ON FLOWS OF MEASURE-PRESERVING TRANSFORMATIONS

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1. INTRODUCTION

The purpose of this paper is to investigate some conditions under which a measure-preserving transformation can be embedded into a "flow" (real-parameter, measurable group) of transformations. The class of transformations to be studied consists of the invertible, measure-preserving, ergodic transformations of the unit interval onto itself which have "discrete spectrum;" that is, of transformations whose proper functions form a basis for L_2. (A convenient summary of terminology and facts concerning such transformations is contained in [2].) By a *d.s. transformation* will be meant a member of this class; its *spectrum* means the set of proper values (point spectrum). A d.s. transformation is determined to within conjugacy by its spectrum, which is a denumerable subgroup of the complex numbers of modulus 1 (and any countable subgroup of the unit circle is the spectrum of a d.s. transformation); therefore the embeddability criteria to be derived will be in terms of spectra.

Halmos has shown in [1] that (in particular) a d.s. transformation has a square root if and only if -1 does not belong to its spectrum. In Section 2, this result is generalized to give the criterion for the existence of nth roots. The result and the proof are analogous to the square root case. In Section 3, a necessary and sufficient condition is derived for the possibility of embedding a d.s. transformation into a flow. The condition is somewhat cumbersome; however, it has corollaries which show that the embedding is possible for many transformations, but that the existence of roots of all orders is not sufficient.

2. Nth ROOTS

LEMMA 1. *Let* G *be a countable subgroup of the unit circle, and* n *an integer* ($n \geq 2$). *In order that there exist another subgroup* G' *of the unit circle, isomorphic to* G *in such a way that the correspondence of* g' *to* g *implies that* $g'^n = g$, *it is necessary and sufficient that* G *contains no* n*th root of unity, except unity itself.*

The proof will be omitted, since it is not difficult and may be carried out in an analogous manner to the proof of Lemma 3 of [1].

THEOREM 1. *A d.s. transformation* T *has an* n*th root if and only if its spectrum contains no* n*th root of* 1 *other than* 1 *itself.*

Proof. Suppose that $T = R^n$. Then the spectrum of T consists of the nth powers of the spectrum of R. Both spectra are groups, and the correspondence of $g' \in Sp(R)$ to $g'^n = g \in Sp(T)$ is one-to-one, as a consequence of ergodicity. Hence, by the lemma, the spectrum of T contains no proper nth root of unity.

Now suppose that the spectrum of T contains no nth root of 1. Any d.s. transformation may be represented as (is conjugate to) a rotation of a compact Abelian group—the group may be taken to be the character group of the spectrum of the transformation, and then the element by which the group is rotated is the character

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$z(g) = g$ which is the identity function on the spectrum. (See [2], page 53.) Let g' be the correspondent (in the group G' guaranteed by the lemma) of the number g in the spectrum of T; let $x(g) = g'$ for all g in Sp(T). Then $x(g)$ is a character of Sp(T), and $x^n(g) = z(g)$ for all g. Hence rotation of the character group by $x(g)$ is a transformation whose nth power is conjugate to T, so that T has an nth root.

COROLLARY 1. *A d.s. transformation* T *has roots of all orders if and only if the spectrum of* T *consists only of irrational rotations (only numbers of the form* $e^{2\pi i a}$, *where* a *is irrational).*

COROLLARY 2. *A d.s. transformation* T *has an* n*th root if and only if* T^n *is ergodic.*

COROLLARY 3. *Let* G *be a compact Abelian group satisfying the second countability axiom, and* g *an element of* G *such that the set of all (positive and negative) powers of* g *is dense in* G. *Then* g *has an* n*th root in* G *if and only if the powers of* g^n *are also dense in* G.

Proof. A rotation of a compact Abelian group always has discrete spectrum; it is ergodic if and only if the set of powers of the element by which the group is rotated is dense in the group ([2], pp. 28-30). The third corollary therefore follows from the second.

3. EXISTENCE OF A FLOW

LEMMA 2. *Let* G *be a compact Abelian group satisfying the second countability axiom. Let* $\{g_n\}$ *be a sequence of elements of* G *such that* $g_n^2 = g_{n-1}$, *and such that* $\{g_n\}$ *converges to the identity. Let* $\{\sigma_n\}$ *be a sequence of integers such that* $\sigma_n/2^n \to 0$. *Then* $\{g_n^{\sigma_n}\}$ *converges to the identity of* G.

