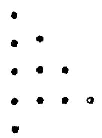
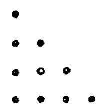
A COMBINATORIAL PROBLEM [1]

by R. M. Thrall

1. Introduction.

Let m be a natural number and let (a) = ($a_1, \ldots,$ a_k), $a_1 > a_2 > \ldots > a_k > 0$, be a partition of m into unequal parts. We associate with the partition (a) a diagram D(a) of m nodes (or places) having a_1 rows and k columns, so arranged that the j-th column has a_j nodes, and that the top node of the j-th column is in the j-th row (numbered from the top down).
For example,

D(5, 3, 2)=  and D(4, 3, 2, 1)=  .

A one-to-one mapping of the nodes of D(a) onto the set of natural numbers 1, ..., m is called a <u>labelling</u> of D(a) and is indicated by writing in each place of D(a) its image under the mapping. A labelling is said be be <u>regular</u> if the "labels" increase in each row when read from left to right and increase in each column when read from top to bottom. Thus the first of the two labellings below is regular whereas the second is not:

```
1                  1
2 4                3 2
3 5 7        ,     4 7 10     ,
6 8 9 10           6 5 8 9
```

We denote by g(a) the number of regular labellings of D(a). The main result in this note is a formula for g(a). (See Theorem 1). The problem of

---

[1] The work on this paper was performed under the sponsorhip of the ONR.

determining g(a) is similar in many respects to the problem of determining the degrees of the irreducible representations of the symmetric group [2.p.67] and the formula obtained for g(a) is analogous to the degree formula. In the final section there is a discussion of the connection between the special case (a) = (k, k-1, ..., 1) of this problem and a phase of the theory of psychological measurement. It was this connection which first led to the consideration of the special case and attempts to handle the special case lead to the general one.

2. Combinatorial preliminaries.

A node which lies at the bottom of its column and at the right end of its row is said to be a <u>corner node</u>. If a corner node lies in the j-th column of D(a), its removal gives the diagram $D(a^{(j)})$ where

$$(a^{(j)} = (a_1, \ldots, a_{j-1}, a_j - 1, a_{j+1}, \ldots a_k),$$

i.e., the j-th component of (a) is decreased by 1. [In particular if $j = k$ and $a_k = 1$, (a^k) consists merely of the first k-1 components of (a)].

Let L(a) be any regular labelling for D(a). The regularity of the labelling requires that the label m shall be assigned to a corner node. Moreover, if removal of this corner node gives $D(a^{(j)})$, the labelling $L(a^{(j)})$ induced by L(a) is regular. Conversely, if $L(a^{(j)})$ is any regular labelling for $D(a^{(j)})$ then addition of a node labelled m at the the bottom of the j-th column will give a regular labelling L(a). This argument establishes the following lemma.

<u>Lemma 1.</u> <u>Let</u> $a = (a_1, \ldots, a_k)$ <u>where</u> $a_1 > \ldots > a_k$. <u>Then</u>

(1) $$g(a) = \sum{}' g(a(j))$$

<u>where $\sum'$ denotes the summation over all j for which the bottom node of the j-th column of D(a) is a corner node.</u>
(The results of this section are precisely analogous to those in the symmetric group case [2. p. 67 ff.])

3. The formula for g(a).

For any vector $x = (x_1, \ldots x_k)$ with components in the field of complex numbers we define

(2) $$\Delta(x) = \prod_{i>j} (x_i - x_j)$$

and

(3) $$\nabla(x) = \prod_{i>j} (x_i + x_j) \ .$$

(The product $\Delta(x)$ can also be characterized as the Van der Monde determinant det $\|a_{ij}\|$ where $a_{ij} = x_i^{j-1}$.)

<u>Theorem 1. Let</u> $a_1, \ldots, a_k$ <u>be natural numbers such that</u> $a_1 > a_2 > \ldots a_k > 0$. <u>Then we have</u>

(4) $$g(a) = \frac{(a_1 + \ldots + a_k)!}{a_1! \ldots a_k!} \frac{\Delta(a)}{\nabla(a)} \ .$$

<u>Proof</u>: We denote the function on the right hand side of (4) by G(a), and consider the domain of G(a) to be all vectors (x) whose components are non negative integers. We first observe that G(x) = 0 if any two components of x are equal. Next, if $(x) = (x_1, \ldots, x_k)$ is any vector we define $x(j)$ to be the vector $(x_1 \ldots x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_k)$, $(j = 1, \ldots, k)$. Now if $a_1 > \ldots > a_k > 0$ then either the node at the bottom of the j-th column of D(a) is a corner node or $a_j = a_{j+1} + 1$ so that $G(a(j)) = 0$. This establishes the following lemma.

