A THEOREM IN PROJECTIVE N-SPACE EQUIVALENT TO COMMUTATIVITY

by

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In any geometry satisfying the rudimentary projective incidence axioms an algebra of points can be introduced on any line, with operations defined by incidences, and by this means an "intrinsic" coordinate system can be introduced. If in this algebra of points, multiplication is commutative, we say the geometry is commutative. It is well known that the Pappus theorem, as an incidence relation on lines in the 2-planes of the space, is a necessary and sufficient condition for the commutativity of the geometry The question arises whether in dimensions above 2 one can state such a condition in terms of the proper elements of the geometry--points and hyperplanes.

For 3-space it has been shown by Reidemeister and Schönhardt (1) that the existence of "Möbius configuration" is such a condition--this being a pair of tetrahedra each of which circumscribes the other. Schönhardt showed by projection that such a configuration implies the existence of the Pappus configuration. The theorem of this paper is a generalization of both the Pappus theorem and a theorem in 3-space (theorem A) equivalent to the existence of the Möbius configuration.

<u>Theorem A</u>. Let T be a plane quadrangle in a commutative projective 3-space, and let a distinct plane be passed through each of the six sides of T. The vertices of T fall into four triangles; let the planes be grouped correspondingly to determine four points--these four points are coplanar.

It is not difficult to see how this theorem is related to the Möbius configuration, but it is susceptible of a simple direct proof, by the Grassman calculus, which is of interest in itself. (This calculus assumes commutativity.)

It can be readily verified that if the six planes are a, b, c, x, y, z so arranged that the vertices of T are the four points [xyz], [xbc], [ayc], [abz] (outer products) then the points required to be dependent are [abc], [ayz], [xbz], [xyc], obtainable from the preceding by simply interchanging a and x, b and y, c and z. By hypothesis the outer product of the vertices of T vanishes; upon expansion this gives the scalar equation

$$[xyza][aycb][cbxz] = [abcx][xbzy][zyac].$$

But one sees that the interchange which would make this equation express conclusion, in fact merely interchanges the two sides.

It was the comparison of theorem A with the following restatement of the Pappus theorem which led to the present paper.

Theorem B. Let H be a line in a commutative plane, and let T be a triple of distinct points on H. Through each point of T let two lines distinct from H be passed, and let these lines be paired by considering each pair of points in T and choosing one line through each point of the pair, in such a way that each line is used exactly once. These three pairs of lines determine a new triple of points T', and also determine a second pairing, made by choosing in each case the alternative line: these pairs determine a further triple T". If T' is a collinear triple, then so is T".

Upon comparing theorems A and B one sees that the hypothesis of B is a lowering of the dimension of that of A, with certain changes. Instead of one line through each point of T we have two, and in consequence two new triples instead of one. We may regard the conclusion as altered to the assertion that the dependence of one of these triples implies that of the other. Our principal result, theorem C, is a direct dimensional generalization of theorem B; we shall see that in odd dimensional spaces the analogous generalization of theorem A appears as a special case.

Theorem C. Let S_n be a commutative projective n-space, $(n > 1)$. In a hyperplane H_o of S_n let $T = \{t_i\}$ $(i = 1 \ldots n+1)$ be a set of $n+1$ points, no proper subset of which are dependent. Let $A_k^m = A_m^k$ $(k \neq m)$ be the subspace determined by $T - t_k - t_m$, and through each A_k^m let there be passed two hyperplanes distinct from H_o, to be denoted H_k^m and H_m^k. For each k the n hyperplanes H_k^m $(m = 1, 2, \ldots k-1, k+1 \ldots n+1)$ determine a point: let it be p_k. Also, for each m the n - hyperplanes H_k^m $(k = 1, 2 \ldots m-1, m+1 \ldots n+1)$ determine a point: let it be q_m. If now the p_k's are dependent, then the q_m's are also and the dependence is of the same rank.

