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# Chisini's Conjecture for Curves with Singularities of Type $x^n = y^m$

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## 1. Introduction

This paper is devoted to a classical problem that can be summarized as follows: Let $S$ be a nonsingular compact complex surface, let $\pi: S \to \mathbb{P}^2$ be a finite morphism having simple branching, and let $B$ be the branch curve; then (cf. [F2]), "to what extent does $B$ determine $\pi: S \to \mathbb{P}^2$"?

The problem was first studied by Chisini [Ch], who proved that $B$ determines $S$ and $\pi$, assuming (i) $B$ to have only nodes and cusps as singularities, (ii) the degree $d$ of $\pi$ to be greater than 5, and (iii) a strong hypothesis on the possible degenerations of $B$. Chisini posed the question of whether the first or the third hypothesis could be weakened. More recently, Kulikov [Ku] and Nemirovski [Ne] proved the result for $d \geq 12$, assuming $B$ to have only nodes and cusps as singularities.

In this paper we weaken the hypothesis about the singularities of $B$: we generalize the theorem of Kulikov and Nemirovski for $B$ having only singularities of type $\{x^n = y^m\}$, using the additional hypothesis of smoothness for the ramification divisor (automatic in the "nodes and cusps" case). Moreover, we exhibit a family of counterexamples showing that our additional hypothesis is necessary.

In order to more precisely state the problem and our results, we need to introduce a bit of notation.

Definition 1.1. A *normal generic cover* is a finite holomorphic map $\pi: S \to \mathbb{C}^2$, which is an analytic cover branched over a curve $B$ such that $S$ is a connected normal surface and the fiber over a smooth point of $B$ is supported on $\deg \pi - 1$ distinct points.

Two normal generic covers $(S_1, \pi_1)$, $(S_2, \pi_2)$ with the same branch locus $B$ are called (analytically) *equivalent* if there exists an isomorphism $\phi: S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.

The main interest in generic covers comes from the well-known fact that, by the Weierstrass preparation theorem, given an analytic surface $S \subset \mathbb{C}^n$, a generic

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projection $\pi\colon S \to \mathbb{C}^2$ is (at least locally, in order to ensure $\deg \pi < \infty$) a normal generic cover branched over a curve (see [GuRo]).

A standard way to study generic covers is as follows. Given a generic cover $\pi\colon S \to \mathbb{C}^2$ with branch curve $B$, define the *monodromy homomorphism* $\rho$: $\pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_{\deg \pi}$ as the action of this fundamental group on the fiber of a regular value.

The pair $(B, \rho)$ gives the "building data" of the cover: one can reconstruct the cover from $(B, \rho)$ (cf. [GrRe]). Despite this explicit construction, understanding the singularity of the cover from the building data is very difficult (except in specific cases). It is, for example, still an open problem to classify all the possible "building data" coming from smooth surfaces.

In [MP] we gave a complete classification of the normal generic covers branched over irreducible curves of type $\{x^n = y^m\}$ in terms of what we called "monodromy graphs"; we will recall briefly this result and the definition of monodromy graphs in the next section. Let us point out that, according to the Puiseux classification, this class of singularities is a natural first step for a complete classification.

Our first result (to which the balance of Section 2 is devoted) is a "more friendly" classification theorem that will be crucial in the following sections. Let $h, k, a, b$ be positive integers with $(h, k) = 1$, and consider the surface $S_{h,k,a,b}$ in $\mathbb{C}^4$ defined by the equations $hz^k + kw^h - (h+k)x^a = zw - y^b = 0$. Let $F\colon S_{h,k,a,b} \to \mathbb{C}^2$ be the projection on the $(x, y)$-plane.

THEOREM 1.2. *The map $F\colon S_{h,k,a,b} \to \mathbb{C}^2$ is a generic cover branched over $x^{a(h+k)} = y^{bhk}$ of degree $h + k$.*

*Conversely, up to exchanging $x$ and $y$, every generic cover $\pi\colon S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over $\{x^n = y^m\}$ with $(n, m) = 1$ is equivalent to one of the previous maps.*

In Section 3 we consider the "global" case of projective generic covers.

