ON THE STRUCTURE OF RECURRENCE RELATIONS

by

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Recurrence relations have long been known for the characteristic solutions of many linear second-order differential equations obtained from the hypergeometric equation. These relations may be found by an examination of the Taylor series for the solutions. Recently a factorization method for deriving characteristic solutions of linear second-order differential equations was reported by L. Infeld and T. E. Hull [1], and their approach includes a partial determination of the coefficients in recurrence relations for the solutions. The equations which may be investigated in this way include the hypergeometric and confluent hypergeometric equations in many forms, as well as a variety of the equations of wave mechanics which, however, are chiefly also of hypergeometric type. Differential equations of the second order with four regular singular points or their confluences present greater difficulty. Of their solutions, only the Mathieu functions, studied by E. T. Whittaker [2], and the radial prolate spheroidal functions, to which Whittaker's method was applied by the writer [3], are known to have recurrence relations. The determination of the coefficients in these relations is complicated by the fact that, for non-hypergeometric equations, neither the characteristic values nor the coefficients in the Taylor series for the characteristic solutions are known functions of the indices.

It is the purpose of this note to present a general framework which appears to include all known recurrence relations, and which may be of assistance in determining coefficients for recurrence relations not yet completely known. The method of Infeld and Hull appears as the simplest case, while the method of Whittaker is seen to be a special example of a very general approach that will yield information about all equations of interest.

The differential equation to be discussed is

$$d/dZ[P(Z)\, dY(Z)/dZ] + [R(Z) + t\,S(Z)]\, Y(Z) = 0. \tag{1}$$

It is assumed that the functions P, R, and S satisfy suitable conditions over a given range of the variable Z, and that a sequence t_n of characteristic values of the parameter t, and a corresponding sequence of characteristic solutions $Y_n(Z)$ of (1), are determined by additional conditions. The functions Y_n are so normalized that a single function $Y(Z, t)$, analytic in t for each fixed Z which is not a singular value of the differential equation, coincides with Y_n for $t = t_n$ and with Y_{n-1} for $t = t_{n-1}$.

By a well-known transformation of the independent and dependent variables in (1), the equation can be replaced by another in which $P \equiv 1$. Other

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well-known transformations make S constant, say $S \equiv 1$, or both $P \equiv 1$ and $S \equiv 1$. In the case of many of the equations of mathematical physics it is also possible to employ the latter transformations to replace the function R by a constant, say $R \equiv 1$ (cf. [1]). Returning to the original notation, one may therefore discuss equation (1) under the assumption that it has any one of the forms

(1a) $$d^2Y/dZ^2 + (R + tS)\cdot Y = 0,$$

(1b) $$d^2Y/dZ^2 + (t + tS) \cdot Y = 0,$$

(1c) $$d^2Y/dZ^2 + (R + t) \cdot Y = 0,$$

(1d) $$d/dZ\,(P\,dY/dZ) + (R + t)\cdot Y = 0,$$

and only (1b) is not a completely general form.

In all known cases, the available recurrence relations include a pair of differential relations of the type

(2) $$P(Z)\,A_n(Z)\,dY_n/dZ + B_n(Z)\,Y_n = Y_{n-1},$$

(3) $$P(Z)\,F_n(Z)\,dY_{n-1}/dZ + G_n(Z)\,Y_{n-1} = Y_n,$$

and all other recurrence relations may be deduced from this pair. Furthermore, in every completely known case relations (2), (3) with the same coefficients hold for a sequence of solutions $W_n(Z)$, $W_{n-1}(Z)$ that are linearly independent of Y_n, Y_{n-1}, respectively, and are normalized so as to be values of the same analytic function $W(Z, t)$. Under the assumption that two linearly independent sequences of solutions of (1) satisfy (2) and (3), the structure of the coefficients A_n, B_n, F_n, G_n is to be determined.

By substituting the left hand member of (2) into the right hand member of (3), one obtains a differential equation for Y_n, and similarly for Y_{n-1}. It is a trivial lemma that these equations must have coefficients proportional to the coefficients in (1) for $t = t_n$ and for $t = t_{n-1}$, respectively. Consequently the recurrence coefficients must satisfy the equations

(4) $$PF_nA_n' + F_nB_n + G_nA_n = 0, \quad PA_nF_n' + A_nG_n + B_nF_n = 0,$$

(5) $$\frac{PF_nB_n' + G_nB_n - 1}{PF_nA_n} = Q_n, \quad \frac{PA_nG_n' + B_nG_n - 1}{PA_nF_n} = Q_{n-1},$$

where $Q_n(Z) = R(Z) + t_n S(Z)$, $Q_{n-1}(Z) = R(Z) + t_{n-1} S(Z)$. From the difference of the two equations in (4), one finds that $A_n = d_nF_n$, where d_n is a constant. [As a general convention in this note, constants are written as lower case letters, while variable quantities are written as capitals.] The two equations in (4) are seen to be identical. If one writes also $C_n(Z) = d_nG_n(Z)$, the recurrence relations take the form

