COHOMOLOGY THEORY IN TOPOLOGICAL GROUPS

by

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Introduction. For a topological group Q, there are two obviously different cohomology theories which have been established in the mathematical literature, namely, the cohomology theory of Q as a topological space and that of Q as an abstract group [6] [2]. The initial purpose of the present work is to study the possible relations between these theories.

In Chapter I, for a topological group Q which operates on a topological abelian group G, three kinds of cohomology groups of Q over the coefficient group G are introduced, namely, the cohomology groups, the cohomology groups with empty supports, and the reduced cohomology groups. The methods used here are more or less analogous to those of Eilenberg and MacLane [6]. The first kind of the cohomology groups of Q over G reduce to the cohomology groups of the abstract group Q over G if Q is discrete, while the reduced cohomology groups of Q over G are closely related to those of Q as a topological space. In fact, it is proved in Chapter II that, if Q is compact and connected and G is a finite dimensional vector group on which Q operates simply, then the reduced cohomology groups of Q over G are isomorphic with the Cech cohomology groups of Q as a topological space over the abstract group G.

In Chapter III, we define a local cohomology theory of a local group Q over a local abelian group G on which Q operates. It is proved that, if a topological group Q has the local extension property with respect to a topological abelian group G (see §12), then the reduced cohomology groups of Q over G are isomorphic with the local cohomology groups of the local group Q over the local group G for the dimensions greater than unity. As an application of this result, we have generalized the classical decomposition theorem of E. Cartan [3] for compact connected Lie groups to the category of compact connected groups with Lie centers, (see Theorems 15.1 and 16.2).

Chapter I. Cohomology Groups of a Topological Group.

Throughout the present chapter we assume that Q is a topological group, written multiplicatively, and G is an abelian topological group, written additively. Further, we shall also assume that Q operates on the left of G. By this we mean that for each $x \in Q$ and $g \in G$ there is determined an element $xg \in G$, subject to the following conditions:

(i) xg is continuous in x and g simultaneously,
(ii) $x(g_1+g_2) = xg_1+xg_2$,
(iii) $x_1(x_2g) = (x_1x_2)g$,
(iv) $1g = g$.

If for every $x \in Q$ and $g \in G$ we have $xg = g$ then we say that Q operates on G simply.

1. The Homogeneous Approach

Q^{p+1} will denote the (p+1)-fold product of Q with itself; that is, Q^{p+1} is the set of points $(x_0, x_1, \ldots, x_p)$ where $x_i \in Q$, $i = 0, \ldots, p$. Q^{p+1} forms a topological group in the usual way.

For each integer $p \geq 0$, let us consider the continuous maps $F: Q^{p+1} \to G$. Such a map F will be called a p-dimensional cochain of Q over G if it satisfies the following homogeneity condition:

(1.1) $$F(xx_0, \ldots, xx_p) = xF(x_0, \ldots, x_p).$$

By means of functional addition, the totality of the p-dimensional cochains of Q over G form an additive abelian group denoted by $C^p(Q, G)$, $p = 0, 1, \ldots$. Throughout the paper, we shall not consider any topology in the groups of cochains or the cohomology groups.

For each $F \in C^p(Q, G)$ we define a continuous map $\delta F: Q^{p+2} \to G$ by taking [3)]

(1.2) $$(\delta F)(x_0, \ldots, x_{p+1}) = \sum_{i=0}^{p+1} (-1)^i F(x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}).$$

The following properties of the operation δ can be easily verified:

(1.3) δF is a (p+1)-dimensional cochain,

(1.4) $$\delta(F_1 + F_2) = \delta F_1 + \delta F_2,$$

(1.5) $$\delta(\delta F) = 0.$$

Hence the operator δ will be called the coboundary operator and the cochain δF will be called the coboundary of F. The p-dimensional cochains F with $\delta F = 0$ are called p-dimensional cocycles; they form a subgroup $Z^p(Q, G)$ of $C^p(Q, G)$. If $p > 0$, the p-dimensional cochains such that $F = \delta F'$ for some $F' \in C^{p-1}(Q, G)$ are called p-dimensional coboundaries: they form a subgroup $B^p(Q, G)$ of $C^p(Q, G)$. If $p = 0$, we define formally $B^0(Q, G) = 0$. The property (1.5) implies that $B^p(Q, G)$ is a subgroup of $Z^p(Q, G)$. The p-dimensional cohomology group $H^p(Q, G)$ of Q over G is defined as the quotient group [4)]

(1.6) $$H^p(Q, G) = Z^p(Q, G) / B^p(Q, G).$$

For each integer $p \geq 0$ and each cochain F $C^p(Q, G)$, we shall define the support of F as follows.

$S(F)$ is the subset of Q defined by the condition:

(1.7) $x \in Q$ is not contained in $S(F)$ if and only if there exists a neighborhood U of x in Q such that $F(x_0, \ldots, x_p) = 0$ whenever $x_i \in U$ for each $i = 0, \ldots, p$.

The homogeneous condition (1.1) implies that $S(F)$ is either Q or the vacuous set $\Box$. It follows also from (1.1) that

(1.8) $S(F) = \Box$ if and only if there exists a neighborhood U of the identity element 1 in Q such that $F(x_0, \ldots, x_p) = 0$ whenever $x_i \in U$ for each $i = 0, 1, \ldots, p$.

The cochains $F \in C^p(Q, G)$ such that $S(F) = \Box$ form a subgroup $C^p_\Box(Q, G)$, called the group of p-dimensional cochains with empty supports. Let $Z^p_\Box(Q, G) = Z^p(Q, G) \cap C^p_\Box(Q, G)$. If $p > 0$, we define $B^p_\Box(Q, G) = \delta C^{p-1}_\Box(Q, G)$; if $p = 0$, we set $B^0_\Box(Q, G) = 0$. It follows immediately from (1.8) that δ maps $C^{p-1}_\Box(Q, G)$ into $C^p_\Box(Q, G)$ for each $p > 0$. Hence $B^p_\Box(Q, G)$ is a subgroup of $Z^p_\Box(Q, G)$ for every $p \geq 0$. The quotient group

(1.9) $$H^p_\Box(Q, G) = Z^p_\Box(Q, G)/B^p_\Box(Q, G)$$

will be called the p-dimensional cohomology group with empty supports of Q over G.

For each integer $p \geq 0$, let $C^p_*(Q, G)$ denote the quotient group of $C^p(Q, G)$ over $C^p_\Box(Q, G)$. Since the coboundary operator δ in $C^p(Q, G)$ maps $C^p_\Box(Q, G)$ into $C^{p+1}_\Box(Q, G)$, it induces a unique coboundary operator $\delta^*: C^p_*(Q, G) \to C^{p+1}_*(Q, G)$ satisfying (1.4) and (1.5). The elements of $C^p_*(Q, G)$ will be called the p-dimensional reduced cochains of Q over G. By the procedures used above, one can define the group $Z^p_*(Q, G)$ of p-dimensional reduced cocycles and the group $B^p_*(Q, G)$ of p-dimensional reduced coboundaries. The quotient group

(1.10) $$H^p_*(Q, G) = Z^p_*(Q, G)/B^p_*(Q, G)$$

will be called the p-dimensional reduced cohomology group of Q over G.

Let us denote respectively by

$$\iota : C^p_{\square}(Q, G) \to C^p(Q, G),$$
$$\pi : C^p(Q, G) \to C^p_{*}(Q, G)$$

the natural inclusion and projection homomorphisms. Since both ι and π commute with the coboundary operator δ, ι and π induce homomorphisms

$$\iota^{*} : H^p_{\square}(Q, G) \to H^p(Q, G),$$
$$\pi^{*} : H^p(Q, G) \to H^p_{*}(Q, G)$$

for each integer $p \geqq 0$. We are going to define a homomorphism

$$\delta^{*} : H^p_{*}(Q, G) \to H^{p+1}_{\square}(Q, G)$$

for every $p \geq 0$ as follows. Let α be an arbitrary element of $H^p_{*}(Q, G)$. Choose a reduced cocycle $F_{*} \in C^p_{*}(Q, G)$ which represents α. Since π maps $C^p(Q,G)$ onto $C^p_{*}(Q, G)$, there is a cochain $F \in C^p(Q, G)$ with $\pi F = F_{*}$. Since $\pi \delta F = \delta \pi F = \delta F_{*} = 0$, we have $\delta F \in Z^{p+1}_{\square}(Q, G)$. Hence δF represents an element β of $H^{p+1}_{\square}(Q, G)$. It is not difficult to see that β depends only on α. We define the homomorphism δ^{*} by taking $\delta^{*}(\alpha) = \beta$.

The following theorem is a special case of a general theorem of Kelley and Pitcher [13].

Theorem 1.11. <u>The sequence of groups and homomorphisms</u>

$$H^0_{\square}(Q, G) \xrightarrow{\iota^{*}} \cdots \xrightarrow{\delta^{*}} H^p_{\square}(Q, G) \xrightarrow{\iota^{*}} H^p(Q, G) \xrightarrow{\pi^{*}}$$
$$H^p_{*}(Q, G) \xrightarrow{\delta^{*}} H^{p+1}_{\square}(Q, G) \xrightarrow{\iota^{*}} \cdots$$

<u>is exact in the sense that the image of each homomorphism coincides with the kernel of the following one.</u>

2. Two Special Cases

For the first special case, let us assume that the topological group Q is discrete. Then, for each $p \geqq 0$, every function $F: Q^{p+1} \to G$ is a continuous map. Hence our definition of the cohomology group $H^p(Q, G)$ reduces to that of Eilenberg and Maclane [6]. In this particular case, the topology of G has no significance. For each $p \geqq 0$, the following assertion is immediate.

(2.1) For a discrete group Q, $F \in C^p_{\square}(Q, G)$ if and only if $F(x_0, \ldots, x_p) = 0$ whenever $x_i = 1$ for each $i = 0, \ldots, p$.

Now, for each $p \geqq 0$, we are going to construct a homomorphism $h_p: G \to C^p(Q, G)$ by taking

(2.2) $$h_p(g)(x_0, \ldots, x_p) = x_0 g.$$

Obviously h_p is an isomorphism of G into $C^p(Q, G)$. Let

$$C^p_{\#}(Q, G) = h_p(G).$$

Then $C^p_{\#}(Q, G)$ is a subgroup of $C^p(Q, G)$ and clearly we have

(2.3) $$C^p_{\square}(Q, G) \cap C^p_{\#}(Q, G) = 0.$$

Lemma 2.4. If Q is discrete, then $C^p(Q, G)$ is the direct sum of $C^p_{\square}(Q, G)$ and $C^p_{\#}(Q, G)$.

Proof. Let $F \in C^p(Q, G)$ be an arbitrary cochain. Call $g = F(1, \ldots, 1)$. It follows from (2.2) and (2.1) that $F - h_p(g)$ is in $C^p_{\square}(Q, G)$. Hence $C^p_{\#}(Q, G)$ is the sum of $C^p_{\square}(Q, G)$ and $C^p_{\#}(Q, G)$. (2.3) implies that the sum is direct. Q.E.D.