DEFINITION 1.3. A *projective generic cover* is a finite morphism $\pi\colon S \to \mathbb{P}^2$ branched over an irreducible curve $B$ such that $S$ is an irreducible projective surface and the fiber over a smooth point of $B$ has cardinality $\deg \pi - 1$.

This is the same as requiring that $\pi^*(B) = 2R + C$, with $R$ irreducible and $C$ reduced, and that $\pi|_R\colon R \to B$ is $1:1$ over smooth points of $B$. As in the previous case, for each irreducible projective surface $S$, a generic projection $\pi\colon S \to \mathbb{P}^2$ is a projective generic cover branched over a (projective plane) curve $B$.

We say that a projective generic cover is *smooth* if the surface $S$ and the ramification divisor $R$ are nonsingular. Actually, when $S$ is nonsingular, a "general" generic projection has a ramification divisor $R$ that is nonsingular. Let us point out that, if $B$ has only nodes and cusps as singularities, then $R$ is automatically smooth.

Again, we will consider projective generic covers up to analytic equivalence: $(S_1, \pi_1)$, $(S_2, \pi_2)$ with the same branch locus $B$ are *equivalent* if there exists an isomorphism $\phi\colon S_1 \to S_2$ such that $\pi_1 = \pi_2 \circ \phi$.

CONJECTURE 1.4 (Chisini). *Let $B$ be the branch locus of a smooth projective cover $\pi: S \to \mathbb{P}^2$ of degree $\deg \pi \geq 5$. Then $\pi$ is unique up to equivalence.*

In other words, if $S$ is smooth and the degree high enough, then the curve $B$ determines the cover.

In fact, Chisini proved this result using the aforementioned additional hypotheses that the branch curve $B$ has only nodes and cusps as singularities and that $B$ has some particular degeneration. In the same paper [Ch], he wondered if these two last hypotheses could be weakened.

The bound for the degree of $\pi$ is needed according to a counterexample (due to Chisini and Catanese; see [Ca]) of a sextic curve with nine cusps that is the branch curve of four nonequivalent smooth projective covers, three of degree 4 and one of degree 3.

More recently, Kulikov [Ku] developed a new approach proving Chisini's conjecture for curves with only nodes and cusps as singularities, using the additional hypothesis that the degree of $\pi$ is greater than a certain function of the degree, genus, and number of cusps of the branch locus. After that, Nemirovski [Ne], using the Bogomolov–Miyaoka–Yau inequalities, found a uniform bound of 12 for the Kulikov function.

Combining these two results yields the following theorem.

THEOREM 1.5 [Ku; Ne]. *Let $\pi_1 : S_1 \to \mathbb{P}^2$ and $\pi_2 : S_2 \to \mathbb{P}^2$ be two smooth projective covers having the same branch curve $B$. Assume that $B$ has only nodes and cusps as singularities and that $\deg \pi_1 \geq 12$. Then $\pi_1$ and $\pi_2$ are equivalent.*

In Section 3, we use Theorem 1.2 to improve on the previous results as follows.

THEOREM 1.6. *Let $\pi_1 : S_1 \to \mathbb{P}^2$ and $\pi_2 : S_2 \to \mathbb{P}^2$ be two smooth projective covers having the same branch curve $B$. Assume that $B$ has only $r$ singular points of type $x^{n_i} = y^{m_i}$, $i = 1, \dots, r$. If*

$$\deg \pi_1 > \frac{4(3d + g - 1)}{2(3d + g - 1) - \sum_{i=1}^{r} (\min(m_i, n_i) - \gcd(m_i, n_i))},$$

*where $2d = \deg B$ and $g = g(B)$ is its genus, then $\pi_1$ and $\pi_2$ are equivalent.*

THEOREM 1.7. *Let $\pi_1 : S_1 \to \mathbb{P}^2$ and $\pi_2 : S_2 \to \mathbb{P}^2$ be two smooth projective covers having the same branch curve $B$. Assume that $B$ has only singularities of type $x^n = y^m$ and that $\deg \pi_1 \geq 12$. Then $\pi_1$ and $\pi_2$ are equivalent.*

Finally, in Section 4, we will construct a family of projective generic covers and will show that the hypothesis of smoothness for $R$ is necessary by finding pairs of nonequivalent projective generic covers of arbitrarily large degree having the same branch curve. More precisely, we prove (we defer the definitions of $\bar{f}_i$, $\bar{g}_j$ to Section 4) the following.