(6) $PA_n dY_n/dZ + B_nY_n = Y_{n-1}, \quad PA_n dY_{n-1}/dZ + C_nY_{n-1} = d_nY_n$,

while the equations (4) and (5) become

$$(7) \qquad PA_n' + B_n + C_n = 0,$$

$$(8) \qquad PA_nB_n' + B_nC_n - d_n - PQ_nA_n^2 = 0 ,$$

$$(9) \qquad PA_nC_n' + B_nC_n - d_n - PQ_{n-1}A_n^2 = 0$$

If A_n, B_n, C_n, d_n are found to satisfy the system (7), (8), (9), the functions Y_n and Y_{n-1} satisfy the recurrence relations (6) by virtue of the differential equations (1) for $t = t_n$ and $t = t_{n-1}$. The function A_n contains a multiplicative parameter, which may be fixed by a normalization of Y_n and Y_{n-1}, and the same normalization fixes the value of d_n. Additional parameters involved in the solution of the system must be so chosen that (7), (8), (9) are satisfied. In the simple case where A_n is constant, the choice of parameters to satisfy the equations has been examined in detail by Infeld and Hull [1]. In the present note, only necessary conditions for a solution will be given.

If $A_n = a_n$ is constant, the recurrence relations (6) will be said to be of type I, and of type Ia, Ib, or Ic if in addition the differential equation is in the form (1a), (1b), or (1c), respectively. Of the most general type I with differential equation in the form (1), no examples are known to the writer, and so this rather complicated case is omitted. The solution of the system for relations of type Ia is still complicated, and since these relations are relatively rare it will also be omitted. A brief discussion of types Ib and Ic follows.

After the appropriate modifications, the equations (8) and (9) for types Ia, Ib, or Ic become

$$(8') \qquad a_nB_n' - B_n^2 - d_n - a_n^2 Q_n = 0 ,$$

$$(9') \qquad a_nB_n' + B_n^2 + d_n + a_n^2 Q_{n-1} = 0 ,$$

since (7) now implies that $C_n = -B_n$. In the case (1b), when $Q_n = 1 + t_n S$, $Q_{n-1} = 1 + t_{n-1} S$, one deduces from the sum of (8') and (9') that

$$(10) \qquad B_n' = \frac{1}{2} (t_n - t_{n-1}) a_n S ,$$

and therefore that

$$(11) \qquad B_n = \frac{1}{2} (t_n - t_{n-1}) a_n T + b_n ,$$

where $T(Z)$ is an integral of $S(Z)$ and b_n is a constant. From the difference of the equations one deduces

(12) $$B_n^2 + a_n^2 [1 + \frac{1}{2}(t_n + t_{n-1})S] = -d_n .$$

A necessary condition that type Ib occur is that the constants a_n, b_n in (11) can be so chosen that the left hand member of (12) is a constant $-d_n$. If the resulting quantities B_n, a_n, d_n satisfy (8') and (9'), the recurrence coefficients have been found.

An alternative approach is to differentiate (12) and eliminate B_n' by use of equation (10):

(13) $$B_n = \frac{1}{2} \frac{t_{n-1} + t_n}{t_{n-1} - t_n} \frac{S'}{S}$$

Comparing the derivative of (13) with (10), one obtains the necessary condition

(14) $$S^{-1} \frac{d}{dZ} (S^{-1} \frac{dS}{dZ}) = \text{constant}.$$

As examples may be cited the functions $S = k (e^{cZ} \pm e^{-cZ})^{-2}$ and $S = kZ^{-2}$, where k and c are real or complex constants. The former of these functions is shown in [1] to appear as the coefficient S of a number of forms of the hypergeometric equation, including the associated Legendre equation; the latter is the coefficient S of the Bessel equation, when that equation is in the form (Ib).

If the recurrence relations are to be of type Ic, the corresponding equations (8') and (9') are easy to integrate explicitly. It is found that the characteristic values must differ by a constant independent of n, $t_n - t_{n-1} = 2k$, and that the coefficient R of the differential equation (1c) must be a quadratic function of the form $R = -k^2x^2 - 2fkx + g$. The remaining recurrence coefficients are given by $B_n = a_n(kx + f)$, $d_n = -a_n^2 (g - f^2 - t_{n-1} - k)$, and the parameter a_n must be fixed by the normalization of Y_n and Y_{n-1}.