Corollary 2.5. If Q is discrete, then the projection homomorphism π maps the subgroup $C^p_{\#}(Q, G)$ isomorphically onto $C^p_{*}(Q, G)$.

Theorem 2.6. If Q is discrete, then we have $H^0_{*}(Q, G) \approx G$ and $H^p_{*}(Q, G) = 0$ for all $p > 0$.

Proof. It follows from (2.5) and the definition of $C^p_{\#}(\overline{Q, G})$ that $C^p_*(Q, G) \approx G$. Let $F_* \in C^o_*(Q, G)$ be arbitrarily given. By (2.5), there is an $F_{\#}$ of $C^o_{\#}(Q, G)$ with $\pi F_{\#} = F_*$. Since

$$\delta F_{\#}(1, 1) = F_{\#}(1) - F_{\#}(1) = 0,$$

it follows that $\delta F_{\#} \in C^1_{\square}(Q, G)$ by (2.1). Hence

$$\delta F_* = \delta \pi F_{\#} = \pi \delta F_{\#} = 0.$$

This implies that $Z^o_*(Q, G) = C^o_*(Q, G)$. Since, by definition, $B^o_*(Q, G) = 0$, we conclude $H^o_*(Q, G) \approx G$.

Now assume $p > 0$. Let $\alpha \in H^p_*(Q, G)$ be an arbitrary element. Choose a reduced cocycle $F_* \in C^p_*(Q, G)$ which represents α. By (2.5), there is an $F_{\#} \in C^p_{\#}(Q, G)$ with $\pi F_{\#} = F_*$. By the definition of $C^p_{\#}(Q, G)$, there is a $g \in G$ such that $F_{\#} = h_p(g)$. Since $\delta F_* = 0$, we have $\pi \delta F_{\#} = \delta \pi F = \delta F_* = 0$. This implies $\delta F_{\#} \in C^{p+1}_{\square}(Q, G)$. If p is odd, then

$$\delta F_{\#}(1, \ldots, 1) = \sum_{i=0}^{p+1} (-1)^i g = g.$$

Since $\delta F_{\#} \in C^{p+1}_{\square}(Q, G)$, we must have $g = 0$. Hence $F_* = \pi F_{\#} = \pi h_p(0) = 0$. This proves $\alpha = 0$ for the case that p is odd. If p is even, let $F' = h_{p-1}(g)$. Then we have $\delta F'(1, \ldots, 1) = g$. This implies that the cochain $F_{\#} - \delta F'$ is in $C^p(Q, G)$ by (2.1). Hence

$$F_* - \delta \pi F' = \pi (F_{\#} - \delta F') = 0.$$

This proves $F_* = \delta \pi F'$ and $\alpha = 0$. Hence $H^p_*(Q, G) = 0$ for every $p > 0$. Q.E.D.

From Theorem 2.6 and the assertion (4.1) given in §4 below, we deduce the following corollary. We denote by G^* the subgroup of G consisting of those elements g of G such that $xg = g$ for all $x \in Q$.

Corollary 2.7. <u>If Q is discrete, the homomorphism</u>

$$\iota^* : H^p_{\square}(Q,G) \to H^p(Q,G)$$

is an isomorphism onto for each $p \geq 2$. When $p = 1$, ι^* is a homomorphism onto with a kernel isomorphic to the quotient group G/G^*.

For the second special case, let us assume that the topological group Q is compact and the topological abelian group G is a finite dimensional vector group with the euclidean topology on which Q operates by means of a representation of Q with G as the representation space.

There is a Haar measure in Q with the measure of Q being 1, [22, p. 38].

Theorem 2.8. If Q is compact and G is a finite dimensional vector group, then we have $H^p(Q,G) = 0$ for all $p > 0$.

Proof. Let α denote an arbitrary element of $H^p(Q,G)$, $p > 0$. Choose a cocycle $F \in Z^p(Q,G)$ which represents α. Then F is a continuous map of Q^{p+1} into G satisfying (1.1). Define a continuous map Φ : $Q^p \to G$ by taking for each point $(x_0, \ldots, x_{p-1}) \in Q^p$

$$\Phi(x_0, \ldots, x_{p-1}) = \int_Q F(x, x_0, \ldots, x_{p-1})dx.$$

Then it follows from (1.1) and the left invariance of the Haar measure that for each $y \in Q$ we have

$$\Phi(yx_0, \ldots, yx_{p-1}) = \int_Q F(x, xy_0, \ldots, yx_{p-1})dx$$

$$= \int_Q F(yx, yx_0, \ldots, yx_{p-1})dx = \int_Q yF(x, xo, \ldots, x_{p-1})dx$$

$$= y\int_Q F(x, x_0, \ldots, x_{p-1})dx = y\Phi(x_0, \ldots, x_{p-1}).$$

Hence Φ satisfies (1.1), that is to say, Φ is a (p-1)-cochain.

Since F is a cocycle, we deduce from (1.2) that, for each $x \in Q$ and $(x_0, \ldots, x_p) \in Q^{p+1}$, we have

$$\delta F(x, x_0, \ldots, x_p) = F(x_0, \ldots, x_p) - \sum_{i=0}^{p}(-1)^i F(x, x_0, \ldots, \hat{x}_i, \ldots, x_p) = 0.$$

Hence we have

$$\begin{aligned} F(x_0, \ldots, x_p) &= \sum_{i=0}^{p}(-1)^i F(x, x_0, \ldots, \hat{x}_i, \ldots, x_p) \\ &= \int_Q \sum_{i=0}^{p}(-1)^i F(x, x_0, \ldots, \hat{x}_i, \ldots, x_p)dx \\ &= \sum_{i=0}^{p}(-1)^i \int_Q F(x, x_0, \ldots, \hat{x}_i, \ldots, x_p)dx \\ &= \sum_{i=0}^{p} \Phi(x_0, \ldots, \hat{x}_i, \ldots, x_p) = \delta\Phi(x_0, \ldots, x_p). \end{aligned}$$

This proves that F is a coboundary. Hence $\alpha = 0$. Q. E. D.

Corollary 2.9. <u>Under the assumptions of Theorem 2.8, the homomorphism $\delta^*: H^p_*(Q, G) \to H^{p+1}_\square(Q, G)$ is an isomorphism onto for each $p \geq 1$. When $p = 0$, δ^* is a homomorphism onto with a kernel isomorphic to G^*.</u>

3. The non-Homogeneous Approach

The cochains in §1 may be called homogeneous because of the homogeneity condition (1.1). The various cohomology groups of Q over G may be defined by means of non-homogeneous cochains which will be described as follows.

For each integer $p > 0$, we define the group $\tilde{C}^p(Q, G)$ of p-dimensional non-homogeneous cochains of Q over G to be the set of all continuous maps $f: Q^p \to G$ with the functional addition as the group operation. For $p = 0$, we set $\tilde{C}^0(Q, G) = G$. For each $p \geq 0$, we shall define a coboundary operator

$$\delta: \tilde{C}^p(Q, G) \to \tilde{C}^{p+1}(Q, G)$$

as follows. If $p > 0$, the coboundary δf of the cochain $f \in \tilde{C}^p(Q, G)$ is a cochain $\delta f \in \tilde{C}^{p+1}(Q, G)$ defined by

(3.1) $$(\delta f)(x_1, \ldots, x_{p+1}) = x_1 f(x_2, \ldots, x_{p+1}) + \sum_{i=i}^{p} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+1}) + (-1)^{p+1} f(x_1, \ldots, x_p).$$

If $p = 0$, a cochain $f \in \tilde{C}^0(Q, G)$ is by definition an element $f \in G$. The coboundary of f is the cochain $\delta f \in \tilde{C}^1(Q, G)$ defined by

(3.2) $$(\delta f)(x_1) = x_1 f - f, \qquad (x_1 \in Q).$$

It can be verified directly that δ is a homomorphism for every $p \geq 0$ and that $\delta\delta = 0$.

For each integer $p \geq 0$, there is a one-one correspondence $F \Leftrightarrow f$ between the p-dimensional homogeneous and non-homogeneous cochains. If $p > 0$, this correspondence is defined by the formulas

(3.3) $$F(x_0, x_1, \ldots, x_p) = x_0 f(x_0^{-1} x_1, x_1^{-1} x_2, \ldots, x_{p-1}^{-1} x_p),$$

(3.4) $$f(x_1, \ldots, x_p) = F(1, x_1, x_2, \ldots, x_1 x_2 \ldots x_p).$$

If $p = 0$, it will be defined by the formulas

(3.5) $$F(x_0) = x_0 f, \qquad f = F(1).$$

As verified by Eilenberg and MacLane [6, p. 54], this correspondence $F \Leftrightarrow f$ establishes for each $p \geq 0$ an (onto) isomorphism

$$C^p(Q, G) \approx \tilde{C}^p(Q, G)$$

which commutes with the coboundary operators δ. To the subgroup $C^p_\square(Q, G)$ of $C^p(Q, G)$, it corresponds the subgroup $\tilde{C}^p_\square(Q, G)$ of $\tilde{C}^p(Q, G)$ defined as follows. Assume $p > 0$. Then $f \in \tilde{C}^p(Q, G)$ is in $\tilde{C}^p_\square(Q, G)$ if and only if there is a neighborhood U of the identity 1 in Q such that $f(x_1, \ldots, x_p) = 0$ whenever $x_i \in U$ for each $i = 1, \ldots, p$. If $p = 0$, we put $\tilde{C}^0_{\square}(Q, G) = 0$. For each $p \geqq 0$, let

$$\tilde{C}^p_*(Q, G) = \tilde{C}^p(Q, G)/\tilde{C}^p_{\square}(Q, G).$$

The correspondence $F \leftrightarrow f$ induces (onto) isomorphisms

$$C^p_{\square}(Q, G) \approx \tilde{C}^p_{\square}(Q, G), \qquad C^p_*(Q, G) \approx \tilde{C}^p_*(Q, G)$$

commuting with the coboundary operators δ. Hence, in the definition of the various cohomology groups of Q over G given in §1, we may replace the homogeneous cochains by the corresponding non-homogeneous ones and obtain the same groups up to isomorphisms.

4. The cases p = 0, 1

In the present section we shall describe the algebraic meaning of the various cohomology groups of Q over G for the cases $p = 0$ and 1.

$p = 0$. Denote by G^* the subgroup of G consisting of those elements $g \in G$ such that $xg = g$ for all $x \in G$. Let $G^{\#}$ denote the subgroup of G defined by the condition that $g \in G^{\#}$ if and only if there is a neighborhood U of the identity 1 in Q such that $xg = g$ for all $x \in U$. Obviously $G^* \subset G^{\#}$. It is also clear that $G^{\#} = G$ if Q is discrete and that $G^{\#} = G^*$ if Q is connected. If Q operates simply on G, then $G^* = G = G^{\#}$.

A cochain $f \in \tilde{C}^0(Q, G)$ is by definition an element $f \in G$. By (3.2) and $\tilde{C}^0_{\square}(Q, G) = 0$, one can easily see the following assertion.

(4.1) $\quad H^0_{\square}(Q, G) = 0, \; H^0(Q, G) \approx G^*, \; H^0_*(Q, G) \approx G^{\#}$.