PROPOSITION 1.8. *Let $t \in \mathbb{N}$ with $t \geq 1$ and let $B$ be the projective plane curve given by the equation*

$$\bar{g}_{4t+1}(x, w)^{2t(2t+1)} = \bar{f}_{2t(2t+1)}(y, w)^{4t+1}.$$

*Then there are two generic covers $\pi'\colon S' \to \mathbb{P}^2$ and $\pi''\colon S'' \to \mathbb{P}^2$, with $S'$, $S''$ smooth and of degrees $4t+2$ and $4t+1$, respectively.*

*The ramification divisor is singular except in the case of the cover $\pi'$ for $t=1$.*

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## 2. Equations

Consider the following surface $S_{h,k}$ in $\mathbb{C}^4$ ($S_{h,k,1,1}$ in Section 1):

$$\begin{cases} hz^k + kw^h = (h+k)x, \\ zw = y, \end{cases} \tag{2.1}$$

where $1 \le h < k$ are coprime integers. The Jacobian matrix is

$$\begin{bmatrix} h+k & 0 & hkz^{k-1} & hkw^{h-1} \\ 0 & 1 & w & z \end{bmatrix},$$

from which we see that $S_{h,k}$ is smooth and that we can choose $z$, $w$ as local coordinates near $(0,0,0,0)$ for $S_{h,k}$.

Consider the map $F_{h,k}\colon S_{h,k} \to \mathbb{C}^2$, which is the restriction to $S_{h,k}$ of the projection of $\mathbb{C}^4$ on the $(x,y)$-plane.

PROPOSITION 2.2. *$F_{h,k}$ is a normal generic cover of degree $h+k$ branched over the curve $x^{h+k} = y^{hk}$.*

*Proof.* We have that $F_{h,k}^{-1}(0,0) = (0,0,0,0)$, and one can easily check that the degree of $F_{h,k}$ is $h+k$.

The equations of the ramification divisor $R$ in the local coordinates $(z,w)$ of $S_{h,k}$ are given by the vanishing of the determinant of the submatrix of the Jacobian matrix

$$\begin{bmatrix} hkz^{k-1} & hkw^{h-1} \\ w & z \end{bmatrix};$$

that is, $z^k = w^h$.

Substituting into the equations of $S_{h,k}$ in $\mathbb{C}^4$, we obtain that the locus defined by the equation $y^{hk} = z^{hk}w^{hk} = x^{h+k}$ in the $(x,y)$-plane contains the branch curve $B$. But this locus is irreducible since $(h,k)=1$, so we have found the equation of the branch curve.

We are left with the "genericness" check. Of course, it is enough (by irreducibility) to check over a smooth point of $B$, and we take the point $(1,1)$. Let $F_{h,k}^{-1}(1,1)$ be the set of points of the form $(1,1,z,w)$ described by the equations $\left\{\frac{hz^k+kw^h}{h+k} = zw = 1\right\}$. Then $z \neq 0$, $w = 1/z$, and (multiplying by $z^h$) we have to compute the solutions of

$$\left(\frac{hz+k}{h+k}\right)^{h+k} = z^h,$$

that is, the roots of the polynomial $P(z) = (hz+k)^{h+k} - (h+k)^{h+k} z^h$.

We must show that $P$ has exactly $h+k-1$ distinct roots. Its first and second derivatives are

$$P'(z) = h(h+k)[(hz+k)^{h+k-1} - (h+k)^{h+k-1} z^{h-1}],$$

$$P''(z) = h(h+k)[h(h+k-1)(hz+k)^{h+k-2} - (h-1)(h+k)^{h+k-1} z^{h-2}].$$

But $P(z) = P'(z) = 0$ implies

$$(hz+k)(h+k)^{h+k-1} z^{h-1} = (h+k)^{h+k} z^h,$$

and since 0 is not a root of $P$ we have $hz + k = (h+k)z$; that is, $z = 1$.