The Mathieu equation

(15) $$\frac{d^2Y}{dZ^2} + (-2h^2 \cos 2Z + t) Y = 0$$

is in the form (1c), but the coefficient R clearly does not satisfy the conditions of the preceding paragraph. For the radial prolate spheroidal equation

(16) $$\frac{d}{dZ} [(Z^2 - 1) \frac{dY}{dZ}] + (c^2Z^2 - \frac{m^2}{Z^2 - 1} - t_{mn}) Y = 0 ,$$

recurrence relations of the form (6) are known to exist [3], both for a change of the index n and for a change of the index m. Still, as is easily checked, these relations are not of type I, and after the differential equation is transformed as in (1a), (1b), or (1c), the relations that correspond are not of types Ia, Ib, or Ic. A method of investigating the structure of the recurrence coefficients for variable A_n is needed, and it is furnished by a generalization of the method used by Whittaker [2] for the equation (15). Relations (6) with variable A_n will be referred to as type II relations.

Returning to equations (8) and (9), one may subtract to derive

$$(17) \qquad B_n' - C_n' - (Q_n - Q_{n-1}) A_n = 0 .$$

Differentiating (7) and combining with (17), one has

$$(18) \qquad 2 B_n' + \frac{d}{dZ}(P A_n') - (Q_n - Q_{n-1}) A_n = 0 ,$$

$$(19) \qquad 2 C_n' + \frac{d}{dZ}(P A_n') + (Q_n - Q_{n-1}) A_n = 0 ,$$

so that B_n and C_n may be determined by quadratures, once A_n is known. A linear equation for A_n may be obtained by differentiation of (8) and elimination of B_n, C_n and their derivatives by use of (7), (17), and (18). If $d(Q_n - Q_{n-1})/dZ = 0$, this equation can be easily identified, and so the differential equation (1) is first transformed to the case (Id), with $Q_n = R + t_n$, $Q_{n-1} = R + t_{n-1}$. The structure of the coefficients can be completely determined from the case (Id), and for relations of type II the differential equation will be considered in this form only. The equation for A_n obtained from the elimination is now

$$(20) \qquad \frac{d^2}{dZ^2}\left[P \frac{d}{dZ}(P A_n')\right] + \frac{d^2}{dZ^2}\left[P(Q_n + Q_{n-1}) A_n\right]$$

$$+ (Q_n + Q_{n-1}) \frac{d}{dZ}(P A_n') + 2 P R' A_n' + (t_n - t_{n-1})^2 A_n = 0 .$$

To identify the equation (20), consider two functions $U(Z)$, $V(Z)$ that respectively satisfy the differential equations

$$\frac{d}{dZ}\left(P \frac{dU}{dZ}\right) + F(Z)\, U = 0, \quad \frac{d}{dZ}\left(P \frac{dV}{dZ}\right) + G(Z)\, U = 0.$$

If $d(F - G)/dZ = 0$, the product $W = U V$ of the two functions is a solution of the differential equation

$$(21) \quad \frac{d^2}{dZ^2}\left[P \frac{d}{dZ}(P W')\right] + \frac{d^2}{dZ^2}\left[P(F + G)W\right] + (F + G) \frac{d}{dZ}(P W')$$

$$+ P\left[\frac{d}{dZ}(F + G)\right] W' + (F - G)^2 W = 0.$$

(This easily derived formula is an adaptation of one given without source by Whittaker [2]). A comparison of (21) and (20) shows that

$$A_n = c_n^{11} Y_n^{(1)} Y_{n-1}^{(1)} + c_n^{12} Y_n^{(1)} Y_{n-1}^{(2)} + c_n^{21} Y_n^{(2)} Y_{n-1}^{(1)} + c_n^{22} Y_n^{(2)} Y_{n-1}^{(2)} \tag{22}$$

where $Y_n^{(1)} = Y_n$ denotes the characteristic solution of (Id) for $t = t_n$ and $Y_n^{(2)} = W_n$ denotes a solution linearly independent of $Y_n^{(1)}$ for each n. The constants c_n^{jk}, (j, k = 1, 2) in (22) must be fixed by additional information about the given equation, e.g., for the Mathieu equation they are determined in [2] from the condition of periodicity. By virtue of (22) and the differential equations (Id) for Y_n and Y_{n-1}, equations (18) and (19) may also be written

$$B_n' = Q_n A_n - P \sum_{j,k} c_n^{jk} (dY_n^{(j)}/dZ)(dY_{n-1}^{(k)}/dZ). \tag{23}$$

$$C_n' = Q_{n-1} A_n - P \sum_{j,k} c_n^{jk} (dY_n^{(j)}/dZ)(dY_{n-1}^{(k)}/dZ). \tag{24}$$

The integration constants in the integrations of (23) and (24) must be such that d_n is a constant in (8) and (9). The possibility of so choosing the constants is a compatibility condition or necessary condition for recurrence relations of type II.

It is expected that a somewhat expanded treatment of the type II procedure as applied to recurrence relations for the radial prolate spheroidal functions will appear in UMM-126, a forthcoming publication of the Willow Run Research Center.

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