$p = 1$. A cochain $f \in \tilde{C}^1(Q, G)$ is a continuous map $f\colon Q \to G$, and

$$(\delta f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1).$$

Hence f is a cocycle if and only if

(4.2) $\quad f(x_1 x_2) = f(x_1) + x_1 f(x_2)$, i.e., f is a continuous crossed homomorphism of Q into G. Therefore, $\tilde{Z}^1(Q, G)$ is the group of all continuous crossed homomorphisms of Q into G. In order that $f \in \tilde{B}^1(Q, G)$, we must have

$$f(x) = xg - g$$

for some constant $g \in G$. These particular continuous crossed homomorphisms are called principal homomorphisms. $\tilde{B}^1(Q, G)$ is the group of all principal homomorphisms of Q into G. Hence

(4.3) The cohomology group $H^1(Q, G)$ is the group of all continuous crossed homomorphisms of Q into G reduced modulo the principal homomorphisms.

A continuous crossed homomorphism $f: Q \to G$ is said to be locally trivial if there exists a neighborhood U of the identity 1 in Q such that $f(x) = 0$ whenever $x \in U$. Then clearly $\tilde{Z}^1_{\square}(Q, G)$ is the group of all locally trivial continuous crossed homomorphisms of Q into G. $\tilde{C}^0_{\square}(Q, G) = 0$ implies $\tilde{B}^1_{\square}(Q, G) = 0$. Hence we have proved the following assertion.

(4.4) The cohomology group $H^1_{\square}(Q, G)$ with empty supports is the group of all locally trivial continuous crossed homomorphisms of Q into G.

The following assertions are easy corollaries of (4.3) and (4.4).

(4.5) If Q operates simply on G, then $H^1(Q, G)$ is the group of all continuous homomorphisms of Q into G and $H^1_{\square}(Q, G)$ is the group of all locally trival continuous homomorphisms of Q into G.

(4.6) If a compact group Q operates simply on a finite dimensional vector group G with euclidean topology, then we have

$$H^1(Q, G) = 0 = H^1_{\square}(Q, G).$$

(4.7) <u>If Q is connected, then</u> $H^1_{\square}(Q, G) = 0$.

A continuous map $f\colon Q \to G$ is said to be <u>locally crossed homomorphic</u> if there is a neighborhood U of 1 in Q such that (4.2) is true whenever x_1 and x_2 are in U. f is said to be <u>locally principal</u> if there exist a neighborhood U of 1 in Q and an element $g \in G$ such that $f(x) = xg - g$ whenever x is in U. The following assertion can be easily proved.

(4.8) <u>The reduced cohomology group</u> $H^1_*(Q, G)$ <u>is isomorphic with the group of all locally crossed homomorphic continuous maps of Q into G reduced modulo the locally principal continuous maps.</u>

5. Topological Group Extensions

The study of the 2-dimensional cohomology groups $H^2(Q, G)$ and $H^2_{\square}(Q, G)$ leads to the topological group extensions of G by Q having cross-sections. The corresponding description of the group $H^2_*(Q, G)$ needs the notion of topological loop prolongations and will be studied in a forthcoming paper.

A <u>topological group extension</u> of the group G by the group Q is a pair (E, ϕ) where E is a topological (multiplicative) group containing the group G as a closed normal subgroup and ϕ is an open continuous homomorphism of E onto Q with the subgroup G of E as the kernel, [1], [17], [19].

A <u>cross-section</u> of a topological group extension (E, ϕ) of G by Q is a continuous map $u\colon Q \to E$ such that $\phi u(x) = x$ for each $x \in Q$. A local cross-section of (E, ϕ) is a continuous map $u\colon U \to E$ defined on a neighborhood U of 1 in Q such that $\phi u(x) = x$ for each $x \in U$. A cross-section u of (E, ϕ) is said to be <u>homomorphic</u> if it is a homomorphism of Q into E. A cross-section or a local cross-section u of (E, ϕ) is said to be <u>locally</u>

homomorphic if there is a neighborhood V of 1 in Q such that $u(x_1x_2) = u(x_1)u(x_2)$ whenever x_1 and x_2 are in V.

A topological group extension (E, ϕ) of G by Q is said to be inessential if it has a homomorphic cross-section. (E, ϕ) is said to be locally inessential if it has a locally homomorphic local cross-section, [2].

Let (E, ϕ) be a topological group extension of G by Q which has a cross-section $u: Q \to E$. Since G is abelian, one can easily see that for each $g \in G$ and $x \in Q$ the element $u(x)gu(x)^{-1} \in G$ does not depend on the choice of the cross-section u. (E, ϕ) is said to be corresponding to the given way in which Q operates on G if

$$u(x)gu(x)^{-1} = xg$$

for all $x \in Q$ and $g \in G$. In particular, if Q operates simply on G, then G is contained in the center of E and (E, ϕ) is called a central extension of G by Q.

Now let us consider the set of all topological group extensions of G by Q corresponding to the given way in which Q operates on G. Any two of such extensions (E_1, ϕ_1) and (E_2, ϕ_2) are said to be equivalent if there exists an open continuous isomorphism $\sigma: E_1 \approx E_2$ such that $\phi_2 \sigma = \phi_1$ and $\sigma(g) = g$ for each $g \in G$. If $u_1: Q \to E_1$ is any cross-section of (E_1, ϕ_1), then $\sigma u_1: Q \to E_2$ is clearly a cross-section of (E_2, ϕ_2).

Let (E, ϕ) be a given topological group extension of G by Q corresponding to the given way in which Q operates on G. Assume that (E, ϕ) has a cross-section $u: Q \to E$. Define a continuous map $f: Q^2 \to G$ by taking

$$f(x_1, x_2) = u(x_1)u(x_2)u(x_1x_2)^{-1}$$

for each $x_1 \in Q$ and $x_2 \in Q$. This cochain $f \in \tilde{C}^2(Q, G)$ is called the factor set corresponding to the cross-section u. The associative law in E imposes the following condition on f; namely,

(5.1) $$x_1 f(x_2, x_3) + f(x_1, x_2 x_3) = f(x_1, x_2) + f(x_1 x_2, x_3)$$

for all x_1, x_2, x_3 in Q. This condition implies that $\delta f = 0$, i.e., that $f \in \tilde{Z}^2(Q, G)$.

Let $u': Q \to E$ to another cross-section of (E, ϕ) and $f': Q^2 \to G$ the corresponding factor set. Define a cochain $h \in \tilde{C}^1(Q, G)$ by taking

$$h(x) = u'(x)u(x)^{-1}, \qquad x \in Q$$

Then it can be verified [6, p. 57] that $f' - f = \delta h$. Hence (E, ϕ) determines uniquely an element α of $H^2(Q, G)$. Further, if $f \in \tilde{Z}^2(Q, G)$ is any representative of the element α, then there is a cross-section $u: Q \to E$ of (E, ϕ) such that f is the factor set corresponding to u.

Let (E^*, ϕ^*) be any extension which is equivalent to (E, ϕ). Choose an arbitrary open continuous isomorphism $\sigma: E \approx E^*$ such that $\phi^* \sigma = \phi$ and $\sigma(g) = g$ for each $g \in G$. Then $u^* = \sigma u$ is a cross-section of (E^*, ϕ^*). It is easily verified that the factor set f^* corresponding to u^* is identical with the factor set f corresponding to u. Hence equivalent extensions of G by Q determine the same element of $H^2(Q, G)$.

For any given cocycle $f \in \tilde{Z}^2(Q, G)$, we shall define a topological group extension (E_f, ϕ_f) of G by Q as follows. The space E_f is the topological product $G \times Q$. The group operation of E_f is defined by the multiplication-rule :

$$(g_1, x_1)(g_2, x_2) = (g_1 + x_1 g_2 + f(x_1, x_2), x_1 x_2).$$

One easily observes that $\delta f = 0$ implies that this multiplication is associative. The homomorphism $\phi_f: E_f \to Q$ is defined by

$$\phi_f(g, x) = x, \qquad (g, x) \in E_f.$$

Since ϕ_f is the projection of $G \times Q$ onto Q, it is open and continuous. The kernel of G_0 of ϕ_f is the set of all pairs $(g, 1)$. G_0 can be identified with G by means of the correspondence $g \Leftrightarrow (g - f(1,1), 1)$.

Since $f \in \tilde{Z}^2(Q, G)$, the condition (5.1) is true. One can easily deduce the following equalities from (5.1):

$$(5.2) \qquad xf(1,1) = f(x,1) = f(1,x) = f(1,1)$$

for each $x \in Q$. (E_f, ϕ_f) has an obvious cross-section $u: Q \to E_f$ defined by

$$u(x) = (0, x), \qquad x \in Q.$$

Then, for each $x \in Q$ and $g \in G$, we have

$$u(x)gu(x)^{-1} = (0,x)(g-f(1,1),1)(0,x)^{-1}$$
$$=(xg-xf(1,1)+f(x,1),x)(0,x)^{-1}=(xg-xf(1,1),1)(0,x)(0,x)^{-1}$$
$$=(xg-xf(1,1),1) = (xg-f(1,1),1) = xg.$$

Hence (E_f, ϕ_f) is corresponding to the given way in which Q operates on G.

For any two elements x_1 and x_2 in Q, we have

$$u(x_1)u(x_2)u(x_1x_2)^{-1} = (0,x_1)(0,x_2)(0,x_1x_2)^{-1}$$
$$= (f(x_1,x_2), x_1x_2)(0,x_1x_2)^{-1}$$
$$=(f(x_1,x_2)-f(1,x_1x_2),1)(0,x_1x_2)(0,x_1x_2)^{-1}$$
$$=(f(x_1,x_2)-f(1,x_1x_2),1) = (f(x_1,x_2)-f(1,1),1)$$
$$= f(x_1,x_2).$$

Hence f is the factor set corresponding to the cross-section u of (E_f, ϕ_f). This proves that every element of $H^2(Q, G)$ is determined by some extension of G by Q which has a cross - section and corresponds to the given way in which Q operates on G.

Let (E, ϕ) be any topological group extension of G by Q which admits cross-section and corresponds to the given way in which Q operates on G. (E, ϕ) determines an element $\alpha \in H^2(Q, G)$. Let $f \in \tilde{Z}^2(Q, G)$

be an arbitrary representative of α. Then there is a cross-section u: $Q \to E$ of (E, ϕ) such that f is the factor set corresponding to u. Define a map σ: $E_f \to E$ by taking

$$\sigma(g, x) = g \cdot u(x), \qquad (g, x) \in E_f.$$

That σ is open and continuous is obvious. σ is a homomorphism because

$$\sigma(g_1, x_1) \cdot \sigma(g_2, x_2) = g_1 \cdot u(x_1) \cdot g_2 \cdot u(x_2)$$
$$= g_1 \cdot u(x_1) g_2 u(x_1)^{-1} \cdot u(x_1) u(x_2) = (g_1 \cdot x_1 g_2 \cdot f(x_1, x_2)) \cdot u(x_1 x_2)$$
$$= \sigma(g_1 + x_1 g_2 + f(x_1, x_2), x_1 x_2) = \sigma((g_1, x_1)(g_2, x_2)).$$

Since $\phi\, \sigma(g, x) = x$, it follows that σ is an isomorphism. Let y be any element of E. Let $x = \phi(y) \in Q$ and $g = yu(x)^{-1} \in G$. Then $\sigma(g, x) = Y$. Hence σ is onto. Thus we have proved that $\sigma: E_f \approx E$, i.e. (E, ϕ) is equivalent with (E_f, ϕ_f). This implies that, if two given extensions (E_1, ϕ_1) and (E_2, ϕ_2) determine the same element of $H^2(Q, G)$, they have to be equivalent.