Since $P(1) = P'(1) = 0$ but $P''(1) \neq 0$, we conclude that $z = 1$ is a double root of $P$ and that all the others are simple roots. □

From the proof of Proposition 2.2 we may also derive the following.

REMARK 2.3. The ramification divisor $R$ is cut (on $S_{h,k}$) by the hypersurface $z^k = w^h$. The preimage of the branch locus $B$ is $2R + C$, where $C$ is the union of the curves cut by the hypersurfaces $z^k = \alpha w^h$ for $\alpha \neq 1$ a root of $P(t) = (ht+k)^{h+k} - (h+k)^{h+k} t^h$.

Now we introduce the complete class of covers that we need for our classification theorem.

Consider the pullback of $F_{h,k}$ under the base change given by the map

$$f_{a,b} \colon \mathbb{C}^2 \to \mathbb{C}^2, \quad f_{a,b}(x, y) = (x^a, y^b).$$

We obtain the surface $S_{h,k,a,b}$ of equations

$$\begin{cases} hz^k + kw^h = (h+k)x^a, \\ zw = y^b, \end{cases} \tag{2.4}$$

and the map $F_{h,k,a,b} \colon S_{h,k,a,b} \to \mathbb{C}^2$ given by the two coordinates $(x, y)$.

Now we can introduce the main result of this section.

THEOREM 2.5. *The maps $F_{h,k,a,b}$ are generic covers of degree $h+k$, branched over $x^{a(h+k)} = y^{bhk}$.*

*Conversely, up to exchanging $x$ and $y$, every generic cover $\pi \colon S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over $\{x^n = y^m\}$ with $(n, m) = 1$ is equivalent to one of the previous maps.*

The first part of the statement is the following lemma.

LEMMA 2.6. *The maps $F_{h,k,a,b}$ are normal generic covers of degree $h+k$, branched over the curve $x^{a(h+k)} = y^{bhk}$.*

*Proof*. The statement follows from Proposition 2.2 using the base change map $f_{a,b}$. The normality of $S_{h,k}$ implies the normality of $S_{h,k,a,b}$ by [MP, Thm. 2.2]. □

In order to prove the second part of Theorem 2.5, we use the well-known fact (already mentioned in Section 1; see [GrRe]) that the pair (branch curve $B$, monodromy homomorphism) determines the cover. We will now introduce precisely the monodromy homomorphisms and the monodromy graphs that represent them, in terms of which we gave (in [MP]) a classification theorem for generic covers branched over irreducible curves of type $\{x^n = y^m\}$, a result that we briefly recall here.

Let $(S, \pi)$ be a normal generic cover of degree $\deg \pi = d$. Every element in the fundamental group $\pi_1(\mathbb{C}^2 \setminus B)$ of the set of regular values of $\pi$ induces a permutation of the $d = \deg \pi$ points of the fiber over the base point and thus a homomorphism $\varphi\colon \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$, called the *monodromy* of the cover. The "generic" condition means that, for each geometric loop (i.e., a simple loop around a smooth point of the curve), its monodromy is a transposition. The homomorphisms with this property are called *generic* monodromies.

So, in order to classify generic covers $\pi : S \to \mathbb{C}^2$ of degree $d$ with $S$ a normal surface branched over some curve $B$, one needs to classify generic monodromies $\varphi\colon \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$. We have done so (in [MP], for curves $B$ of type $\{x^n = y^m\}$), representing the monodromy of a normal generic cover of degree $d$ branched on the curve $\{x^n = y^m\}$ by a labeled graph $\Gamma$, called a *monodromy graph*. We will denote by $\mathrm{Gr}_{d,n}$ the set of all (isomorphism classes) of graphs with $d$ vertices and $n$ labeled edges.

Note that the monodromies of equivalent generic covers differ by an inner automorphism of $\mathcal{S}_d$, so we will say that two monodromies $\varphi_1, \varphi_2\colon \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$ are *equivalent* if there exists a $\sigma \in \mathcal{S}_d$ such that

$$\varphi_1(\gamma) = \sigma\varphi_2(\gamma)\sigma^{-1}$$

for all $\gamma \in \pi_1(\mathbb{C}^2 \setminus B)$.

