Unusual Generating Functions for Ultraspherical Polynomials

by

Fred Brafman

1. Introduction. A generating function equation for a set of polynomials $\{g_n(x)\}$ is an equation of type

$$(1) \qquad G(x, t) = \sum_{n=0}^{\infty} A_n g_n(x) t^n ,$$

where the coefficients A_n do not involve either x or t but may depend on the index of summation n, parameters (if any) of the set $\{g_n(x)\}$, and numerical factors. In many classical generating functions, the A_n is made up of ratios and products of factorial functions $(\alpha)_n$, where by definition

$$(2) \qquad (\alpha)_n = (\alpha + 1) \ldots (\alpha + n - 1) = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}$$

$$\text{for } n = 1, 2, 3, \ldots ; \ (\alpha)_0 = 1 \ \text{ if } \alpha \neq 0.$$

There has however been some interest paid to the case where the index of the factorial symbols making up A_n may be either n or $\left[\frac{n}{2}\right]$, the greatest integer in n/2. [1]

It is also obvious that some attention must be paid to the form permitted for G(x, t). One of the most useful forms is that G(x, t) consist of a finite sum of terms, each term of which is a product of a finite number of generalized hypergeometric functions,

(3) $$ {}_pF_q \left[\begin{matrix} a_1, & a_2, & \ldots, & a_p; \\ b_1, & b_2, & \ldots, & b_q; \end{matrix} \; z \right] = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n \, z^n}{(b_1)_n (b_2)_n \ldots (b_q)_n \, n!} $$

Such a restriction will be adhered to in this note.

2. Statement of result. This work will be concerned with the ultraspherical polynomial set $\left\{ P_n^{(\alpha,\alpha)}(x) \right\}$, where [2]

(4) $$ P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \, x^n \, {}_2F_1 \left[\begin{matrix} -n/2, & \frac{-n+1}{2}; \\ & 1+\alpha; \end{matrix} \; \frac{x^2-1}{x^2} \right]. $$

The ultraspherical polynomials are a special case of the Jacobi polynomials

(5) $$ P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1 \left[\begin{matrix} -n, & 1+\alpha+\beta+n; \\ & 1+\alpha; \end{matrix} \; \frac{1-x}{2} \right]. $$

The following notation will hold throughout:

(6) $$ \rho = (1 - 2xt + t^2)^{1/2} $$

$$ \mu = (1 + 2xt + t^2)^{1/2} $$

$$ k = \left[\frac{n}{2} \right] $$

where the branch convention is

$$\rho \to 1 \text{ as } t \to 0 \text{ and } \mu \to 1 \text{ as } t \to 0 .$$

The main result to be presented here is

$$(7) \quad \rho^{-1}\left(\frac{1 - t^2 + \rho\mu}{2}\right)^{-\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, (1+\alpha)_k}{(1+\alpha)_n\, k!}\, t^n .$$

It will be noted that for $\alpha = 0$, there is obtained one of the classical generating function equations for Legendre polynomials $P_n(x) \equiv P_n^{(0,0)}(x)$:

$$(8) \quad \rho^{-1} = \sum_{n=0}^{\infty} P_n(x)\, t^n .$$

3. <u>Derivation of result (7).</u> Begin by observing the identity

$$(9) \quad \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, (1+\alpha)_k\, (A)_k\, t^n}{(1+\alpha)_n\, k!\, (B)_k}$$

$$= \sum_{n=0}^{\infty} \frac{P_{2n}^{(\alpha,\alpha)}(x)\, (2n)!\, (1+\alpha)_n\, (A)_n\, t^{2n}}{(1+\alpha)_{2n}\, n!\, (B)_n}$$

$$+ \sum_{n=0}^{\infty} \frac{P_{2n+1}^{(\alpha,\alpha)}(x)\, (2n+1)!\, (1+\alpha)_n\, (A)_n\, t^{2n+1}}{(1+\alpha)_{2n+1}\, n!\, (B)_n} .$$

The symbols A and B are merely representative of numerator and denominator parameters and more (or none) can be inserted by repeating (or omitting) the behavior of A and B.

Next recall [3] the relations

$$P_{2n}^{(\alpha,\alpha)}(x)$$

$$= \frac{(1+\alpha)_{2n}(-1)^n\, n!\, P_n^{(-1/2,\alpha)}(1-2x^2)}{(2n)!\,(1+\alpha)_n},$$

$$P_{2n+1}^{(\alpha,\alpha)}(x)$$

$$= \frac{(1+\alpha)_{2n+1}\, n!\,(-1)^n\, x\, P_n^{(1/2,\alpha)}(1-2x^2)}{(1+\alpha)_n\,(2n+1)!}$$

where $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials, defined by (5). The identity (9) now becomes

$$(11)\quad \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\,(1+\alpha)_k\,(A)_k\, t^n}{(1+\alpha)_n\, k!\,(B)_k}$$

$$= \sum_{n=0}^{\infty} \frac{P_n^{(-1/2,\alpha)}(1-2x^2)\,(A)_n\,(-t^2)^n}{(B)_n}$$

$$+\; xt \sum_{n=0}^{\infty} \frac{P_n^{(1/2,\alpha)}(1-2x^2)\,(A)_n\,(-t^2)^n}{(B)_n}$$