So far we have proved that there is a one-one correspondence between the elements of $H^2(Q, G)$ and the equivalence classes of topological group extensions of G by Q having cross-sections and corresponding to the given way in which Q operates on G. By means of this correspondence, these equivalence classes form an abelian group. Hence

(5.3) The cohomology group $H^2(Q, G)$ is isomorphic with the group of those topological group extensions of G by Q having cross-sections and corresponding to the given way in which Q operates on G.

To study the group $H^2_\square(Q, G)$, let us consider the set of all topological group extensions of G by Q having locally homomorphic cross-sections and corresponding to the given way in which Q operates on G. Let (E, ϕ) be one of these extensions. Two locally homomorphic cross-sections $u, v\colon Q \to E$ of (E, ϕ) are said to be equivalent if there is a neighborhood U of 1 in Q such that $u(x) = v(x)$ for $x \in U$. The locally homomorphic cross-sections of (E, ϕ) are divided into disjoint equivalence classes. Each of these equivalence classes is called a slice of (E, ϕ). The extension (E, ϕ) together with a given slice s is called a sliced topological group extension of G by Q denoted by (E, ϕ, s), [19].

Two sliced extensions (E_1, ϕ_1, s_1) and (E_2, ϕ_2, s_2) are said to be equivalent if there exist an open continuous isomorphism $\sigma\colon E_1 \approx E_2$ and representatives $u_i\colon Q \to E_i$ of s_i $(i = 1, 2)$ such that $\phi_2 \sigma = \phi_1$, $\sigma u_1 = u_2$, and $\sigma(g) = g$ for each $g \in G$. Equivalent sliced extensions may be considered as identical. By an analogous investigation as above, one can prove the following assertion.

(5.4) The cohomology group $H^2_\square(Q, G)$ with empty supports is isomorphic with the group of the sliced topological group extensions of G by Q corresponding to the given way in which Q operates on G.

The following assertions are corollaries of (5.3) and (5.4).

(5.5) If Q is simply connected and locally connected, then $H^2_\square(Q, G) = 0$.

Proof: Let (E, ϕ, s) be any sliced extension of G by Q corresponding to the given way in which Q operates on G. Choose an arbitrary locally homomorphic cross-section $u\colon Q \to E$ which represents s. Then there is a neighborhood U of 1 in Q such that $u|U$ is a local homomorphism of Q into E, [4, p. 48]. Since Q is locally connected, we may assume that U is connected. Since Q is simply connected, it follows from a classical theorem [4, p. 49] that $u|U$ can be extended to a continuous homomorphism $u^*\colon Q \to E$.

Since Q is connected, it is generated by U, [4, p. 35]. Let x be an arbitrary element of Q, then there are a finite number of elements $x_1, \ldots, x_n$ in U such that $x = x_1 \ldots x_n$. Hence we have

$$\phi u^*(x) = \phi(u^*(x_1) \ldots u^*(x_n)) = \phi(u(x_1) \ldots u(x_n))$$
$$= \phi u(x_1) \ldots \phi u(x_n) = x_1 \ldots x_n = x.$$

This proves that u^* is a cross-section of (E, ϕ). Since u^* is homomorphic, the corresponding factor set $f^* = 0$. Hence (E, ϕ, s) determines the zero element of $H^2_{\square}(Q, G)$. This completes the proof.

(5.6) If Q is a free topological group [16, p. 17], then we have $H^2(Q, G) = 0$.

Proof. By definition Q is the free topological group of some completely regular space X which is a closed subset of Q, [16, p. 16]. Let (E, ϕ) be any topological group extension of G by Q with a cross-section $u: Q \rightarrow E$. Since Q is the free topological group of X, it follows that [16, p. 16] the continuous map $u|X$ can be extended to a continuous homomorphism $u^*: Q \rightarrow E$. Since Q is generated topologically by X [16, p. 13], it is easy to show that u^* is a cross-section of (E, ϕ). Since u^* is homomorphic, the corresponding factor set is $f^* = 0$. Since (E, ϕ) is arbitrary, we have $H^2(Q, G) = 0$. Q. E. D.

(5.7) If G is a finite dimensional toroidal group and Q is the direct product of a simply connected compact Lie group and a finite dimensional toroidal group, then $H^2(Q, G) = 0$.

Proof. Let (E, ϕ) be any topological group extension of G by Q with a cross-section. By a theorem of Calabi and Ehresmann [1, Prop. 5], (E, ϕ) is the inessential extension of G by Q. This implies that $H^2(Q, G) = 0$. Q. E. D.

Chapter II. Cohomology Groups of a Space with Operators.

Throughout the present chapter, we shall assume that a multiplicative topological group Q is acting as a group of transformations on a topological space Y. By this we mean that for each $x \in Q$ a transformation

$$T_x: Y \to Y$$

is given such that

(i) $T_x(y)$ is continuous in x and y simultaneously.

(ii) $$T_{x_1 x_2} = T_{x_1} T_{x_2},$$

(iii) T_1 is the identity transformation of Y.

If for every $x \in Q$ and $y \in Y$ we have $T_x(y) = y$ then we say that Q operates on Y simply.

We shall also assume that G is a finite dimensional vector group with the euclidean topology and Q operates on G by means of a representation P of Q with G as representation space. For each $x \in Q$ a linear transformation $P_x : G \to G$ is then defined satisfying conditions analogous to (i) - (iii).

6. Cohomology Groups of Y

Y^{p+1} will denote the (p+1)-fold topological product of Y with itself; that is Y^{p+1} is the set of points $(y_0, y_1, \ldots, y_p)$ where $y_i \in Y$, this set being topologized in the usual way. For each $p \geq 0$, a continuous p-map of Y into G is a continuous map $\phi: Y^{p+1} \to G$. The continuous p-maps of Y into G form an abelian group $M^p(Y, G)$ with functional addition as the group operation.

For each continuous p-map ϕ of Y into G, we shall define the support of ϕ to be the subset $S(\phi)$ of Y defined by the condition that $y \notin S(\phi)$ if and only if there is a neighborhood U of y in Y such that $\phi(y_o, \ldots, y_p) = 0$ whenever $y_i \in U$ for all $i = 0, \ldots, p$. The continuous p-maps of Y into G with empty supports form a subgroup $M^p_\square(Y, G)$ of $M^p(Y, G)$. The cosets of $M^p_\square(Y, G)$ in $M^p(Y, G)$ are called continuous p-cochains of Y with coefficients in G. They form an abelian group

$$C^p(Y, G) = M^p(Y, G)/M^p_\square(Y, G).$$

For each continuous p-map $\phi \in M^p(Y, G)$, we define a continuous (p+1)-map $\delta\phi \in M^{p+1}(Y, G)$ by taking

(6.1) $$(\delta\phi)(y_o, \ldots, y_{p+1}) = \sum_{i=o}^{p+1} (-1)^i \phi(y_o, \ldots, \hat{y}_i, \ldots, y_{p+1}).$$

The following properties of the operation $\delta: M^p(Y, G) \to M^{p+1}(Y, G)$ can be easily verified.

(6.2) δ is a homomorphism,

(6.3) $\delta\delta = 0$,

(6.4) δ maps $M^p_\square(Y, G)$ into $M^{p+1}_\square(Y, G)$.

Hence the operator δ will be called the coboundary operator and $\delta\phi$ will be the coboundary of ϕ. (6.4) implies that δ induces a coboundary operator δ: $C^p(Y, G) \to C^{p+1}(Y, G)$.

As in §1, one can define the subgroups $Z^p(Y, G)$ and $B^p(Y, G)$ of $C^p(Y, G)$, namely, the group of continuous p-cocycles and that of continuous p-coboundaries. The p-dimensional continuous cohomology group $H^p(Y, G)$ of Y over G is defined as the quotient group

(6.5) $$H^p(Y, G) = Z^p(Y, G)/B^p(Y, G).$$

The following theorem is rather well-known. For a proof, one may refer to [11, §5].

Theorem 6.6. *If Y is a compact Hausdorff* space and G a finite dimensional vector group with the euclidean topology, then the continuous cohomology group $H^p(Y, G)$ is isomorphic with the corresponding Alexander cohomology group and hence also with the corresponding Cech cohomology group.

Now let Y and Z be topological spaces and T: $Y \to Z$ a continuous map. For an arbitrary $\phi \in M^p(Z, G)$, define a continuous p-map $T^{\#}\phi \in M^p(Y, G)$ by taking

(6.7) $$(T^{\#}\phi)(y_0, y_1, \dots, y_p) = \phi(Ty_0, Ty_1, \dots, Ty_p)$$

for every point $(y_0, y_1, \dots, y_p)$ of Y^{p+1}. The correspondence $\phi \to T^{\#}\phi$ defines a homomorphism

$$T^{\#}: M^p(Z, G) \to M^p(Y, G)$$

which commutes with the coboundary operator δ and maps $M^p_{\square}(Z, G)$ into $M^p_{\square}(Y, G)$. Hence $T^{\#}$ induces a homomorphism

$$T^*: H^p(Z, G) \to H^p(Y, G).$$

7. Equivariant Cohomology Groups of Y

For each $x \in Q$, the linear transformation P_x: $G \to G$ of the vector group G induces an endomorphism

$$P^{\#}_x: M^p(Y, G) \to M^p(Y, G)$$

as follows. For each $\phi \in M^p(Y, G)$, the continuous p-map $P^{\#}_x\phi$ is defined by

(7.1) $$(P^{\#}_x\phi)(y_0, y_1, \dots, y_p) = P_x(\phi(y_0, y_1, \dots, y_p))$$

for every point $(y_0, y_1, \dots, y_p)$ of Y^{p+1}. Obviously, $P^{\#}_x$ commutes with the coboundary operator δ and maps $M^p_{\square}(Y, G)$ into itself.