The representation is done as follows. Let $\varphi\colon \pi_1(\mathbb{C}^2 \setminus B) \to \mathcal{S}_d$ be a generic monodromy; if $\gamma_1, \dots, \gamma_n$ is a set of geometric loops that generates

$$\pi_1(\mathbb{C}^2 \setminus B) \cap \{y = 1\}$$

(in particular, they generate $\pi_1(\mathbb{C}^2 \setminus B)$; see e.g. [MP; O] for a more detailed description of this fundamental group), then we write $d$ vertices labeled $\{1, \dots, d\}$. Now $\mathcal{S}_d$ acts naturally on the set of our vertices, and then, for all $i \in \{1, \dots, n\}$, we draw the edge labeled $i$ between the two points exchanged by $\varphi(\gamma_i)$. Finally, we must delete the labeling of the vertices (this corresponds to considering $\varphi$ up to the equivalence relation introduced previously). Observe that the monodromy graph does not carry all the information needed to reconstruct the cover; $\Gamma$ has $n$ edges, but we have lost $m$.

For a fixed $\Gamma \in \mathrm{Gr}_{d,n}$, we say that $m$ is *compatible* with $\Gamma$ if $\Gamma$ defines a normal generic cover branched over $x^n = y^m$. Then, a pair $(\Gamma \in \mathrm{Gr}_{d,n}, m)$, where $m$ is compatible with $\Gamma$, determines the cover.

Finally, note that this construction is not symmetric in the two variables $x, y$. Hence, simply exchanging $m$ and $n$ yields a natural involution that sends compatible pairs $(\Gamma \in \mathrm{Gr}_{d,n}, m)$ to compatible pairs $(\Gamma' \in \mathrm{Gr}_{d,m}, n)$; we call this operation "duality".

We need a last definition, as follows.

DEFINITION 2.7. A polygon with $d$ vertices, valence $a$, and increment $j$, where $j$ and $d$ are relatively prime, is a graph with $n = ad$ and $d$ vertices such that, for all $s, t$, the edges labeled $s$ and $t$ have:

two vertices in common if and only if $s - t = \lambda d$;
one vertex in common if and only if $s - t = \lambda d + j$ or $s - t = \lambda d - j$;
no vertices in common otherwise.

This complicated definition is probably better explained by the example shown in Figure 1.

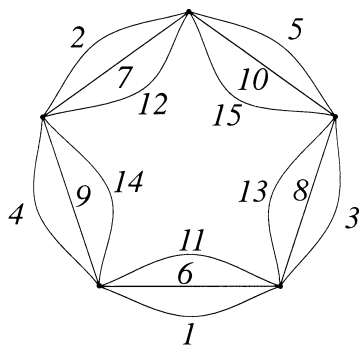


**Figure 1** A polygon with 5 vertices, valence 3, and increment 2

Now we are able to introduce the main result of [MP].

THEOREM 2.8. *The monodromy graphs for generic covers $\pi : S \to \mathbb{C}^2$ of degree $d \geq 3$ branched over the curve $\{x^n = y^m\}$, with $(n, m) = 1$, are the following.*

(1) *"Polygons" with $d$ vertices, valence $n/d$ (resp., $m/d$), and increment $j$, where $(j, d) = 1$, $j < d/2$, and $j(d - j)|m$ (resp., $j(d - j)|n$). Moreover, $d$ must divide $n$ (resp., $m$).*
(2) *"Double stars" of type $(j, d - j)$ and valence $n/j(d - j)$ (resp., $m/j(d - j)$), where $(j, d) = 1$, $j < d/2$, and $j(d - j)|n$ (resp., $j(d - j)|m$). Moreover, $d$ must divide $m$ (resp., $n$).*

*Duality takes graphs of type (1) to graphs of type (2), and vice versa.*

We skip here the definition of the double stars (cf. [MP]) that we do not need.

Briefly, in Theorem 2.8 we have shown that generic covers branched over an irreducible curve of type $\{x^n = y^m\}$ are classified by pairs (polygon in $\mathrm{Gr}_{d,n}$,

$m$ multiple of $j(d-j)$), up to exchanging $x$ and $y$. Recall that such pairs describe generic covers also when the hypothesis $(n,m)=1$ fails, but in this case we have examples of covers that cannot be described in this way (with monodromy graphs of different type). Given Theorem 2.8, proving the balance of Theorem 2.5 requires only the following.