Suppose now there is no A or B. Then (11) becomes

$$(12)\quad \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, (1+\alpha)_k\, t^n}{(1+\alpha)_n\, k!}$$

$$= \sum_{n=0}^{\infty} P_n^{(-1/2,\alpha)}(1-2x^2)\,(-t^2)^n$$

$$+ \, xt \sum_{n=0}^{\infty} P_n^{(1/2,\alpha)}(1-2x^2)\,(-t^2)^n .$$

Let the right side of (12) be denoted by

$$h_1(xt) + xt\, h_2(xt) .$$

Now consider [4] the generating function equation

$$(13)\quad \left(\frac{1-t+\rho}{2}\right)^{-\beta}\left(\frac{1+t+\rho}{2}\right)^{-\alpha}\rho^{-1}$$

$$= \sum_{n=0}^{\infty} P_n^{(\beta,\alpha)}(x)\, t^n .$$

By adapting (13) to the special requirements of $h_1(xt)$ and $h_2(xt)$ separately it is easily shown that

$$(14)\quad h_1(xt) + xt\, h_2(xt)$$

$$= \left(\frac{1-t^2+\rho\mu}{2}\right)^{-\alpha}\rho^{-1}\mu^{-1}\, C(x,t) ,$$

where $C(x,t) = \dfrac{\mu(\mu+\rho)}{\sqrt{2(1+t^2+\rho\mu)}}$.

Taking $\sqrt{C^2(x,t)}$, with the branch convention adopted above, shows that $C(x,t) = \mu$. Combining (12) and (14) now gives the result (7).

4. Conclusions and other results. The proof of (7) above used the generating function (13) for Jacobi polynomials. This is of course not necessary as the right side of (11) can be summed directly. However, use of (13) is a convenience and points out other results. By using other generating functions for Jacobi polynomials and adopting the necessary parameters A and B in (11) for each generating function used, other results follow immediately. The use [5] of generating functions

$$(15)\quad {}_0F_1\left[\begin{matrix} - ; & \frac{t(x-1)}{2} \\ 1+\alpha ; & \end{matrix}\right] {}_0F_1\left[\begin{matrix} - ; & \frac{t(x+1)}{2} \\ 1+\beta ; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\beta)}(x)\, t^n}{(1+\alpha)_n (1+\beta)_n} ,$$

$$(16)\quad (1+t)^{-1-\alpha-\beta}\, {}_2F_1\left[\begin{matrix} \frac{1+\alpha+\beta}{2}, & \frac{2+\alpha+\beta}{2} ; & \frac{2t(x+1)}{(1+t)^2} \\ & 1+\alpha ; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n}{(1+\alpha)_n} P_n^{(\beta,\alpha)}(x)\, t^n ,$$

$$(17) \quad {}_2F_1\left[\begin{matrix} a,\ 1+\alpha+\beta-a; \\ 1+\beta; \end{matrix}\ \frac{1-t-\rho}{2}\right]$$

$$\times\ {}_2F_1\left[\begin{matrix} a,\ 1+\alpha+\beta-a; \\ 1+\alpha; \end{matrix}\ \frac{1+t-\rho}{2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n\,(1+\alpha+\beta-a)_n}{(1+\alpha)_n\,(1+\beta)_n}\, P_n^{(\beta,\alpha)}(x)\, t^n ,$$

allows sums to be made of

$$(18) \quad f_1(x,t) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, t^n}{(1+\alpha)_n\, k!\, (1/2)_k} ,$$

$$(19) \quad f_2(x,t) = \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, (3/2+\alpha)_k\, t^n}{(1+\alpha)_n\, k!} ,$$

$$(20) \quad f_3(x,t)$$

$$= \sum_{n=0}^{\infty} \frac{P_n^{(\alpha,\alpha)}(x)\, n!\, (a)_k\, (3/2+\alpha-a)_k\, t^n}{(1+\alpha)_n\, k!\, (1/2)_k} .$$

The notation $k = [\frac{n}{2}]$ of (6) has been used here; in (17) and (20), a is an arbitrary parameter. The functions f_1, f_2, f_3, have been obtained by the author in the required form; each consists of a finite sum of terms each of which is a product of a finite number of hypergeometric functions. However, f_1, f_2, f_3 do not have the simplicity of the left side of (7) and the actual functions will not be given here.

References and Footnotes

1. A Hermite polynomial generating function which involves the greatest integer symbol is given, with reference to Doetsch, by Szegö, Orthogonal Polynomials, A. M. S. Colloquium Publications, Vol. XXIII, 1939, p. 371. A generalization of that formula is given by Brafman, "Generating Functions of Jacobi and Related Polynomials", Proc. of the Amer. Math. Soc., Vol. 2, 1951, p. 949.

2. The transformations possible on ${}_pF_q$'s generally and on ${}_2F_1$'s in particular permit many other forms of expression for the various functions mentioned, but all are equivalent to those given.

3. Szegö, op. cit., p. 58.

4. Szegö, op. cit., p. 68.

5. Equation (15) is equivalent to that by Bateman, H., "A generalization of the Legendre polynomial", Proc. of the Lond. Math. Soc. (2), 3, (1905), 111-123. Equation (16) is listed in several works, but apparently the first is Watson, G. N, "Notes on generating functions of polynomials: (4) Jacobi polynomials", Jour. Lond. Math. Soc., 9 (1934), 22-28. Equation (17) is given by Brafman, op. cit, p. 944. In (16) and (17), the α and β are inverted from the standard order for ease of insertion in (19) and (20).