A continuous p-map $\phi \in M^p(Y, G)$ is said to be equivariant if

(7.2) $$T_x^{\#}\phi = P_x^{\#}\phi$$

for each $x \in Q$. A continuous p-cochain $c \in C^p(Y,G)$ is said to be equivariant if it can be represented by an equivariant continuous p-map. The equivariant continuous p-cochains form a subgroup $C_e^p(Y,G)$ of $C^p(Y,G)$. Since both $T_x^{\#}$ and $P_x^{\#}$ commute with the coboundary operator δ, it follows that the coboundary $\delta\phi$ of an equivariant continuous p-map ϕ is equivariant. This implies that δ maps $C_e^p(Y,G)$ into $C_e^{p+1}(Y,G)$. Then we may define the subgroups $Z_e^p(Y,G)$ and $B_e^p(Y,G)$ of $C_e^p(Y,G)$ in the usual way, namely, the group of equivariant continuous p-cocycles and the group of equivariant continuous p-coboundaries. The p-dimensional equivariant cohomology group $H_e^p(Y,G)$ of Y over G is defined as the quotient group

(7.3) $$H_e^p(Y,G) = Z_e^p(Y,G)/B_e^p(Y,G).$$

The inclusion map $\pi: C_e^p(Y,G) \to C^p(Y,G)$ induces a natural homomorphism of the cohomology groups

(7.4) $$\pi^* : H_e^p(Y,G) \to H^p(Y,G).$$

The following theorem will be proved in §8.

Theorem 7.5. *If Q is compact then π^* maps $H_e^p(Y,G)$ isomorphically into $H^p(Y,G)$.*

Following the method of Chevalley and Eilenberg [5, p. 89], one can deduce a more detailed analysis of (7.4) from the decomposition of the representation P into irreducible components. Such a decomposition always exists if Q is compact. Let $G = G_1 + \ldots + G_n$ be a direct decomposition of the vector space G into irreducible invariant subspaces and let P_i be the corresponding representation of Q in G_i. There result direct sum decompositions

$$H_e^p(Y, G) = \sum_{i=0}^{n} H_e^p(Y, G_i), \quad H^p(Y, G) = \sum_{i=0}^{n} H^p(Y, G_i)$$

and an appropriate decomposition of (7.4). Hence we may concentrate our attention on irreducible representations.

The following theorem will be proved in §8.

Theorem 7.6 <u>If the space Y is compact, the group Q is compact and connected, and the representation P of Q in G is irreducible and non-trivial, then $H_e^p(Y, G) = 0$ for each $p \geq 0$.</u>

The following theorem is a restatement of the Theorem 10.2 in [11].

Theorem 7.7. <u>If the space Y is compact, the group Q is compact and connected, the group G is the additive group of real numbers, and the representation P is trivial, then π^* is an isomorphism onto, i.e., $H_e^p(Y, G) \approx H^p(Y, G)$ for each $p \geq 0$.</u>

8. The Averaging Process

In the present section, we assume Q to be compact. This implies the existence of a Haar measure with the measure of Q being 1, [22, p. 38].

Given a continuous p-map $\phi \in M^p(Y, G)$ of Y into G, consider the family of continuous p-maps $P^{\#}_{x^{-1}} T^{\#}_x \phi$ for all $x \in Q$. For each point $(y_0, \ldots, y_p)$ of Y^{p+1}, we have

$$(P^{\#}_{x^{-1}} T^{\#}_x \phi)(y_0, \ldots, y_p) = P_{x^{-1}} \phi(T_x y_0, \ldots, T_x y_p).$$

It follows that $(P_{x^{-1}} T_x \phi)(y_0, \ldots, y_p)$ is continuous in x and $(y_0, \ldots, y_p)$ simultaneously. Hence we may define a p-map $I\phi$ by taking

$$(I\phi)(y_0, \ldots, y_p) = \int_Q (P^{\#}_{x^{-1}} T^{\#}_x \phi)(y_0, \ldots, y_p) dx$$

for each point $(y_o, \ldots, y_p)$ of Y^{p+1}. The p-map $I\phi$ thus obtained as the following properties:

(8.1) $I\phi$ is continuous,

(8.2) $\delta(I\phi) = I(\delta\phi)$,

(8.3) $I\phi$ is equivariant,

(8.4) if ϕ is equivariant than $I\phi = \phi$,

(8.5) If ϕ is of empty support then so is ϕ.

These properties can be verified by the methods analogous to those used in §3 of [5] and in §8 of [11].

Proof of Theorem 7.5. Let α be an arbitrary element of $H^p_e(Y, G)$ such that $\pi^*(\alpha) = 0$. Choose an equivariant p-cocycle $c \in C^p_e(Y, G)$ which represents α. Then there is an equivariant continous p- map ϕ such that c is the coset of $M^p_\square(Y, G)$ in $M^p(Y, G)$ containing ϕ. $\pi^*(\alpha) = 0$ implies $c \in B^p(Y, G)$. If $p = 0$, this implies $c = 0$ and hence $\alpha = 0$. If $p > 0$, this means that there exist $\xi \in M^{p-1}(Y, G)$ and $\eta \in M^p_\square(Y, G)$ such that $\phi = \delta\xi + \eta$. Obviously the operation I defined above is homomorphic. Hence, by (8.4) and (8.2), we have

$$\phi = I\phi = I\delta\xi + I\eta = \delta(I\xi) + I\eta.$$

By (8.3), $I\xi$ is equivariant. By (8.5), $I\eta \in M^p_\square(Y, G)$. Hence $c \in B^p_e(Y, G)$, that is, $\alpha = 0$. This completes the proof.

Proof of Theorem 7.6. Let α be an arbitrary element of $H^p_e(Y, G)$. Choose an equivariant p-cocycle $c \in Z^p_e(Y, G)$ which represents α. Then there is an equivariant continuous p-map $\phi \in c$. By definition, we have $T^\#_x\phi = P^\#_x\phi$ for every $x \in Q$. Define a continuous p-map ϕ^* by taking

$$\phi^*(y_o, \ldots, y_p) = \int_Q T^\#_x\phi(y_o, \ldots, y_p)dx = \int_Q P^\#_x\phi(y_o, \ldots, y_p)\,dx$$

for each point $(y_o, \ldots, y_p)$ of Y^{p+1}. Let z be an arbitrary element of Q. Then we have

$$P_z(\phi^*(y_o, \ldots, y_p)) = P_z \int_Q P_x(\phi(y_o, \ldots, y_p))dx$$

$$= \int_Q P_{zx}(\phi(y_o, \ldots, y_p))dx = \int_Q P_x(\phi(y_o, \ldots, y_p))dx$$

$$= \phi^*(y_o, \ldots, y_p).$$

Since this holds for every $z \in Q$ and since the representation P is irreducible and non-trivial, it follows that $\phi^* = 0$. According to the fundamental lemma 9.1 of [11], there exist $\xi \in M^{p-1}(Y, G)$ and $\eta \in M^p_{\square}(Y, G)$ such that

$$\phi = \phi - \phi^* = \delta\xi + \eta \ .$$

This implies that $\pi^*(\alpha) = 0$. It follows from Theorem 7.5 that $\alpha = 0$. Hence $H^p_e(Y, G) = 0$. Q.E.D.

9. Reduced Cohomology Groups of Topological Groups

In the present section, we shall consider the special case where Y = Q and Q operates on Y by means of left translations. It turns out naturally that the reduced cohomology group $H^p_*(Q, G)$ is isomorphic with the equivariant cohomology group $H^p_e(Y, G)$.

Let α be an arbitrary element of $H^p_*(Q, G)$ in the homogeneous approach given in §1. Choose a reduced cocycle $c \in C^p_*(Q, G)$ which represents α. Let $F \in C^p(Q, G)$ be an arbitrary p-dimensional cochain in the coset c. (1.1) implies that F is an equivariant continuous p-map of Y into G. $F \in c \in Z^p_*(Q, G)$ implies that δF is in $M^{p+1}_{p\square}(Y, G)$. Hence F represents an element $\kappa(\alpha)$ of $H^p_e(Y, G)$ which clearly depends only on α. The correspondence $\alpha \to \kappa(\alpha)$ defines a natural homomorphism.

$$\kappa : H^p_*(Q,G) \to H^p_e(Y,G).$$

(9.1) <u>The natural homomorphism κ maps</u> $H^p_*(Q,G)$ isomorphically onto $H^p_e(Y,G)$ for each $p \geqq 0$.

<u>Proof.</u> Let $\beta \in H^p_e(Y,G)$ be arbitrarily given. Choose an equivariant cocycle $d \in C^p_e(Y,G)$ which represents β. Then there is an equivariant continuous p-map F contained in the coset d. That F is equivariant and $F \in d \in Z^p_e(Y,G)$ implies that $F \in C^p(Q,G)$ and $\delta F \in C^{p+1}_{\square}(Q,G)$. Hence F determines an element $\alpha \in H^p_*(Q,G)$ and obviously $\kappa(\alpha) = \beta$. This proves that κ is onto.

Let α be any element of $H^p_*(Q,G)$ such that $\kappa(\alpha) = 0$. Choose a representative $F \in C^p(Q,G)$ of α as in the definition of $\kappa(\alpha)$. $\kappa(\alpha) = 0$ implies that the equivariant continuous p-map F represents and equivariant coboundary $[F] \in B^p_e(Y,G)$. If $p = 0$, this means $[F] = 0$ and hence $F \in M^p_{\square}(Y,G)$. This implies $F \in C^p_{\square}(Q,G)$ and hence $\alpha = 0$. If $p > 0$, then there exist an equivariant $\xi \in M^{p-1}(Y,G)$ and an $\eta \in M^p_{\square}(Y,G)$ such that $F = \delta\xi + \eta$. It follows that $\eta = F - \delta\xi$ is also equivariant. Since ξ and η are equivariant, we have $\xi \in C^{p-1}(Q,G)$ and $\eta \in C^p_{\square}(Q,G)$. $F = \delta\xi + \eta$ implies $\alpha = 0$. Hence κ is an isomorphism. Q.E.D.

The following theorem is an easy consequence of (9.1) and the theorems in §7.

Theorem 9.2. <u>If a compact connected group Q operates on a finite dimensional vector group G with the euclidean topology, then the reduced cohomology group $H^p_*(Q,G)$ of Q over G is isomorphic with the Cech cohomology group $H^p(Q,G_*)$ of the topological group over the vector group G_* which consists of the totality</u>

of the elements $g \in G$ such that $xg = g$ for all $x \in Q$.

10. Compact Connected Semi-Simple Groups.

A topological group Q is said to be semi-simple [5] if it contains no connected solvable normal subgroup other than the trivial subgroup 1. If Q is a Lie group, then this definition reduces to the classical one, [18, p. 267].

Lemma 10.1. *A compact connected semi-simple group Q is the projective limit of an inverse system $\{Q_\alpha, h_{\beta\alpha}\}$ of compact connected semi-simple Lie groups Q_α, that is, $Q = \varprojlim \{Q_\alpha, h_{\beta\alpha}\}$.*

Proof. Let $\{N_\alpha\}$ denote the collection of all closed normal subgroups N_α of Q such that the quotient group $Q_\alpha = Q/N_\alpha$ is a Lie group. Since Q is compact and connected, so is Q_α. We define a partial order in the indices $\{\alpha\}$ by $\beta > \alpha$ if and only if $N_\beta \subset N_\alpha$. If $\beta > \alpha$, then there is a natural open continuous homomorphism $h_{\beta\alpha} : Q_\beta \to Q_\alpha$ of Q_β onto Q_α defined by the inclusion $N_\beta \subset N_\alpha$. It is well-known [22, p. 89] that $Q = \varprojlim\{Q_\alpha, h_{\beta\alpha}\}$. It remains to prove that Q_α is semi-simple for every index α.