PROPOSITION 2.9. *The normal generic cover branched over $x^{an}=y^{bm}$ associated to the polygon with $n$ edges, increment $h$, and valence $a$ is the cover $F_{h,n-h,a,b}$.*

*Proof.* We have to compute the monodromy graphs of the covers $F_{h,k,a,b}$. Let us start by considering the case $a=b=1$, that is, the covers $F_{h,k}$. Recall that $F_{h,k}$ is a normal generic cover branched over $B=\{x^n=y^m\}$ with $n=h+k$ and $m=hk$. Notice that the assumption $(h,k)=1$ implies $(n,m)=1$.

By Theorem 2.8, the monodromy graph $\Gamma$ is, up to exchanging $x$ and $y$, a polygon. In fact, we do not need to exchange $x$ and $y$; otherwise, we would have $d|m$ while $\deg F_{h,k}=n$ and $(n,m)=1$. Hence $\Gamma$ must be a polygon of valence 1 $(d=n)$ and some increment $h'$. Set $k'=n-h'$.

By [MP, Cor. 4.2], the smoothness of $S_{h,k}$ forces $m=h'k'$ (the minimal compatible integer for $\Gamma$). But now $h'+k'=h+k$ and $h'k'=hk$, so $\{h',k'\}=\{h,k\}$.

Summing up, we have proved that the monodromy graph of $F_{h,k}$ is a polygon with valence 1 and increment $h$ (or $k$). Of course, the corresponding $m$ is $hk$. We now remark that, for all $a,b$, $F_{h,k,a,b}$ can be obtained by fiber product from $F_{h,k}$ and the map $f_{a,b}\colon\mathbb{C}^2\to\mathbb{C}^2$ defined by $f_{a,b}(x,y)=(x^a,y^b)$. As shown in [MP], this fiber product acts on the "building data" of the cover, multiplying the valence by $a$ and the compatible $m$ by $b$. The corresponding monodromy graph is thus a polygon with $d=h+k$ vertices, valence $a$, and increment $h$ (or $k$). Conversely, the cover associated to a pair ("polygon with $n$ edges, valence $a$, and increment $h$", $m$) is $F_{h,n-h,a,m/h(n-h)}$, as stated. □

This concludes the proof of Theorem 2.5.

One immediately obtains the following corollary, whose first statement completes Corollary 4.2 in [MP].

COROLLARY 2.10. *The cover $F_{h,k,a,b}$ is smooth if and only if $a=b=1$ or $h=b=1$. The cover and the ramification divisor are both smooth if and only if $h=a=b=1$.*

*Proof.* The first statement follows from equations (2.4), whence the second can be easily checked in local coordinates as in Remark 2.3. □

In the next section we will use the following consequence.

COROLLARY 2.11. *Let $n$ and $m$ be coprime integers. Then there exists a nonsingular normal generic cover $\pi\colon S\to\mathbb{C}^2$ branched over $x^n=y^m$ for which the ramification divisor is nonsingular if and only if* (i) $|m-n|=1$ *or* (ii) $d=2$ *and* $n=1$.

*In the first case, the cover is unique of degree $d = \max(m, n)$ and its monodromy graph is the polygon with $d$ edges, increment 1, and valence 1. In the second case, the cover is given by the projection on the $(x, y)$-plane of the surface $z^2 = x - y^m$.*

*Proof.* For $d \geq 3$, by previous corollary, we have only the covers having as monodromy graph the polygon with $d$ edges, increment 1, and valence 1. For $d = 2$, the result is given immediately by the remark that, for every curve $\{f(x, y) = 0\}$, there is exactly one double cover given by projection on the $(x, y)$-plane of the surface $z^2 = f$. $\square$

We conclude this section with a direct computation of the monodromy graph associated to $\pi = F_{h,k,a,b}$, although we don't need it in the rest of the paper; the uninterested reader can skip directly to the next section. We refer to [MP] for notation.