The proof generalizes the following proof for $n = 2$ (theorem B). In S_2, let $e_i (i = 1, 2, 3)$ be the vertices of a triangle of reference of a homogeneous coordinate system; we write $e_1 = (1, 0, 0)$ etc. Let H_o be the line of e_1 and e_2, and let T be the following triple:

$$t_1 = e_1, \; t_2 = e_2 \cdot t_3 = \bar{e} = e_1 + e_2 \; .$$

The A_k^m's are in this case individual points; $A_1^2 = A_2^1 = \bar{e}$, etc. The H_k^m's are lines: we shall consider them to be determined by the p_k's. Since no p_k is on H_o we can write

$$\begin{aligned} p_1 &= (p_{11}, \; p_{12}, \; 1) \\ p_2 &= (p_{21}, \; p_{22}, \; 1) \\ p_3 &= (\;0\;, \;\;0\;, \; 1) \;\; = e_3 \; . \end{aligned}$$

Now H_k^m is the line represented in row k and column m of the following array:

$$\begin{pmatrix} 0 & p_1\bar{e} & p_1e_2 \\ \bar{e}p_2 & 0 & e_1p_2 \\ e_2e_3 & e_1e_3 & 0 \end{pmatrix}$$

What must be shown is that if the points determined by the rows (the p_k's) are dependent, then so are those determined by the columns (the q_m's); and, that the dependence is of the same rank. If one treats the point symbols as Grassman extensives, the q_m's are readily computed: e.g., $q_1 = [\bar{e}p_2e_3]e_2 - [\bar{e}p_2e_2]\, e_3$, which in coordinates is $(0,\ p_{21} - p_{22},\ -1)$. One finds the matrix of the q_m's to be

$$(q_{mi}) = \begin{pmatrix} 0 & p_{21}-p_{22} & -1 \\ p_{12}-p_{11} & 0 & -1 \\ p_{11} & p_{22} & 1 \end{pmatrix}.$$

It may be described as derived from the transpose of (p_{ki}) by replacing each element in the last row by the corresponding diagonal element, and then subtracting this row from each of the others. Obvious elementary transformations make this matrix identical with (p_{ki}) and the theorem follows. We shall see now that the same result is obtained for arbitrary n.

In S_n let $e_i (i = 1 \ldots n + 1)$ be the basic vectors of a homogeneous coordinate system. Let H_o be the hyperplane $[e_1e_2 \ldots e_n]$. Let $t_i = e_i (i = 1 \ldots n)$ and $t_{n+1} = \sum_{i=1}^{n} e_i = \bar{e}$.
Let

$$p_k = \sum_{i=1}^{n} p_{ki}e_i + e_{n+1} \quad (k \le n)$$

$$p_{n+1} = e_{n+1}\ .$$

The hyperplanes H_k^m are given by the outer products $H_k^m = [A_k^m\, p_k]$. For $k \le n$, $m \le n$, we can write $H_k^m = [e_1e_2 \ldots e_{k-1}p_ke_{k+1} \ldots e_{m-1}\bar{e}\, e_{m+1} \ldots e_n]$. But all the e_i's which appear in $\bar{e}$ are already present with the exception of e_k and e_m, hence we can substitute $e_k + e_m$ for $\bar{e}$ and get the array:

$$H_k^m = [e_1 \cdots e_{k-1}p_ke_{k+1} \cdots e_{m-1}(e_k + e_m)\, e_{m+1} \cdots e_n]\ (k \le n,\ m \le n)$$