According to E. Cartan [3, §52], for each α, there exist in Q_α a compact connected abelian subgroup A_α which is the connected component of the center of Q_α and a closed normal semi-simple subgroup S_α such that $Q_\alpha = A_\alpha S_\alpha$ and $A_\alpha \cap S_\alpha$ is a finite group. Further, the homomorphism $h_{\beta\alpha} : Q_\beta \to Q_\alpha$ maps A_β onto A_α and S_β onto S_α. If we denote by $k_{\beta\alpha} : A_\beta \to A_\alpha$ the partial homomorphism of $h_{\beta\alpha}$ restricted on A_β, then we obtain a inverse system $\{A_\alpha, k_{\beta\alpha}\}$ of compact connected abelian groups A_α. The projective limit $\{A = \varprojlim A_\alpha, k_{\beta\alpha}\}$ is a compact connected abelian group and is contained in the center of Q, [22, p. 89]. Since Q is semi-simple, A contains only the identity of Q. Let $\pi_\alpha : A \to A_\alpha$ be the natural projection. Let $a_\alpha \in A_\alpha$ be an arbitrary element. Since the groups A_β are compact and the homomorphisms $k_{\gamma\beta}$, $\gamma > \beta$, are onto, it follows from a theorem of Steenrod [14, p. 32] that there is an element $a \in A$ such that $\pi_\alpha(a) = a_\alpha$. Since A contains only one element, so does A_α for all α. Hence we obtain $Q_\alpha = S_\alpha$ and Q_α is semi-simple for all α. Q. E. D.

Theorem 10. 2. <u>If a compact connected semi-simple group Q operates on a finite dimensional vector group G with the euclidean topology, then the reduced cohomology group</u> $H^p_*(Q, G) = O$ <u>when</u> $p = 1, 2, 4$.

<u>Proof.</u> Let G_* denote the subgroup of G consisting of all elements $g \in G$ such that $xg = g$ for all $x \in Q$. By Theorem 9. 2, $H^p_*(Q, G)$ is isomorphic with the Cech cohomology group $H^p_\#(Q, G_*)$. According to Lemma 10. 1, Q is the projective limit of an inverse system $\{Q_\alpha, h_{\beta\alpha}\}$ of compact connected semi-simple Lie groups Q_α. Since the groups Q_α are compact, it follows from the Continuity Axiom [20, p. 419] that $H^p_\#(Q, G_*)$ is isomorphic with the limit group $\varinjlim \{H^p_\#(Q_\alpha, G_*), h^*_{\beta\alpha}\}$ of the direct system of cohomology groups $H^p_\#(Q_\alpha, G_*)$ with the homomorphisms $h^*_{\beta\alpha} : H^p_\#(Q_\alpha, G_*) \to H^p_\#(Q_\beta, G_*)$ induced by $h_{\beta\alpha}: Q_\beta \to Q_\alpha$. Since G_* is a finite dimensional vector group and Q_α is a compact connected semi-simple Lie group, we have $H^p_\#(Q_\alpha, G_*) = O$ when $p = 1, 2, 4$, [5, p. 109]. Hence

$$H^p_*(Q, G) \approx H^p_\#(Q, G_*) \approx \varinjlim \{H^p_\#(Q_\alpha, G_*), h^*_{\beta\alpha}\} = O$$

when $p = 1, 2, 4$. This completes the proof.

Chapter III. <u>Cohomology Groups of Local Groups</u>

Throughout the present chapter we assume that Q is a local group [18, p. 83], written multiplicatively, and G is an abelian local group, written additively. Further, we shall also assume that Q operates on the left of G. By this we mean that there exist a neighborhood Q_1 of 1 in Q and a neighborhood G_o of O in G such that, for each $x \in Q_1$ and $g \in G_o$, there is determined an element $xg \in G$ subject to the following conditions:

(i) xg is continuous in x and g simultaneously.

(ii) If $g_1 \in G_o$ and $g_2 \in G_o$ are such that $g_1 + g_2$ is defined and is in G_o, then, for every $x \in Q_1$, $xg_1 + xg_2$ is defined and

$$x(g_1 + g_2) = xg_1 + xg_2.$$

(iii) If $x_1 \in Q_1$, $x_2 \in Q_1$, $g \in G_o$ are such that $x_1 g \in G_o$ and $x_2 x_1$ is defined and is in Q_1, then

$$x_2(x_1 g) = (x_2 x_1)g.$$

(iv) For each $g \in G_o$, we always have $1g = g$.

If for every $x \in Q_1$ and $g \in G_o$ we have $xg = g$ then we say that Q operates on G simply.

11. Local Cohomology Groups

For each integer $p > O$, Q^p will denote the p-fold topological product of Q with itself. Throughout this chapter, we are interested only in the case $p > O$.

A local p-map of Q into G is a continuous map $\phi: V^p \to G$ defined on the subset V^p of the space Q^p for some neighborhood V of 1 in Q such that $\phi(x_1, \ldots, x_p) = O$ whenever $x_1 = \ldots = x_p = 1$. Two local p-maps $\phi_1: V_1^p \to G$ and $\phi_2: V_2^p \to G$ of Q into G are said to be equivalent (notation; $\phi_1 \equiv \phi_2$) if there is a neighborhood V of 1 contained in the intersection $V_1 \cap V_2$ such that

$$\phi_1(x_1, \ldots, x_p) = \phi_2(x_1, \ldots, x_p)$$

whenever x_i is in V for all $i = 1, \ldots, p$. The totality of local p-maps of Q into G are thus divided into disjoint equivalence classes, called the local p-cochains of Q over G. We shall denote by $C_L^p(Q,G)$ the set of all local p-cochains of Q over G. The local p-cochain that contains the local p-map ϕ will be denoted by $[\phi]$ and ϕ will be called a representative of $[\phi]$.

Let c_1 and c_2 be any two local p-chains of Q over G. Choose two local p-maps $\phi_1: V_1^p \to G$ and $\phi_2: V_2^p \to G$ such that $c_1 = [\phi_1]$ and $c_2 = [\phi_2]$. Choose a neighborhood U of O in G such that $g_1 + g_2$ is defined whenever g_1 and g_2 are in U. Choose a neighborhood V of 1 in Q such that $V^p \subset \phi_1^{-1}(U) \cap \phi_2^{-1}(U)$. Define a local p-map ϕ: $V^p \to G$ by taking

(11.1) $\phi(x_1, \ldots, x_p) = \phi_1(x_1, \ldots, x_p) + \phi_2(x_1, \ldots, x_p)$

whenever $x_i \in V$ for all $i = 1, \ldots, p$. It is easily verified that the local p-cochain $[\phi]$ does not depend on the choice of the representatives ϕ_1 and ϕ_2 for c_1 and c_2. Hence we may define an addition in $C_L^p(Q,G)$ by means of $c_1 + c_2 = [\phi]$. In this way, it is easy to see that $C_L^p(Q,G)$ becomes an abelian group.

For each $p > O$, we are going to define a coboundary operator

$$\delta : C_L^p(Q,G) \to C_L^{p+1}(Q,G)$$

as follows. Let $c \in C_L^p(Q.G)$. Choose a local p-map $\phi: V^p \to G$ such that $c = [\phi]$. Let U_o be a neighborhood of O in G such that

$$\sum_{i=0}^{p+1} (-1)^i g_i$$

is defined whenever $g_i \in U_o$ for each $i = 0, 1, \ldots, p+1$. Choose a neighborhood V_o of 1 in Q and a neighborhood of U_1 of O in G such that $xg \in U_o$ whenever $x \in V_o$ and $g \in U_1$. Choose a neighborhood V_1 of 1 in Q such that $V_1 \subset V_o$ and $V_1^p \subset \phi^{-1}(U_1)$. Let V_2 be a neighborhood 1 in Q such that $xy \in V_1$ for all $x \in V_2$ and $y \in V_2$. Define a local (p+1) -map ψ: $V_2^{p+1} \to G$ by the formula

$$(11.2) \qquad \psi(x_1, \ldots, x_{p+1}) = x_1\phi(x_2, \ldots, x_{p+1}) + \sum_{i=o}^{p} (-1)^i \phi(x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+1}) + (-1)^i \phi(x_1, \ldots, x_p)$$

for each point $(x_1, \ldots, x_{p+1})$ of V_2^{p+1}. It can be easily verified that the local (p+1) -cochain $[\psi]$ depends only on the given local p-cochain c. Hence we may define the coboundary of c by setting $\delta c = [\psi]$. It can be verified directly that the operation δ is a homomorphism and $\delta\delta = O$.

As in §1, one can define for each $p > O$ subgroups $Z_L^p(Q, G)$ and $B_L^p(Q, G)$ of $C_L^p(Q, G)$, namely the group of <u>local</u> p-<u>cocycles</u> and that of <u>local</u> p - <u>coboundaries</u>. Here one has to put $B_L^1(Q, G) = O$. The p-dimensional <u>local cohomology group</u> $H_L^p(Q, G)$ of Q over G is defined to be the quotient group

$$(11.3) \qquad H_L^p(Q, G) = Z_L^p(Q, G)/B_L^p(Q, G).$$

12. A Natural Homomorphism

In the present section, we shall assume that Q and G satisfy the assumptions of the first chapter and hence they also satisfy the assumptions of this chapter which are obviously weaker. Therefore, for each integer $p > O$, both of the groups $H_*^p(Q, G)$ and $H_L^p(Q, G)$ are defined. We are going to establish a natural homomorphism

$$\lambda : H_*^p(Q, G) \to H_L^p(Q, G),$$

for $p \geqq 2$, or $p = 1$ and Q operates on G simply.

With this purpose, let us first prove the following

Lemma 12. 1. Every element α of $H_*^p(Q, G)$, $p > 0$, can be represented by a non-homogeneous p-cochain $f: Q^p \to G$ such that $\delta f \in \tilde{C}_\square^{p+1}(Q, G)$ and $f(x_1, \ldots, x_p) = 0$ when $x_1 = \ldots = x_p = 1$.

Proof. α is represented by a cochain $f' \in \tilde{C}^p(Q, G)$ with $\delta f'$ in $\tilde{C}_\square^{p+1}(Q, G)$. Call $f'(1, \ldots, 1) = g \in G$. First let us assume that p is odd. Then we have

$$\delta f'(1, \ldots, 1) = \sum_{i=0}^{p+1} (-1)^i g = g.$$

Since $\delta f' \in \tilde{C}_\square^{p+1}(Q, G)$, this implies that $g = 0$. Hence we may take $f = f'$. Next assume p to be even. Define an $h \in \tilde{C}^{p-1}(Q, G)$ by taking

$$h(x_1, \ldots, x_{p-1}) = g, \quad (x_1, \ldots, x_{p-1}) \in Q^{p-1}.$$

Take $f = f' - \delta h$, Then f represents α and satisfies the requirements. Q. E. D.

