The Intersection of a Linear Subspace with the Positive Orthant

by

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1. This note discusses the geometry of those convex polyhedral cones[1] in Euclidean n-space E^n which are the intersection of the positive orthant P^n with some linear subspace. Cones of this sort occur in linear programming problems (cf. e.g. G. B. Dantzig, AAPA, p. 360). However, the present work was motivated by its application to a geometrical description of the frame of an arbitrary convex polyhedral cone, which description will appear in a subsequent paper.

Terminology and notation used in this paper will be defined only where they depart from those of Gerstenhaber (AAPA , Chap. XVIII).

2. In any Euclidean space E^m, the (closed) positive orthant P^m is the set of all vectors a such that $a \geqq 0$.

Given any $n \times k$ matrix A, that is, any linear transformation on E^k to E^n, the image AP^k of P^k under A is evidently the set of all positive linear combinations of the columns of A, considered as vectors in E^n. The class of all AP^k, for all k and A, is therefore by definition exactly the class of all convex polyhedral cones.

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1. See Cowles Commission, Activity analysis of production and allocation, ed. T. C. Koopmans (cited hereafter as AAPA), particularly Chaps. SVII (by D. Gale) and XVIII (by M. Gerstenhaber).

AP^k does not in general determine A uniquely. However, if AP^k is pointed the condition that A be a frame of AP^k (that is, that the columns of A be a frame of AP^k) determines A to within the following changes: any number of interchanges of columns and any number of multiplications of a column by a positive scalar.

A prime following the symbol for a matrix will denote the transpose. Vectors in Euclidean space will be thought of as column vectors, with the corresponding row vectors indicated by primes, and with the usual notation for the inner product.

By the "geometric polar" of a cone C will be meant its positive polar when it is imbedded in $D\{C\}$. For pointed cones, the geometric polar of the geometric polar is the original cone; and the relation of geometric polarity is the most natural duality relation between such cones.

The main results can now be stated.

THEOREM 1. *If* $AP^k = P^n \cap D\{AP^k\}$, *then* $A'P^n = P^k \cap D\{A'P^n\}$.

THEOREM 2. *Under the same hypothesis,* $A'P^n$ *is affine isomorphic to the geometric polar of* AP^k.

These two theorems will turn out to be closely related.

3. Let us begin by proving the first theorem.

$AP^k \subset P^n$ implies $A'P^n \subset P^k$, since both are equivalent to the statement that every matrix element of A is $\geqq 0$.

But $A'P^n \subset D\{A'P^n\}$ holds by definition, therefore $A'P^n \subset P^k \cap D\{A'P^n\}$. The non-trivial part is to prove inclusion in the other direction.

Consider therefore an arbitrary $x \in P^k \cap D\{A'P^n\}$. Is it the image under A' of some element of P^n? Surely $x = A'a$ for some $a \in E^n$, for it is clear that $D\{A'P^n\}$ is exactly the range $A'E^n$ of A'. Then also $x = A'(a + b)$ for any $b \in (AE^k)^{\perp}$, for $(AE^k)^{\perp}$, being the orthogonal complement of the range of A, is the nullspace of A'. The object will be to choose b so $a + b \in P^n$.

I point out next that $a \in (AP^k)^+$, the positive polar of AP^k. For, $x \in P^k = P^{k+}$, that is, for any $y \in P^k$, $0 \leq y'x = y'A'a = a'Ay$, and Ay takes on all values in AP^k.

The proof of the theorem is therefore reduced to that of the following

LEMMA. Let S be a linear subspace of E^n. Let $a \in (S \cap P^n)^+$. Then $a + S^{\perp}$ intersects P^n.

The proof is indirect.

Supposing the two sets in question do not intersect; it is not hard to see that there must be a pair of points for which the minimum distance is attained. There is no loss in generality in letting $a \in a + S^{\perp}$ and $d \in P^n$ be one such pair of points. Let $e = d - a$; $|e|$, the length of e, is the distance between the two sets, and is not zero.

e may be decomposed (uniquely) as $e = e_1 + e_2$, with $e_1 \in P^n$, $e_2 \in -P^n$, $e_1'e_2 = 0$. (e_1 is simply e with any negative components replaced by zeros.) If $e_2 \neq 0$, then $|e_1|^2 = |e|^2 - |e_2|^2 < |e|^2$, and $d - e_2 \in P^n$ is closer to a than d is. Therefore

$e_2 = 0$, $e \in P^n$.