$$H_k^{n+1} = [e_1 \ldots e_{k-1}p_ke_{k+1} \ldots e_n]$$

$$H_{n+1}^m = [e_1 \ldots e_{m-1}e_{m+1} \ldots e_{n+1}]\ .$$

We assert that

$$q_m = \sum_{i=1}^{n} (p_{im} - p_{ii})\, e_i - e_{n+1} \quad (m \le n)$$

$$q_{n+1} = \sum_{i=1}^{n} p_{ii}\, e_i + e_{n+1}\ .$$

We shall justify this assertion, and the theorem will then follow, as in the case n = 2, by the evident equivalence of the matrices (p_{ki}) and (q_{mi}). It is therefore only required to show that in all cases, with q_m as written, $[H_k^m q_m] = 0$. (That is, q_m must be shown to be incident with each hyperplane in column m of the array). Let us take first the case $k \leq n$, $m \leq n$ and consider the term of H_k^m containing e_k. Let us interchange p_k and e_k, which changes the sign. Since e_m has a zero coefficient in q_m the only non-vanishing term is

$$[e_1 \cdots e_{m-1} p_k e_{m+1} \cdots e_{n+1}] = p_{km} \ .$$

We consider next the term of H_k^m containing e_m. From the summation term of q_m we have a non-vanishing product only if $i = k$. Interchanging e_k and p_k now puts p_k in the $n + 1$ position and gives -1 as the coefficient of $(p_{km} - p_{kk})$. The product with $-e_{n+1}$ is $-p_{kk}$. Altogether, for these values of k and m we have

$$[H_k^m q_m] = p_{km} - (p_{km} - p_{kk}) - p_{kk} = 0.$$

The product $[H_{n+1}^m q_m]$ vanishes trivially. There remains $[H_k^{n+1} q_{n+1}]$. Arguing as above we find that the summation term of q_{n+1} gives $-p_{kk}$ while the remaining term gives $+p_{kk}$. Thus the theorem is proved.

It may be remarked that the theorem does not require the hyperplanes H_k^m and H_m^k to be distinct, and the p_k's can be chosen quite arbitrarily. If they are so chosen that the first n-rowed minor of (p_{ki}) is skew-symmetric, then $q_k = p_k$, for the transformation of (p_{ki}) which gives (q_{mi}), has, in this case, the effect of simply multiplying each row by -1. This makes $H_k^m = H_m^k$ for all k and m: that is, we have but one hyperplane through each A_k^m. In odd dimensions this skew symmetry also implies the dependence of the points represented, and thus shows that the configuration is a generalization of that of theorem A, i. e., a generalization of Möbius. This does not prove the generalizad theorem, however, which we state as follows.

<u>Theorem A'</u>. Let S_n be a commutative projective n-space with n odd. In a hyperplane H_0 of S_n let $T = \{t_i\}$ $(i = 1 \ldots n + 1)$ be a set of $n + 1$ points no proper subset of which are dependent. Let A_k^m be the subspace determined by $T - t_k - t_m$ $(k \neq m)$. Through each A_k^m let a hyperplane H_k^m, distince from H_0 be passed, and let q_m be the intersection of the n hyperplanes H_k^m $(k = 1 \ldots m - 1 \ m + 1 \ldots n + 1)$. The q_m's are dependent.

The proof is suggested by the remark above; that is, we have only to prove that however the H_k^m are chosen the matrix of the points determined has the form described, with the required skew-symmetric minor. In the terminology of theorem C, we may use n of the p's to determine the H_k^m's and then compute the q's. We shall have what we want if we let H_k^m be defined as in theorem C when $m \nless k$; and for $m > k$, put p_m for p_k. The array then becomes symmetric. We display it for $n = 3$.

$$\begin{pmatrix} 0 & [p_2\bar{e}e_3] & [p_3e_2\bar{e}] & [e_4e_2e_3] \\ [\bar{e}p_2e_3] & 0 & [e_1p_3\bar{e}] & [e_1e_4e_3] \\ [\bar{e}e_2p_3] & [e_1\bar{e}p_3] & 0 & [e_1e_2e_4] \\ [e_4e_2e_3] & [e_1e_4e_3] & [e_1e_2e_4] & 0 \end{pmatrix}$$