Let α be any element of $H_*^p(Q, G)$, $p > 0$. According to Lemma 12. 1, α can be represented by an $f \in \tilde{C}^p(Q, G)$ such that δf is in $\tilde{C}_\square^{p+1}(Q, G)$ and $f(1, \ldots, 1) = 0$. The condition $f(1, \ldots, 1) = 0$ assures that f is a local p-map of Q into G. Hence f represents a local p-cochain $[f] \in C_L^p(Q, G)$. The condition $\delta f \in \tilde{C}_\square^{p+1}(Q, G)$ implies that $_L\delta[f] = 0$. Therefore, [f] is a local p-cocycle of Q over G and represents an element $\beta \in H_L^p(Q, G)$.

Lemma 12. 2. If $p \geqq 2$, or $p = 1$ and Q operates simply on G, then the element $\beta \in H_L^p(Q, G)$ depends only on $\alpha \in H^p(Q, G)$ and the correspondence $\alpha \to \beta = \lambda(\alpha)$ defines a homomorphism of $H_*^p(Q, G)$ into $H_L^p(Q, G)$.

Proof. Let $f' \in \tilde{C}^p(Q, G)$ be another representative of α such that $\delta f' \in \tilde{C}_{\square}^{p+1}(Q, G)$ and $f'(1, \ldots, 1) = O$. If p = 1 and Q operates simply on G, then $\tilde{B}^1(Q, G) = O$. Hence $f' - f \in \tilde{C}_{\square}^1(Q, G)$. This implies $[f'] = [f]$ and hence the element β depends only on α. Now assume $p \geqq 2$. Then there exist $\xi \in \tilde{C}^{p-1}(Q, G)$ and $\eta \in \tilde{C}_{\square}^p(Q, G)$ such that

$$f' - f = \delta\xi + \eta .$$

This equality implies that $(\delta\xi)(1, \ldots, 1) = O$. By the method used in the proof of Lemma 12.1, we may choose ξ so that $\xi(1, \ldots, 1) = O$. Hence both ξ and η are local maps. $\eta \in \tilde{C}_{\square}^p(Q, G)$ implies $[\eta] = O$. Therefore $[f'] - [f] = \delta[\xi]$. This proves that the element β depends only on α. It is obvious from the definition of β that λ is a homomorphism of $H_*^p(Q, G)$ into $H_L^p(Q, G)$. Q. E. D.

Q is said to have the <u>local extension property</u> with respect to G provided that, for each integer $p > O$, every local p-map ϕ: $V^p \to G$ of Q into G is equivalent to a local p-map $\phi^*: Q^p \to G$ defined throughout Q^p.

Let us denote by R the real line and by I the closed unit interval in R. A space U is said to be <u>solid</u> if U is homeomorphic with the product space $P_{\alpha \in A} U_\alpha$ where each factor space U_α is either R or I. There is no restriction on the index set A; it might be non-countable.

Lemma 12.3. <u>If Q is locally compact and G has a solid neighborhood U of O, then Q has the local extension property with respect to G.</u>

<u>Proof.</u> Let ϕ: $V^p \to G$ be any local p-map of Q into G. Choose a neighborhood $V_o \subset V$ of 1 in Q such

that $\phi(V_0^p) \subset U$. Since Q is locally compact, there is an open neighborhood V_1 of 1 in Q such that its closure $W_1 = \mathrm{Cl}\, V_1$ is compact and contained in V_0. Choose a closed neighborhood W_2 of 1 in Q contained in V_1. Let

$$C = W_2^p, \qquad D = W_1^p \setminus V_1^p .$$

Then C and D are disjoint closed sets of W_1^p. Define a continuous map $\phi' : C \cup D \to U$ by taking $\phi' = \phi$ on C and $\phi' = 0$ on D. As a compact Hausdorff space, W_1^p is normal. It follows from Tietze's extension theorem, [14, p. 28], ϕ' has a continuous extension $\phi^* : W_1^p \to U$. ϕ^* can be extended throughout Q^p by taking $\phi^* = 0$ on $Q^p \setminus W_1^p$. The continuity of ϕ^* over Q^p follows from the fact that $\phi^*(D) = 0$. Since $\phi^* = \phi$ on W_2^p, ϕ and ϕ^* are equivalent. Q. E. D.

Lemma 12. 4. <u>If Q has the local extension property with respect to G, then the homomorphism λ in Lemma 12. 2 is an isomorphism onto.</u>

<u>Proof.</u> Let β be an arbitrary element of $H_L^p(Q, G)$. Since Q has the local extension property with respect to G, β can be represented by a local p-map $\phi : Q^p \to G$ with $\delta[\phi] = 0$. Hence we have $\phi \in \tilde{C}^p(Q, G)$ and $\delta\phi \in \tilde{C}_\square^{p+1}(Q, G)$. ϕ represents an element α of $H_*^p(Q, G)$ and clearly $\lambda(\alpha) = \beta$. This proves that λ is onto.

Let α be an arbitrary element of $H_*^p(Q, G)$ with $\lambda(\alpha) = 0$. Choose a representative $f \in \tilde{C}^p(Q, G)$ of α with $\delta f \in \tilde{C}_\square^{p+1}(Q, G)$ and $f(1, \ldots, 1) = 0$. $\lambda(\alpha) = 0$ implies that $[f] \in B_L^p(Q, G)$. If $p = 1$, then $[f] = 0$. It follows that $f \in \tilde{C}_\square^p(Q, G)$ and hence $\alpha = 0$. Assume that $p \geq 2$. Then there is a local $(p-1)$-map ϕ of Q into G such that $[f] = \delta[\phi]$. Since Q has the local extension property with respect to G, we may assume that ϕ is defined through Q^{p-1}. Hence $\phi \in \tilde{C}^{p-1}(Q,G)$. $[f] = \delta[\phi]$ implies that $f - \delta\phi \in \tilde{C}^p_{\square}(Q,G)$. This proves that $\alpha = 0$. Hence λ is an isomorphism and the proof is complete.

The following theorems are now obvious.

Theorem 12.5. _If Q has the local extension property with respect to G, then the reduced cohomology group $H^p_*(Q,G)$ is a local property of Q and G when $p \geq 2$ or when $p = 1$ and Q operates simply on G._

Theorem 12.6. _The Cech cohomology groups $H^p_{\#}(Q,G)$, $p \geq 0$, of a compact connected group Q with coefficients in a finite dimensional vector group G are local properties of Q, that is, they are determined by any neighborhood U of 1 in Q as a local group._

Theorem 12.7. _If Q is a local group locally isomorphic with a compact connected semi-simple group and G is an abelian local Lie group on which Q operates, then $H^2_L(Q,G) = 0 = H^1_L(Q,G)$. If Q operates simply on G, then we also have $H^1_L(Q,G) = 0$._

13. Crossed Local Homomorphisms

Let Q and G be assumed as at the beginning of the chapter. To describe algebraically the first local cohomology group $H^1_L(Q,G)$ of Q over G, it leads to the continuous crossed local homomorphisms.

A continuous map ϕ: $V \to G$ defined on a neighborhood V of 1 in Q is called a _continuous crossed local homomorphism_ of Q into G if there exists a neighborhood V_0 of 1 in Q such that

$$\phi(x_1x_2) = \phi(x_1) + x_1\phi(x_2) \tag{13.1}$$

has meaning and is true for all $x_1 \in V_0$ and $x_2 \in V_0$. Put $x_1 = 1 = x_2$ in (13.1), then we get $\phi(1) = 0$. Hence ϕ is also a local 1-map of Q into G. (13.1) implies that the local 1-cochain [ϕ] is a local cocycle. Conversely, if ϕ: V → G is a local 1-map of Q into G such that [ϕ] is a local 1-cocycle, then δ [ϕ] = O implies that (13.1) holds for some V_0.

Two continuous crossed local homomorphisms of Q into G will be call <u>equivalent</u> if they are equivalent as local 1-maps. It is clear that the equivalence classes of the continuous crossed local homomorphisms are identical with the local 1-cocycles $Z^1_L(Q, G)$. Since $B^1_L(Q, G)$ = O by definition, we obtain the following assertion.

(13.2) <u>The first local cohomology group $H^1_L(Q, G)$ is the group of equivalent classes of the continuous crossed local homomorphisms of Q into G.</u>

If Q operates simply on G, then a crossed local homomorphism reduces to a local homomorphism. Hence we have the following assertion, which is a consequence of (12.7)

(13.3) <u>Every continuous local homomorphism ϕ of a compact connected semi-simple group Q into an abelian Lie group G is locally trivial, that is, there is a neighborhood W of 1 in Q such that ϕ (W) = O.</u>

14. Local Group Extensions

For simplicity, a neighborhood of the neutral element of a local group will be called a <u>neutral neighborhood</u>.

An <u>open continuous local homomorphism</u> of a local group X onto a local group Y is a continuous Map ϕ: U → Y defined on a neutral neighborhood U of X into Y such that there exists a neutral neighborhood $U_0 \subset U$ of X satisfying the condition that

$$\phi(x_1x_2) = \phi(x_1)\phi(x_2)$$

for all x_1 and x_2 in U_o and the image $\phi(W)$ of every open neutral neighborhood $W \subset U_o$ of X is an open neutral neighborhood of Y. ϕ is called an open continuous local isomorphism of X onto Y if U_o can be so chosen that $\phi \mid U_o$ is one-to-one.

A local group extension of the local group G by the local group Q is a triple (E, ϕ, θ) where E is a local (mulitplicative) group, ϕ is an open continuous local homomorphism of E onto Q, and θ is an open continuous local isomorphism of G onto the kernel of ϕ.

A local cross-section of a local group extension (E, ϕ, θ) of G by Q is a continuous map u: $V \to E$ defined on a neutral neighborhood V of Q into E such that $\phi u(x) = x$ for each $x \in V$. A local group extension (E, ϕ, θ) is said to be fibered [1] if it has a local cross-section; it is said to be inessential [2] if it has a local cross-section u which is a local homomorphism of Q into E, [18, p. 85].

Let (E, ϕ, θ) be a fibered local group extension of G by Q. Choose a local cross-section u: $V \to E$ of (E, ϕ, θ), Take a neutral neighborhood $V_o \subset V$ of Q and a neutral W_o of G such that both $\theta(xg)$ and $u(x)\theta(g)u(x)^{-1}$ are defined whenever $x \in V_o$ and $g \in W_o$. (E, ϕ, θ) is said to be corresponding to the given way in which Q operates on G if V_o and W_o can be so chosen that

$$u(x)\theta(g)u(x)^{-1} = \theta(xg) \tag{14.1}$$

for all $x \in V_o$ and $g \in W_o$. In particular, if Q operates simply on G, then $\theta(W_o)$ is contained in the center of E and (E, ϕ, θ) is called a central extension of G by Q. Here the center C of a local group E is defined by the condition that $x \in E$ is in C if and only if there is a neutral neighborhood U of E such that $xy = yx$ for all $y \in U$.