In order to see how the minimal standard generators act on the preimage of $(1 - \varepsilon, 1)$, we examine the inverse image of the path $(\lambda\beta, 1)$ for $0 \leq \lambda \leq 1$ in the $(x, y)$-plane, where $\beta^{h+k} = 1$. Since $zw = 1$, we can substitute for $w$ in the first equation (2.1) to obtain

$$hz^{h+k} - (h + k)z^h \lambda\beta + k = 0. \tag{2.12}$$

We claim that, if $\lambda \neq 0, 1$, then $z^{h+k}$ is real if and only if $h + k$ is odd and $z = s\beta^{-1}$ with $s$ negative. Indeed, if $z^{h+k}$ is real, then $z = s\beta^{-1}$ for some real $s$ and $s$ is a zero of the real polynomial function

$$f(s) = hs^{h+k} - (h + k)\lambda s^h + k.$$

Since $f'(s) = h(h + k)s^{h-1}(s^k - \lambda)$, $f$ will have $s = 0$ as a critical point if $h > 1$ and either one other critical point $s = \sqrt[k]{\lambda}$ if $k$ is odd or two other critical points $s = \pm\sqrt[k]{\lambda}$ if $k$ is even.

If $s^k = \lambda$ then $f(s) = k(1 - \lambda s^h)$, which is strictly positive because either $s < 0$ or $0 < \lambda$ and $s < 1$. Thus $f$ has only strictly positive critical values and hence has at most one zero $s_0$; if $f$ does have a zero, then $h + k$ is odd and $s_0 < 0$. If $\lambda = 1$, the same argument shows that $z^{h+k}$ is real if and only if $s = 1$ or $h + k$ is odd and $s < 0$. Note that, since $(h, k) = (h, h + k) = (k, h + k) = 1$, the equations $z^k = \beta$ and $z^k = \beta^{-1}$ for $\beta \neq 1$ and $\beta^{h+k} = 1$ have a unique common solution: $z = \beta^s$, where $sk \equiv -sh \equiv 1 \bmod h + k$.

Now, if $z_0$ is a root of (2.12) with $\beta = 1$ then $z = z_0/\beta^s$ is also a root of (2.12), so we may restrict to the case $\beta = 1$. Observe that if $\lambda = 0$ then $z^{h+k} = -k/h$, whereas if $\lambda = \beta = 1$ then (2.12) has $z = 1$ as a double root, a real negative root if $h + k$ is odd, and no other real roots.

Set

$$\beta_0 = (h + k)^{1/(h+k)} \exp\left\{-i\tfrac{\pi}{h+k}\right\} \quad \text{and} \quad \alpha_0 = \exp\left\{i\tfrac{2\pi}{h+k}\right\}.$$

Then each component of $F_{h,k}^{-1}(\lambda, 1)$ will start from one of the points $\alpha_0^r\beta_0$ (each component from a different point) for $r = 0, \ldots, h + k - 1$. Call $c_r$ the component of $F_{h,k}^{-1}(\lambda, 1)$ that starts from $\alpha_0^{r-1}\beta_0$. Then $c_1$ is contained in the region

$-\pi < (h+k)\arg(z) < 0$, $c_2$ is contained in the region $0 < (h+k)\arg(z) < \pi$, and they both have $z = 1$ as ending point. Also, $c_{h+k+3-r} = \overline{c_r}$ for $3 \le r < (h+k+3)/2$ are complex conjugated paths and $c_r$ must be contained in one of the two regions $(2r-4)\pi < (h+k)\arg(z) < (2r-3)\pi$ or $(2r-3)\pi < (h+k)\arg(z) < (2r-4)\pi$ (see Figure 2). Note that if $h+k$ is odd then $c_{[(h+k)/2]+2}$ is contained in the negative real half-line.