For $m > k$ and $m \leq n$, $k \leq n$, we have

$$H_k^m = [e_1 \cdots e_{k-1}p_m e_{k+1} \cdots e_{m-1}(e_k+e_m)e_{m+1} \cdots e_n].$$

and

$$H_k^{n+1} = H_{n+1}^k \equiv [e_1 \cdots e_{k-1}e_{n+1}e_{k+1} \cdots e_n].$$

In q_m we now put p_m for p_i when $m > i$. That is, for justification by computation, we write

$$q_m = \sum_{i=1}^{m} (p_{mm}-p_{mi}) + \sum_{i=m+1}^{n}(p_{im} - p_{ii})e_i - e_{n+1} (m \leq n).$$

$$q_{n+1} = e_{n+1}.$$

This makes the matrix (q_{mi}) take the required form. It remains to show that with these values, $[H_k^m \; q_m] = 0$.

For q_{n+1} this is trivial. For $m < k$ the computation is identical with that in theorem C. For $m > k$ we get a non-zero result from the first summation term of q_m, with $i = k$, taken with the terms of H_k^m containing e_m. The interchange of e_k and p_n shows the coefficient of $(p_{mm}-p_{mk})$ to be -1. Since the only other non-vanishing term, the product with $-e_{n+1}$, gives $p_{mm}-p_{mk}$, the theorem is proved.

It remains to show that commutativity is necessary for theorem C. For this purpose let the p_{ki} be elements of a division ring, and consider the special case

$$p_k = p_{n+1} = e_{n+1} \quad (k < n - 1)$$
$$p_{n-1} = ae_{n-1} + ce_n + e_{n+1}$$
$$p_n = e_{n-1} + be_n + e_{n+1}$$

Using the left dependence only, it can be verified that the q_m's have the form given in the proof of the theorem. That is,

$$q_m = e_{n+1} \quad (m < n - 1), \text{ and}$$
$$q_{n-1} = (1 - b)e_n - e_{n+1}$$
$$q_n = (c-a)e_{n-1} - e_{n+1}$$
$$q_{n+1} = ae_{n-1} + be_n + e_{n+1}$$

One sees that this is essentially the case $n = 2$ transferred to the plane of e_{n-1}, e_n, e_{n+1}. Thus, for example, q_n must be shown to be left dependent with p_{n-1} and $\overline{e}$. This reduces to showing the left dependence of the vectors (a, c, 1), (1, 1, 0) and (c-a, 0, -1), which is a simple matter. Similarly for the other combinations which can be read off from the array for n = 2. We assume now, in accordance with the theorem, that the left dependence of the p_k's implies that of the q_m's. This amounts to the assertion of the Pappus theorem in this plane, and our conclusion follows. However, a few more sentences will make the discussion complete. The p_k's are left dependent if there exist elements x, y such that xa + y = 0 and xc + yb = 0. This gives xc = xab whence c = ab. For the q_m's we must find x, y, z such that y(c - a) + za = 0, x(1 - b) + zb = 0 and x + y - z = 0. Eliminating z we have y c = -xa and x = -yb, whence c = ba and therefore ab = ba.

Finally, it should be remarked that theorem C is not all that one would wish, since it is not self-dual. In four-space for example, it involves 15 points and 23 hyperplanes. One feels that there should be a self-dual theorem of this type.

Bibliography

(1) Jahreshericht der Deutschon Mathematiker Vereinigung, 1931; 4 Band, 9-12 Heft., p. 48.

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Note: The restatement of the Möbius theorem (Theorem A above) which led to this paper is to be found in Baker, PRINCIPLES OF GEOMETRY, Vol. 1, 1922, p. 62. It was arrived at independently by Mr. Wassel Al-Dhahir, a graduate student at the University of Michigan. As this paper goes to press, Mr. Al-Dhahir informs me that he has a promising n-dimensional self-dual generalization which specializes to give Theorem C of this paper, but he is not yet ready to announce a proof.