Let us consider the set of all local group extensions of G by Q corresponding to the given way in which Q operates on G. Any two of such extensions (E_1, ϕ, θ_1) and (E_2, ϕ_2, θ_2) are said to be equivalent if there exists an open continuous local isomorphism $\sigma: E_1 \approx E_2$ such that there are a neutral neighborhood U of E_1 and a neutral neighborhood W of G satisfying the condition that

$$\sigma\theta_1(g) = \theta_2(g), \phi_2\sigma(y) = \phi_1(y)$$

for all $y \in U$ and $g \in W$, If $u_1: V_1 \to E$ is any local cross-section of (E_1, ϕ_1, θ_1), then the continuous map $u_2: V_2 \to E$ defined by taking $u_2(x) = \sigma u_1(x)$ for all $x \in V_2 = u_1^{-1}(U) \subset V_1$ is clearly a local cross-section of (E_2, ϕ_2, θ_2).

Let (E, ϕ, θ) be a given fibered local group extension of G by Q corresponding to the given way in which Q operates on G. Choose a local cross-section $u: V \to E$ of (E, ϕ, θ). Take a sufficiently small neutral neighborhood $V_o \subset V$ of Q and define a continuous 2-map $f: V_o^2 \to G$ by taking

(14.2) $$f(x_1, x_2) = \theta^{-1}[u(x_1)u(x_2)u(x_1x_2)^{-1}]$$

for each $x_1 \in V_o$ and $x_2 \in V_o$. Following the procedures of §5, one can show that the local 2-cochain [f] is a local 2-cocycle of Q over G which represents an element α of $H_L^2(Q, G)$ independent of the choice of the local cross-section u. α is called the element determined by (E, ϕ, θ). Equivalent extensions determine the same element. On the other hand, given a local 2-map $f: V_o^2 \to G$ of Q into G such that $\delta[f] = O$, we can construct a fibered extension (E_f, ϕ_f, θ_f) as in §5 which determines the element of $H_L^2(Q, G)$ represented by [f]. Finally, if f is the local 2-map defined by (14.2), then (E, ϕ, θ) is equivalent with (E_f, ϕ_f, θ_f). Hence we have the following assertion.

(14.3) The second local cohomology group $H^2_L(Q, G)$ is isomorphic with the group of the equivalence classes of the fibered local group extensions of G by Q corresponding to the given way in which Q operates on G. In particular, the zero element of $H^2_L(Q, G)$ corresponds to the class of inessential extensions of G by Q.

With reference to Theorem 12.7, (14.3) gives immediately the following theorem.

Theorem 14.4 If Q is a local group locally isomorphic with a compact connected semi-simple group and G is an abelian local Lie group, then every fibered local group extension of G by Q is inessential.

Let us assume in the remainder of this section that Q and G are (global) groups as in the first chapter, then every topological group extension of G by Q which has a local cross-section is, by definition, also a fibered local group extension of G by Q. It is natural to raise the question whether or not every equivalence class of the fibered local group extensions of G by Q can be obtained in this way. This question is answered negatively by the example which follows.

Take Q to be the 2-dimensional toroidal group and G the group of real numbers mod 1. Let Q operate simply on G. It follows from (12.2) and (9.2) that $H^2_L(Q, G)$ is isomorphic with the additive group of real numbers. Hence, by (14.3), there are essential central local group extensions of G by Q. On the other hand, by a proposition of Calabi and Ehresmann [1], every topological group extension of G by Q is trivial and hence it gives always an inessential local group extension of G by Q.

15. Compact Groups with Lie Centers

Let E be a compact connected group and assume that the center C of E is a Lie group. Denote by G the connected component of C which contains the neutral element. Then G is clearly a finite dimensional toroidal group.

Since every compact connected solvable group is abelian [12, p. 517] and every compact abelian normal subgroup of a connected group is contained in the center [12, p. 515], G is the radical of E, that is, the uniquely determined maximal closed connected solvable normal subgroup of E, [12, p. 553]. It follows easily that the quotient group $Q = E/G$ is a compact connected semi-simple group. Let $\phi: E \rightarrow Q$ be the natural projection of E onto Q and $\theta: G \rightarrow G$ the identity isomorphism. Then (E, ϕ, θ) form a local group extension of G by Q. According to a theorem of Gleason [8, p. 39], since G is a compact Lie group, (E, ϕ, θ) has a local cross-section. Hence (E, ϕ, θ) is a fibered local group extension of G by Q. By Theorem 14.4, (E, ϕ, θ) is inessential, that is, there is a local cross-section $u: V \rightarrow E$ of (E, ϕ, θ) which is a continuous local homomorphism of Q into E. Since $\phi u(x) = x$ for every $x \in V$, u is actually an open continuous local isomorphism of Q onto a locally closed local subgroup $S = u(V) \subset E$. Since G is contained in the center of E, it is not difficult to see that S is a normal local subgroup of E [18, p. 84] and that the local group E decomposes into direct product of the normal local subgroups G and S, [18, p. 85]. Hence we obtain the following theorem.

Theorem 15.1. If the connected component G of the center of a compact connected group E which contains the neutral element is a Lie group, then E is locally isomorphic with the direct product $G \times Q$ of G and the semi-simple quotient group $Q = E/G$.

In case E is a compact connected Lie group, Theorem 15.1 reduces to the classical theorem [3, p.42] of locally decomposing a compact connected Lie group into it abelian part and a semi-simple part. This classical decomposition theorem is used only once in the above, namely, in the proof of Lemma 10.1 which is trivial if Q is a Lie group. Hence our argument reveals the fact that this classical decomposition is essentially only a consequence of the fact that the second Betti number of a compact connected semi-simple Lie group vanishes.

If, in addition to the hypotheses of Theorem 15.1, we assume that the quotient group Q is simply connected and locally connected, then by means of the arguments used in the prood of (5.5) we can prove that there is a cross-section $u^*: Q \to G$ of the (global) extension (E, ϕ) which is a homomorphism of Q into E. Call $S = u^*(Q)$, then $S \approx Q$. Since G is contained in the center of E, it is easy to see that E decomposes into direct product of the normal subgroups G and S. Hence we have the following

Theorem 15.2. Under the hypothesis of Theorem 15.1, if the quotient group $Q = E/G$ is simply connected and locally connected, then E is isomorphic with the direct product $G \times Q$ of G and Q.

16. Extensions with a non-Abelian Kernel

Analogous to a work of Eilenberg and MacLane [7], one can formulate a theory of topological group extensions and local group extensions of a non-abelian kernel with appropriate modifications and elaborations to meet with the topology of the groups and the fact that the groups are defined locally. However we shall only indicate partially a particular case of the local group extensions in such a way that we are just able to state the assertion (16.1) which is essential in the proofs of the theorems to follow.

Let K be a local group with an abelian local subgroup G as center and let Q be a local group which operates simply on G. A local group extension of K by Q is a triple (E, ϕ, θ) where E is a local group, ϕ is an open continuous local homomorphism of E onto Q, and θ is an open continuous local isomorphism of K onto the kernel of ϕ. Local cross-sections of (E, ϕ, θ) are defined as in §14 and, similarly, one can define the fibered and the inessential extensions of K by Q.

Let (E, ϕ, θ) be a fibered local group extension of K by Q. Choose a local cross-section $u: V \to E$ of (E, ϕ, θ). Take a neutral neighborhood $V_0 \subset V$ of Q and a neutral neighborhood W_0 of G such that $\theta^{-1}[u(x)\theta(k)u(x)^{-1}]$ is defined whenever $x \in V_0$ and $k \in W_0$. (E, ϕ, θ) is called an inner local group extension of K by Q if we can choose V_0 and W_0 in such a way that, for each $x \in V_0$, the map $k \to \theta^{-1}[u(x)\theta(k)u(x)^{-1}]$ defines an inner automorphism of the local group K.

Let us consider the set of all inner local group extensions of K by Q. One can define an equivalence relation between these extensions as in §14 after replacing G by K. The following assertion can be proved by suitable modifications of the proof of Theorem 11.1 in the work of Eilenberg and MacLane [7].

(16.1) The equivalence classes of the inner local group extensions of K by Q are in a one-one correspondence with the elements of the second local cohomology group $H_L^2(Q, G)$. In particular, the zero element of $H_L^2(Q, G)$ corresponds to the class of all inessential inner extensions of K by Q.

Let E be a compact connected group and K be a closed normal subgroup of E which is a Lie group and contains the radical of E, [12, p.553]. Let G denote the center of K. It follows from a theorem of Gotô [10, p.428] that the quotient group $Q = E/K$ is a compact connected semi-simple group. Let ϕ: $E \to Q$ denote the natural projection of E onto Q and θ: $K \to K$ the identity isomorphism. The (E, ϕ, θ) is a local group extension of K by Q. According to a theorem of Gleason [8, p.39], (E, ϕ, θ) has a local cross-section u: $V \to E$. For each $x \in V$, the correspondence $k \to u(x)ku(x)^{-1}$ defines a continuous automorphism $\alpha(x)$ of K onto itself. The correspondence $x \to \alpha(x)$ defines a continuous map α: $V \to A(K)$ of V into the topological group A(K) of all continuous automorphisms of K, [12, p.508]; the inner automorphisms of K form a closed normal subgroup I(K) of A(K). According to a lemma of Iwasawa [12, p.509], since K is a compact Lie group, I(K) is an open subgroup of A(K). Since $\alpha(1) \in I(K)$, there is a neutral neighborhood $V_o \subset V$ of Q such that $\alpha(x)$ is in I(K) for all $x \in V_o$. Hence (E, ϕ, θ) is an inner local group extension of K by Q. Since Q is a compact connected semi-simple group and G is an abelian Lie group, we have $H_L^2(Q,G) = O$ according to Theorem 12.7. Then it follows from (16.1) that there is only one equivalence class of the inner local group extensions of K by Q. Hence, (E, ϕ, θ) is equivalent with the trivial extension of K by Q, namely, the direct product $K \times Q$. We have proved the following theorem.

Theorem 16.2. If a closed normal subgroup K of a compact connected group E is a Lie group and contains the radical of E, then E is locally isomorphic with the direct product $K \times Q$ of K and the semi-simple quotient group $Q = E/K$.

Now let E be any topological group and K be a normal subgroup of E which is a compact semi-simple Lie group. Consider the quotient group $Q = E/K$ and the natural projection $\phi: E \to Q$. Let $\theta: K \to K$ denote the identity isomorphism. Once again we obtain an inner local group extension (E, ϕ, θ) of K by Q. Since the center G of K is discrete [18, p. 282], we deduce $H^2_L(Q, G) = 0$ by definition. Hence as above, we obtain the following theorem which is essentially a lemma of Gleason, [9, p. 89].

Theorem 16.3. If a compact normal subgroup K of a topological group E is a semi-simple Lie group, then E is locally isomorphic with the direct product $K \times Q$ of K and the quotient $Q = E/K$.

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Footnotes

1) Presented to the American Mathematical Society, September, 1951.

2) Numbers in square brackets refer to the bibliography at the end of the paper.

3) The circumflex over x_i indicates that x_i is omitted.

4) This cohomology group has been studied by A. Heller in an unpublished work with some other purpose. Our assertion (5.3) is also known to him.

5) Our definition of the semi-simplicity of a topological group Q coincides with that of M. Gotô [10] defined for the L-groups but differs with that of A. M. Gleason [9, p. 98]. However, Gleason's definition differs also with the classical one for Lie groups.

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