A SIMPLE PROOF OF THE ERDÖS-MORDELL INEQUALITY FOR TRIANGLES

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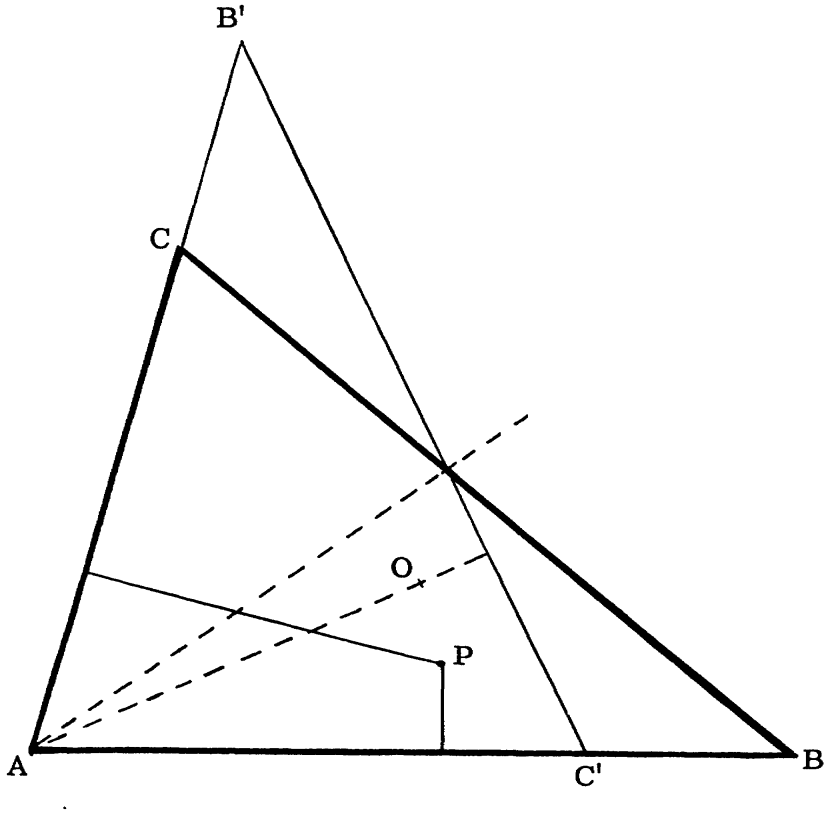
THEOREM (Erdös-Mordell). *For any triangle and any point* P *interior to it or on its boundary, the sum* S_1 *of the distances from* P *to the sides and the sum* S_2 *of the distances from* P *to the vertices satisfy the inequality*

$$S_2 \geq 2S_1;$$

equality holds only in the case where the triangle is equilateral and P *is its centroid.*

For a discussion of this theorem, see [2, pp. 12-14, 28]. My proof of it is based on the following theorem of Pappus [3; Book 4, Proposition 1]: Let ABC be any triangle, and let ABDE and ACFG be two parallelograms of which either both or neither lies entirely outside of ABC. Let H be the point of intersection of DE and FG (extended), and let BCKL be a parallelogram whose side CK is a translate of the vector AH. Then the sum of the areas of ABDE and ACFG is equal to the area of BCKL. I also use the well-known fact (see [1, p. 53]) that the bisector of any angle of a triangle is also the bisector of the angle made by the diameter of the circumcircle and the altitude drawn through the vertex.

Let a, b, and c be the lengths of the sides of the triangle ABC opposite the respective vertices A, B, and C, and let p_a, p_b, and p_c be the lengths of the



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perpendiculars from P to these sides. Let O be the center of the circumcircle. Consider the bisector of the angle at A, and replace the given triangle by its mirror image B'AC' with respect to this bisector. Now apply Pappus' theorem to B'AC', noting that OA is perpendicular to B'C', with the result that

$$\overline{AP}\cos(AP, AO)\,\overline{B'C'} = \overline{AC'}p_c + \overline{AB'}p_b .$$

In other words,

$$a\overline{PA}\cos(AP, AO) = bp_c + cp_b .$$

It follows that

$$a\overline{PA} \geq bp_c + cp_b ,$$

and, similarly,

$$b\overline{PB} \geq cp_a + ap_c ,$$

$$c\overline{PC} \geq ap_b + bp_a .$$

Thus

$$S_2 \geq \left(\frac{c}{b} + \frac{b}{c}\right)p_a + \left(\frac{a}{c} + \frac{c}{a}\right)p_b + \left(\frac{b}{a} + \frac{a}{b}\right)p_c .$$

Since $\cos(AP, AO) = 1$ only if P is on the diameter of the circumcircle which passes through A, equality holds above only if P is the center of the circumcircle. Each quantity in parentheses is greater than or equal to 2, and equality holds only if $a = b = c$. This proves the theorem. It should be remarked that the proof holds even if P lies outside the triangle, provided it remains inside the circumcircle. Of course, one must adopt a convention as to the signs of the p's if P is exterior to ABC.

REFERENCES

1. N. A. Court, *College Geometry*, Johnson, Richmond, 1925.

2. L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, Springer, Berlin, 1953.

3. Pappus d'Alexandrie, *La collection mathématique*, edited by P. VerEecke; Brouwer, Paris, 1933.

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