Proof. It suffices to show that $\phi(g_n^{\sigma_n}) \to 1$ for every character ϕ of G. But the numbers $\phi(g_n)$ form a sequence of successive square roots of a number of modulus one which approach unity. Now if the numbers $z_n = e^{2\pi i \theta_n}$ $(0 \leq \theta_n < 2\pi)$ satisfy $z_n^2 = z_{n-1}$ and $z_n \to 1$, then either $\theta_n = \theta_{n-1}/2$ for all large enough n, or else $\theta_n = \theta_{n-1}/2 + \pi$ for large enough n. Whichever case holds for the $\phi(g_n)$, in view of the condition on the σ_n it is true that

$$\arg \phi(g_n^{\sigma_n}) = \arg \phi^{\sigma_n}(g_n) \to 0 \bmod (2\pi),$$

and the lemma follows.

THEOREM 2. *A d.s. transformation* T *can be embedded into a flow if and only if its spectrum can be embedded in a subgroup* H *of the unit circle which does not contain* -1 *and which has the property that each number in the spectrum has an infinite sequence of successive square roots in* H *which converge to* 1. (Since $-1 \notin H$, square roots are unique in H.)

Proof. Suppose that the transformation T $(T = T(1))$ is embedded into a measurable flow T(t). Let H consist of the union of the spectra of T, T(1/2), ⋯, $T(1/2^n)$, ⋯. No root of T can have -1 in its spectrum, for if it did it would not itself have a square root; therefore $-1 \notin H$. Since the spectrum of $T(1/2^n)$ consists of the squares of the elements of the spectrum of $T(1/2^{n+1})$, each member g of the spectrum of T has an infinite sequence of (unique) successive square roots in H. Now, according to a theorem of M. Stone [3],

$$U_t = \int_{-\infty}^{\infty} e^{it\theta}\, dE(\theta),$$

where U_t is the unitary operator on L_2 induced by the transformation $T(t)$, and where $E(\theta)$ is a spectral measure for U_1. From this representation it is easily seen that the sequence of those successive square roots of $g \in Sp(T)$ which belong to H does converge to 1; therefore the group H has the properties required in the statement of Theorem 2.

Next suppose that Sp(T) can be embedded into such a group H. A 2^nth root of T can be formed immediately by rotating the character group of Sp(T) by the character which maps $g \in Sp(T)$ into its 2^nth root in H. By the assumptions on H, each character of the character group of Sp(T) (any such character is formed by evaluating the characters of Sp(t) at a fixed element of Sp(T)) takes values on these 2^nth root characters which converge to 1, and so the constructed sequence of root characters approaches the identity character. By means of these 2^nth roots of T, a group of transformations can be defined for dyadic rational values of the parameter; it is convenient to continue to think of these transformations $T(\alpha_n/2^n)$ as characters of Sp(T) by which the character group is rotated.

Now if $\alpha_n/2^n \to t$, then $T(\alpha_n/2^n)$ has at least one limit point, since the character group is compact. Suppose that $\alpha_n/2^n \to t$ and $\beta_n/2^n \to t$, and that the sequences $T(\alpha_n/2^n)$ and $T(\beta_n/2^n)$ both converge to limits. Then

$$T(\beta_n/2^n) = T(\alpha_n/2^n)\, T^{\beta_n - \alpha_n}(1/2^n), \qquad \text{where} \qquad (\beta_n - \alpha_n)/2^n \to 0.$$

Therefore by Lemma 2 the limiting transformations must be the same; this limit is defined to be T(t). Since T(t) is multiplicative for dyadic rational t and continuous, the group property must hold for all t, and so the theorem is proved.

COROLLARY 1. *If the spectrum of a d.s. transformation has an independent set of generators (in particular, if it has a finite set of generators and contains no rational rotations), then the transformation may be embedded into a flow.*

Proof. For each generator, choose an infinite sequence of successive square roots which converges to 1; the group H generated by the union of these sequences satisfies the conditions of the theorem.

COROLLARY 2. *There exist d.s. transformations which possess roots of all orders but are not embeddable into a flow.*

Proof. Let G be the group generated by $e^{2\pi i a}$ ($a < 1$, irrational), together with an infinite sequence of successive square roots such that alternately

$$\arg \sqrt{z} \equiv \frac{1}{2} \arg z \qquad \text{and} \qquad \arg \sqrt{z} \equiv \frac{1}{2} \arg z + \pi,$$

where $0 \leq \arg z < 2\pi$. There exists a d.s. transformation with G for its spectrum, which by Corollary 1 to Theorem 1 has roots of all orders, but which by Theorem 2 can not be embedded in a flow.

REFERENCES

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2. ———, *Ergodic theory*, mimeographed lecture notes, University of Chicago, 1955.
3. M. H. Stone, *On one-parameter unitary groups in Hilbert Space*, Ann. of Math. (2) 33 (1932), 643-648.

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