH)$, where $xH = \{xh \mid h \in H\}$. The choice of H ensures that g is well-defined.

To show that g is continuous, let $\bar{F} = g\alpha$, let $\{x_i\}$ be a sequence of points of G converging to $x \in G$, and let $\epsilon > 0$. There exists a $\gamma > 0$ such that $x_ih \in N_\gamma(xh)$ implies $f(x_ih) \in N_\epsilon[f(xh)]$, for each $h \in H$. Let M_γ be the γ-neighborhood of a point $y_0 = xh_0$ $(h_0 \in H)$. Then $y_i = x_ih_0 \to xh_0 = y_0$ and, for some integer I, it is true that $i > I$ implies $x_ih_0 \in M_\gamma$. If, denoting by y_h the point xh, we translate M_γ by $y_hy_0^{-1}$, then, for each $h \in H$, $y_hy_0^{-1}M_\gamma$ contains the points $y_hy_0^{-1}y_i$, for $i > I$. But $y_0^{-1}y_i = h_0^{-1}x^{-1}x_ih_0 = x^{-1}x_i$, and hence $y_hy_0^{-1}y_i = xx^{-1}x_ih = x_ih$. Therefore, x_iH is contained in a γ-neighborhood of xH and $f(x_iH)$ is contained in an ϵ-neighborhood of $f(xH)$, for $i > I$.

Let xh_1 and xh_2 of xH be such that $f(xh_1)$ and $f(xh_2)$ are the endpoints of the smallest arc of S containing $f(xH)$. There exists an integer I_1 such that $i > I_1$ implies $f(x_ih_1) \in N_\epsilon[f(xh_1)]$ and $f(x_ih_2) \in N_\epsilon[f(xh_2)]$. Now, if $i > I + I_1$, then $f(x_iH) \subset N_\epsilon[f(xH)]$, $f(x_iH) \cdot N_\epsilon[f(xh_1)] \neq \phi$, and $f(x_iH) \cdot N_\epsilon[f(xh_2)] \neq \phi$. Hence the midpoint of the smallest arc containing $f(x_iH)$ is within ϵ of the midpoint of the smallest arc containing $f(xH)$, when i is large enough, and $\bar{F}$ is seen to be continuous. Since α is open, this implies that g is continuous also.

Furthermore, $g\alpha \sim f$ on G because $\phi(x) = [f(x), g\alpha(x)]$ is continuous and $\exp[i\phi(x)] = g\alpha(x)/f(x)$, for all $x \in G$. By (2.6), there exists an $h \in (G/H)^*$ such that $h \sim g$; i.e., there exists a $\phi': G/H \to R$ such that $h(y)/g(y) = \exp[i\phi'(v)]$, for all $y \in G/H$. Therefore, $h\alpha(x)/g\alpha(x) = \exp[i\phi'\alpha(x)]$, for all $x \in G$ and $h\alpha \sim g\alpha \sim f$. Since $h\alpha$ is a homomorphism, this proves that F is onto. By (2.5), F is an isomorphism, and the proof is complete.

(2.71) COROLLARY. *Every mapping of a compact, connected, commutative topological group satisfying the second axiom of countability into* S *is homotopic to an interior mapping.*

Proof. By (2.7), if $f \in S^G$, then there exists an $h \in G^*$ such that $h \sim f$ on G. Then h is interior, since it is defined on the compact G; and by a previous remark it is homotopic to f.

REFERENCES

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