<u>Lemma 2.</u> <u>For any</u> $a_1 > \ldots > a_k > 0$ <u>we have</u>

$$(5) \qquad {\sum}' G(a^{(j)}) = \sum_{j=1}^{k} G(a^{(j)}) .$$

The equality $g(b) = G(b)$ is readily checked for small values of $b_1 + \ldots + b_\ell$. We now take as induction hypothesis that $g(b) = G(b)$ for all $b_1 > \ldots > b_\ell > 0$, having $b_1 + \ldots + b_\ell < m$, and let $(a) = (a_1, \ldots, a_k)$, $a_1 > \ldots > a_k > 0$, be a fixed partition of m. Now by Lemma 1 and Lemma 2 we have

$$(6) \qquad g(a) = {\sum}' g(a^{(j)}) = {\sum}' G(a^{(j)}) = \sum_{j=1}^{k} G(a^{(j)}).$$

Our theorem will then follow if we can show that

$$(7) \qquad G(a) = \sum_{j=1}^{k} G(a^{(j)}) .$$

<u>Lemma 3.</u> <u>For all vectors</u> $(x) = (x_1, \ldots, x_k)$ <u>with distinct (complex) components we have</u>

$$(8) \qquad \Delta(x^{(j)}) = \Delta(x) \prod_{v \neq j} \frac{x_v - x_j + 1}{x_v - x_j}$$

and

$$(9) \qquad \nabla(x^{(j)}) = \nabla(x) \prod_{v \neq j} \frac{x_v + x_j - 1}{x_v + x_j}$$

Using (8) and (9) we get

$$(10) \qquad G(a^{(j)}) = \frac{(m-1)!\, a_j}{a_1! \ldots a_k!} \cdot \frac{\Delta(a)}{\nabla(a)} \prod_{v \neq j} \frac{(a_v - a_j + 1)(a_v + a_j)}{(a_v - a_j)(a_v + a_j - 1)}$$

$$= G(a) \frac{a_j}{a_1 + \ldots + a_k} \prod_{v \neq j} \frac{(a_v - a_j - 1)(a_v + a_j)}{(a_v - a_j)(a_v + a_j - 1)}$$

$$= G(a)\, h_j(a)$$

Next set

$$(11) \qquad h(x) = \frac{1}{x_1 + \ldots + x_k} \sum_{j=1}^{k} x_j \prod_{v \neq j} \frac{(x_v - x_j + 1)(x_v + x_j)}{(x_v - x_j)(x_v + x_j - 1)}$$

$$= \sum_{j=1}^{k} h_j(x),$$

and we have from (10) that

(12) $$\sum_{j=1}^{k} G(a^{(j)}) = G(a)h(a)$$

Hence to complete the proof of (7) and therefore of Theorem 1 it is sufficient to establish the following lemma.

Lemma 4. For all vectors (x) with complex components

(13) $$h(x) = 1 .$$

We first remark that h(x) is a rational symmetric function of degree zero in $x_1, \ldots, x_k$. We next show each potential factor in the denominator for h(x) is actually cancelled out when the summands $h_j(x)$ are added together.

For a pair i, j with $i \neq j$ the factors $(x_i - x_j)$ and $(x_i + x_j - 1)$ appear only in the denominators of the two summands $h_i(x)$ and $h_j(x)$. Let

$$f_{ij}(x) = \prod_{v \neq i, j} \frac{(x_v - x_i + 1)(x_v + x_j)}{(x_v - x_i)(x_v + x_j - 1)} .$$

The identity

$$(z - x_i + 1)(z + x_i) - (z - x_j + 1)(z + x_j) = -(x_i - x_j)(x_i + x_j - 1)$$

shows that $f_{ij}(x) - f_{ji}(x)$ is divisible by $(x_i - x_j)(x_i + x_j - 1)$, say,

$$f_{ij}(x) - f_{ji}(x) = (x_i - x_j)(x_i + x_j - 1) f_{ij}^*(x)$$

where $f_{ij}^*(x)$ is a rational function whose denominator does not contain either $(x_i - x_j)$ or $(x_i + x_j - 1)$ as a factor. Now we have

$$(x_1 + \ldots x_k)[h_i(x) + h_j(x)]$$

$$= \frac{x_i(x_j-x_i+1)(x_i+x_j)}{(x_j-x_i)(x_j+x_i-1)} f_{ij}(x) + \frac{x_j(x_i-x_j+1)(x_i+x_j)}{(x_i-x_j)(x_i+x_j-1)} f_{ji}(x)$$

