REMARKS ON A PREVIOUS PAPER[1]

by

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1. Raoul Bott suggested that a proof of Theorems 1 and 2 of [3] which avoided metrical considerations might show the duality relation of Theorem 2 as a consequence of standard duality theorems for convex cones. This method is followed here[2]; it will be seen to shorten the proofs almost to nothing.

It is convenient to take Theorem 1 in a slightly generalized form: <u>Let the cones</u> $U \subset E^k$ <u>and</u> $V \subseteq E^n$ <u>satisfy</u> $D\{U\} = E^k$ <u>and</u> $(D\{V^+\}) = E^n$. <u>Then</u> $AU = V \cap D\{AU\}$ <u>implies</u> $A'V^+ = U^+ \cap D\{A'V^+\}$.

Proof. First, $AU = V \cap D\{AU\}$ gives $(AU)^+ = V^+ + (D\{AU\})^+$; therefore $A'(AU)^+ = A'V^+ + A'(D\{AU\})^+$. Now for any $W \subseteq E^k$ one has $A'(AW)^+ = W^+ \cap A'E^n$. Also in the present case $D\{AU\} = AD\{U\} = AE^k$ and $D\{A'V^+\} = A'D\{V^+\} = A'E^n$. Therefore $A'(AU)^+ = U^+ \cap D\{A'V^+\}$ and $A'V^+ + A'(D\{AU\})^+ = A'V^+ + E^{k+} \cap A'E^n = A'V^+$. The statement has been proved.

Furthermore, $A'V^+$ <u>is affine-equivalent to the geometric polar of</u> AU.

Proof: The latter is $(AU)^+ \bmod (D\{AU\})^+ = (AU)^+ \bmod (AE^k)^+$. Now A' can be considered as defined on $E^n \bmod (AE^k)^+$, since $(AE^k)^+$ is its null-space; so considered, it is one-one, and an affine isomorphism.

Thms. 1 and 2 follow from the above by setting $U = P^k = U^+$, $V = P^n = V^+$; indeed, $D\{P^k\} = E^k$, $D\{P^n\} = E^n$, as required.

2. The following simple construction seems, surprisingly enough, to be new.

Theorem 3. Every pointed convex polyhedral cone is affine-equivalent to the intersection of the positive orthant (in space of appropriate dimension) with a linear subspace.

Proof. Let the cone be $AP^k \subseteq E^n$ (where A may be chosen so that extreme rays of P^k go into extreme rays of AP^k). Consider $A'E^n \cap P^k$. This cone may be represented in the form BP^m (with extreme rays of P^m going into extreme rays of BP^m). Now $BP^m = P^k \cap D\{BP^m\}$, so $B'P^k$

[1] See [3]. The present note follows the terminology and notation of [1] and [3]. Numbers in brackets refer to the Bibliography.

[2] The positive polar of a cone may be regarded as a cone in the dual space, in which case all the proofs in this paper are of affine character. But the dual space is not distinguished notationally.

$= P^m \cap D\{B'P^k\}$ by Theorem 1. The latter cone will be shown affine-equivalent to AP^k. For this it is enough to show A and B' have the same null-space, that is, to show A' and B have the same range: $BE^m = A'E^n$.

Now $BP^m = A'E^n \cap P^k$ implies $BE^m = D\{BP^m\} \subseteq A'E^n$, and equality fails only if some element of $A'E^n$ fails to be a difference of elements of $A'E^n \cap P^k$. This will be ruled out if $A'E^n$ intersects the interior of P^k, as one can check without difficulty; that is, if for some x we have that $y \in P^k$ implies $y'A'x > 0$. But the requirement that AP^k be pointed implies the existence of x such that $Ay \in AP^k$ implies $x'Ay > 0$. This completes the proof.

The point of interest in Theorem 3 is not the fact, but the construction. The fact is essentially the dual of the standard

"Theorem" 4. Every convex polyhedral cone is affine-equivalent to a projection of a positive orthant; that is, it is a linear map of a positive orthant.

Duality proof of Theorem 3. Write the positive polar of the given cone as $CP^m \subseteq E^n$. The fact that the given cone $(CP^m)^+$ is pointed means that $E^n = D\{CP^m\} = CE^m$, which in turn means C' has zero null-space. But C' carries $(CP^m)^+$ onto $P^m \cap C'E^n$, a cone of the desired sort. This completes the proof.

The construction in the first proof is actually simpler. To clarify this it will be stated in terms of matrices. If the columns of the matrix A, considered as vectors, lie along the extreme rays of the given cone, then the desired cone is constructed as follows. Find a matrix B whose columns are linear combinations of the rows of A (and vice versa) and which satisfies conditions (1), (2), (3) of [3]. The rows of B, considered as vectors, lie along the extreme rays of the given cone. This construction involves (in finding B) an inequality system in which every inequality says a number is non-negative, and only the equations are non-trivial. This is simpler than the more general inequality system to be solved in finding C in the second proof.

3. These theorems about pointed cones can be restated as theorems about convex polytopes in the usual way: the intersection of a pointed cone in E^n with an (n-1)-space intersecting the interior of each of its facets, is a convex polytope, and all convex polytopes are obtained in this way. Thus we have these elementary remarks, which seem not to have been made before[3]:

Theorem 3'. Every convex polytope is congruent to a section of a simplex. To get (metric) congruence here one may have to take a simplex which is not regular. The simplex may be chosen to have the same number

[3] Proofs are given for special cases in [4], §§ 1.6, 1.13 and elsewhere.

of faces as the polytope.

Theorem 4'. Every convex polytope is obtainable by orthogonal projection of a simplex. Again the simplex may not be regular. It may be chosen to have the same number of vertices as the polytope.

Bibliography

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