NOTE ON MY PAPER "INTRINSIC RELATIONS SATISFIED BY THE VORTICITY AND VELOCITY VECTORS IN FLUID FLOW THEORY"

by

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The purpose of this note is: (1) to relate the papers of several other authors to the above paper[1] of the present author; (2) to provide some further details for the derivation of one equation of the above paper and to generalize a theorem of S. S. Byušgens; (3) to indicate extensions[2] of our previous results to more general types of gases.[3]

1. Related Papers by other Authors. Our previous equation (2.19) for the decomposition of the vorticity vector is equivalent to the relation 2.3 (6) of Bjørgum's paper.[4] This can be seen with the aid of the following computation. If e_{ijk}, e^{ijk} denote the permutation tensor in an orthogonal Cartesian coordinate system, x^j, $j = 1, 2, 3$, then it is well known that

$$e^{ijk}e_{imp} = \delta^j_m \delta^k_p - \delta^j_p \delta^k_m \tag{1.1}$$

where δ^j_n is the Kronecker tensor. From (1.1), it follows that

$$(b^j n^k - b^j n^k)\, \partial_j t_k = (e_{imp} b^m n^p)(e^{ijk}\, \partial_j t_k) . \tag{1.2}$$

The left hand side of (1.2) can be written in form

$$n^k \frac{\partial t_k}{\partial b} - b^k \frac{\partial t_k}{\partial n} ,$$

Further, the first term of the right hand side of (1.2) is the cross-product of the unit vectors, $\tilde{b}$ and $\tilde{n}$, or the negative of the unit tangent vector, $\tilde{t}$; similarly, the second term of the right hand side of (1.2) is the curl of $\tilde{t}$. Thus, (1.2) reduces to

$$b^k \frac{\partial t_k}{\partial n} - n^k \frac{\partial t_k}{\partial b} = \tilde{t} \cdot \text{curl}\ \tilde{t} . \tag{1.3}$$

The relation (1.3) and 2.3 (6) of Bjørgum's paper lead to our relation (2.19). Similarly, our equation (2.21), which is a generalization of (2.19), is equivalent to Bjørgum's equation 2.6 (17). In fact, Bjørgum's dyadic decomposition of Section 2.6 and our tensor decomposition of Section 2 are closely related.

Our previous equation (4.16) results from applying the divergence to the vector relation.

$$\tilde{v} = q\, \tilde{t} \tag{1.4}$$

and showing that

$$\text{div.}\ \tilde{t} = h_1 + h_2 \tag{1.5}$$

where h_1, h_2 are the two principal values of the symmetric part of the projection of the tensor, grad $\tilde{t}$, in the plane locally perpendicular to $\tilde{t}$. When

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$\tilde{t}$ is orthogonal to ∞^1 surfaces, then

$$h_1 + h_2 = M , \tag{1.6}$$

where M is the mean curvature of these surfaces. Thus in this case

$$\text{div.} \tilde{t} = M . \tag{1.7}$$

The relation (1.7) is basic for the theorems[5] proved by L. Castoldi for incompressible fluids, by S. S. Byušgens and M. Giqueaux for compressible isentropic fluids, and which hold in the case of a compressible non isentropic polytropic gas[6] as noted by the present author. This relation was first stated by J. Challis.[7] A similar result was used by H. Weyl,[8] who expressed the divergence of the electric field on a conductor in terms of the mean curvature of the conductor surface and the normal component of the field.

R. C. Prim[9] has recently obtained a related result. He has worked with the reduced velocity vector and has generalized a result of G. Hamel[10] on the geometry of the ∞^1 surfaces orthogonal to $\tilde{t}$, when these surfaces are minimal ($M = 0$).

2. The Derivation of Equation (4.25) and the Generalization of Byušgens' Theorem. The following computation is needed to clarify the validity of our equation (4.25). For an ideal gas, the following two relations are valid:

$$p = R\rho T , \tag{2.1}$$

$$c^2 = \gamma RT , \tag{2.2}$$

where R is the universal gas constant. In the case of ideal gases, γ is a function of the absolute temperature, T; for polytopic gases, γ is the ratio of the specific heats of the gas and is a constant. Forming the differential of (2.1), we find

$$dp = RT\, d\rho + R\rho\, dT . \tag{2.3}$$

Eliminating T in (2.3) through use of (2.2), we obtain the result

$$dp = d\left(\frac{\rho c^2}{\gamma}\right) . \tag{2.4}$$

From (2.4), the relation (4.25) of our previous paper is easily obtained.