I show next that $d'e = 0$. Only the case $d \neq 0$ needs proof. e may be decomposed (uniquely) as $e = \lambda d + e_3$, with $d'e_3 = 0$. If $\lambda \neq 0$, $\lambda \leqq 1$, then $|e_3| < |e|$ and $d - \lambda d \in P^n$ is closer to a than d is; while if $\lambda > 1$ then still $0 \in P^n$ is closer to a than d is. Therefore $\lambda = 0$, $d'e$ 1 0.

By a still more standard argument, e is orthogonal to $S^{\perp}$.

Since $e \in S \cap P^n$ has now been proved, $a'e \geqq 0$. But $a'e = (d - e)'e = 0 - |e|^2 < 0$, a contradiction. The proof is complete.

4. It may be interesting to state this result in more algebraic form. Proofs of all statements in this section are simple and will be omitted.

Consider the following three properties of matrices.

(1) All elements are non-negative.

(2) Any linear combination of the columns having only non-negative elements is also a positive linear combination of the columns.

(3) No column is a positive combination of the others.

For any $n \times k$ matrix A, properties (1) and (2) together are equivalent to the hypothesis of Theorem 1, while property (3) is equivalent to saying A is a frame (of AP^k).

Denote the duals of the three properties by (1'), (2'), (3'). Now (1) is evidently self-dual. Theorem 1 states that (1) and (2) imply (2').

(3) does not imply (3') (even in the presence of (1) and (2)). In fact any A may be broken down as

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $\begin{pmatrix} A_1 \\ A_3 \end{pmatrix}$ has property (3), $(A_1 \quad A_2)$ has property (3'), and A_1 has both, by a simple rearrangement of rows and columns. This decomposition, in case both AP^k and $A'P^k$ are pointed, is in fact unique, in the obvious sense.

5. Proof of Theorem 2: Identify $E^n \bmod (AE^k)^\perp$ with AE^k in the usual way. Then the natural mapping of E^n onto $E^n \bmod (AE^k)^\perp$ is replaced by the projection B onto AE^k. One checks from definitions that $B((AP^k)^+)$ is exactly the geometric polar of AP^k.

Now since the null-space of B is the same as that of A', BA'^{-1} is an affine isomorphism of $A'E^n$ onto AE^k. It must be shown to map $A'P^n$ onto $B((AP^k)^+)$; that is, BP^n must be shown to be the same as $B((AP^k)^+)$.

But this is easy to see. In the first place, $P^n \subset (AP^k)^+$ gives $BP^n \subset B((AP^k)^+)$. On the other hand, any $a \in (AP^k)^+$ satisfies the hypothesis of the Lemma, with S again taken as AE^k; hence $Ba \in BP^n$. The proof is complete.

Since the conclusion of Theorem 2 seems independent of the hypothesis that $AP^k = P^n \cap D\,\{AP^k\}$, it is worthwhile to give counterexamples showing this is really required.

Example 1. Take

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then A is a frame of a half-space in E^2. Its geometric polar is a half-line. However $A'P^2$ is a right angle (with interior, of course) in E^3.

The cone AP^3 in this example is intrinsically different from those discussed in Theorem 2, in that it has a non-zero lineality space. Accordingly I give also an example showing that the failure of the conclusion of the theorem can occur also when intrinsic properties are the same.

Example 2. Start with the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

This satisfies $AP^4 = P^4 \cap D\{AP^4\}$, and indeed one verifies the conclusion of Theorem 2.[2] For A is a frame of a regular tetrahedral angle, and A' is identical with A.

However, if

$$B = \begin{pmatrix} 1 & \sqrt{1/2} & \sqrt{1/2} & 0 \\ -1 & \sqrt{1/2} & \sqrt{1/2} & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix},$$

Then $B'P^4$ is actually congruent to AP^4, yet $B\,P^4$ is a trihedral angle.

2. The accident that $k = n$ in this example should not cause any difficulty in interpreting the symbols.