$$= \frac{(x_i+x_j)}{(x_i-x_j)(x_i+x_j-1)} + \Big\{ -x_i(x_j-x_i+1)[f_{ji}(x)+(x_i-x_j)(x_i+x_j-1)$$

$$f_{ij}^*(x)] + x_j(x_i-x_j+1) \; f_{ji}(x) \Big\}$$

$$= (x_i+x_j)\, x_i(x_j-x_i+1) \; f_{ij}^*(x) + (x_i+x_j) \; f_{ji}(x),$$

which shows that $(x_i-x_j)(x_i+x_j-1)$ cancels from the denominator of $h_i(x) + h_j(x)$ and, therefore, from the denominator of $h(x)$.

Thus we have

$$h(x) = \frac{h^*(x)}{x_1+\ldots+x_k}$$

where $h^*(x)$ is a symmetric polynomial of degree one and therefore $h^*(x) = c(x_1+\ldots+x_k)$ and hence $h(x) = c$ for all (x). To caluclate c consider $\lim\limits_{x_1\to\infty} h(x)$. We have $\lim\limits_{x_1\to\infty} h_i(x) = 0$ if $i \neq 1$ and $\lim\limits_{x_1\to\infty} h_1(x) = 1$. Hence, $c = 1+0+0 \ldots +0 = 1$. This completes the proof of Lemma 4 and Theorem 1 follows.

4. A special case and a related geometric problem.

If (a) is the partition $(k, k-1, \ldots, 1)$ then $m = k(k+1)/2$, $\Delta(a) = 1!2! \ldots (k-1)!$ and

$$\nabla(a) \cdot 1!2! \ldots k! = (2k-1)! \cdot (2k-3)! \ldots 3!1!$$

Hence*,

---

*Formula (13) was suggested by D. Mela on the basis of empirical study.

(13) $$g(k, \ldots, 1) = g_k = \frac{[k(k+1/2]!(k-1)! \ldots 1!}{(2k-1)!(2k-3) \ldots 1!} .$$

Now consider the following situation. Let $(x) = (x_o, x_1, \ldots, x_k)$ be a vector with real components such that $x_o < x_1 < \ldots < x_k$, and let P_i be the point on the real line with coordinate $x_i (i = 0, \ldots, k)$. Let Q_{ij} be the midpoint of P_i and P_j $(0 \leq i \leq j \leq k)$, and let x_{ij} be the coordinate of Q_{ij}. We suppose the x_j so chosen that the $m = k(k+1)/2$ points Q_{ij} are distinct, and let them be enumerated as they lie from left to right, say $d_x(i, j)$ is the rank order of Q_{ij}.

Next consider the diagram D_k of the partition $(k, \ldots, 1)$. We may designate the j-th node in the (i+1)-st column of D_k by the symbol (i, j) $(0 \leq i \leq j \leq k)$. Then the mapping $(i, j) \rightarrow d_x(i, j)$ is a regular labelling for D_k and hence g_k is an upper bound for the number of possible orderings for the midpoints Q_{ij}. It is an open question whether every regular labbeling for D_k can be obtained as a "midpoint" labelling.

The midpoint problem had its origins in psychological research and was proposed by C. H. Coombs [1]. Suppose that k+1 individuals called stimuli possess some attribute, the amount of which can be measured by a real number, say, stimulus i has amount x_i of the given attribute. We suppose the stimuli so ordered that $x_o < x_1 < \ldots < x_k$. A further individual, called a judge, who possesses x of the given attribute is asked to rank order the stimuli by mapping $i \rightarrow d_i$ such that the numbers $|x - x_{d_i}|, \ldots, |x - x_{d_k}|$ will be in increasing order. Suppose $i < j$. Clearly $|x - x_i| <$

$|x-x_j|$ or $|x-x_i| > |x-x_j|$ according as $x_{ij} > x$ or $x_{ij} < x$, respectively. The basic problem is to see how much information can be obtained about the numbers $x_0, \ldots, x_k$ given only the rank orders provided by some set of judges, and the midpoint problem is one of the sub-problems of the basic problem.

Bibliography

1. Coombs, C.H., "Psychological Scaling Without a Unit of Measurement", Psychological Review, 1950, Vol. 57, pp. 145-158.

2. Littlewood, D.E., The Theory of Group Characters, Oxford (1940).

University of Michigan November, 1951.