The relations (4.18), (4.19) of our previous paper lead to a generalization of Byušgen's theorem[11]. These equations follow from vector decomposition of the acceleration vector with respect to the unit normal and binormal vectors of the stream lines and may be written as

$$\frac{\partial p}{\partial n} = - \rho q^2 \kappa , \quad \frac{\partial p}{\partial b} = 0 . \tag{2.5}$$

This type of decomposition is a common one in analytical mechanics. However, the significance of this procedure for fluid dynamics does not seem to have been noted. Thus, the equations (2. 5) show that: for any gas, with no external force acting, if the stream lines are straight lines, then they are orthogonal to the ω^1 surfaces, p = constant; and conversely, if the stream lines are orthogonal to the ω^1 surfaces, p = constant, then the stream lines are straight lines.

3. Extensions of the Previous Results to More General Types of Gases. In our previous paper, the theory was developed for polytropic gases;[12] however, many of the equations are valid for more general types of gases.

First, the equations (4. 23) are deduced from (4. 19) by assuming that

$$p = p(\rho, S) \tag{3.1}$$

and S, the entropy, is constant. Thus, our equations (4. 23) are valid for an arbitrary type of gas which is isentropic. In fact, the first relation of (4. 23) holds even when S is constant along a stream line but may vary from one stream line to another.

Secondly, if we insert γ inside the parenthesis of the left hand sides of (4. 25), so that (4. 25) is replaced by

$$\frac{\partial}{\partial s}\left(\frac{\rho c^2}{\gamma}\right) = -\rho q \frac{\partial q}{\partial s}, \quad \frac{\partial}{\partial n}\left(\frac{\rho c^2}{\gamma}\right) = -\rho p^2 \kappa, \tag{3.2}$$

then these equations are valid for any ideal gas,[13] instead of for merely polytropic gases.

Finally, it should be noted that our previous equations (4. 24), (4. 26) are valid for any arbitrary gas, as long as S is constant along a stream line.

FOOTNOTES

1. N. Coburn, Intrinsic relations satisfied by the vorticity and velocity vectors in fluid flow theory, Michigan Math. J. vol. 1, (1952), pp. 113-130.
2. The author is indebted to Professor C. Truesdell for calling his attention to the papers by Bjørgum, Challis, and Prim, and also for his comments on the third topic.
3. In this and the previous paper, the terms "gas" and "perfect fluid" are considered equivalent.
4. O. Bjørgum, On Beltrami vector fields and flows, Bergen Univ. Årbok 1951, Naturv. Rekke, no. 1, pp. 1-86.
5. See Section 5 of Reference 1.
6. In fact, these theorems are valid for any gas for which entropy is constant along a stream line.

7. J. Challis, A general investigation of the differential equations applicable to the motion of fluids, Trans. Cambridge Philos. Soc. vol. 7, (1842) pp. 371-393 (see pp. 373-375).

8. H. Weyl, Über die Randwertaufgabe der Strahlungstheorie und asymptotische Spektralgesetze, J. Reine Angew. Math. vol. 143 (1913) pp. 177-202 (see p. 181). The result is credited to T. Levi-Civita.

9. R. C. Prim, 3rd, Steady rotational flows of ideal gases, J. Rational Mech. and Anal., vol. 1 (1952) pp. 425-497 (see pp. 484-485).

10. G. Hamel, Potentialströmungen mit konstanter Geschwindigkeit, S.-B. Preuss. Akad. Wiss., 1937, pp. 5-20.

11. S. S. Byušgens, The geometry of the stationary flow of an ideal incompressible fluid, Izvestiya Akad. Nauk, SSSR. Ser. Mat. vol. 12 (1948) pp. 481-512.

12. See p. 121 of the previous paper.

13. See equation (2.4) of the present paper.

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