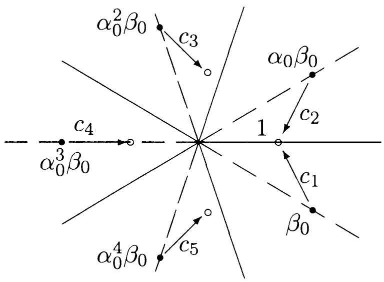


**Figure 2** Configuration of the paths $F_{h,k}^{-1}(\lambda, 1)$ in case $h+k=5$

Number the points $z_1, \dots, z_{h+k}$ in $F_{h,k}^{-1}(1-\varepsilon, 1)$ by the path $c_r$ to which they belong. It is clear that $z_1$ and $z_2$ are near $z = 1$ and that the action of $\gamma_1$ exchanges $z_1$ and $z_2$. In order to see which is the action of $\gamma_{h+1}$, follow the motion of the points over the path $((1-\varepsilon)(1-t), 1)$ and the path $(t(1-\varepsilon)\alpha_0^h, 1)$ for $0 \le t \le 1$. Recall that the paths over $(t(1-\varepsilon)\alpha_0^h, 1)$ are obtained from the paths $c_r$ by multiplying by $\alpha_0^{-sh} = \alpha_0$; thus, the action of $\gamma_{1+h}$ exchanges $z_2$ and $z_3$.

By the same argument, the action of $\gamma_{1+rh}$ will exchange $z_{r+1}$ and $z_{r+2}$, where indices are taken to be cyclical (mod $h+k$); that is, the monodromy graph associated to $F_{h,k}$ is the polygon with $h+k$ edges, increment $h$, and valence 1 (cf. Definition 2.7).

## 3. Chisini's Conjecture

In this section we will obtain results similar to those in [Ku] and [Ne] for curves with singularities of type $x^n = y^m$.

Let $B \subset \mathbb{P}^2$ be an irreducible curve with singularities of type $\{x^n = y^m\}$ only. Throughout this section, for every such curve we write

$$\mathrm{Sing}(B) = \{p_1, \dots, p_r\},$$

where locally (near $p_i$) $B$ is equivalent to $x^{s_i n_i} = y^{s_i m_i}$ for all $i = 1, \dots, r$ and with $(n_i, m_i) = 1$; we set $n_i < m_i$ (unless $n_i = m_i = 1$).

PROPOSITION 3.1. *Suppose $B$ is the branch curve of a smooth projective generic cover (cf. Definition 1.3) $\pi : S \to \mathbb{P}^2$, and let $R$ be the ramification locus of $\pi$.*

*Then, when restricted to the preimage of a small neighborhood of $p_i$, $\pi$ is given by* $\deg\pi - n_i s_i$ *connected components $U_1, \ldots, U_{s_i}$ and $V_1, \ldots, V_{\deg\pi-(n_i+1)s_i}$ such that* (i) *when restricted to one of the $U_j$, $\pi$ gives a generic cover of degree $n_i + 1$ branched over one of the $s_i$ local irreducible components of $B$* (*different components for different $j$*), *and* (ii) *when restricted to each $V_k$, $\pi$ is an isomorphism. Moreover, if $n_i \geq 2$ then $m_i = n_i + 1$ and, locally, $\pi$ restricted to $U_j$ is equivalent to the cover $F_{1,n_i,1,1}$ for each $j = 1, \ldots, s_i$.*

*Proof*. Because we assumed $R$ to be nonsingular, it is locally irreducible; thus, for each $p \in R$ there exists a neighborhood $U \ni p$ such that $\pi(R \cap U)$ is irreducible and hence $\pi|_U$ is a smooth normal generic cover branched over an irreducible curve. Since the image of an irreducible curve is still an irreducible curve, the cover splits locally as disjoint union of covers, each branched over one of the (local) irreducible components of $B$.

In order to prove the first part of the statement, we still must compute the degrees of the cover restricted to the "relevant" components, which may be done directly via Corollary 2.11. In case $n_1 \geq 2$, the assumptions of smoothness of the surface and of the ramification divisor $R$ mean that Corollary 2.11 forces $m_i = n_i + 1 =$ the (local) degree of the cover. The local equation for these covers comes from Proposition 2.9. $\square$

REMARK 3.2. By the degrees computed in the previous proposition, we have $\deg\pi \geq \max\{s_i(n_i + 1)\}$.

We now introduce some notation. Let $\pi : S \to \mathbb{P}^2$ be a smooth projective generic cover, $B$ the branch curve, $B^*$ the dual curve to $B$, $R$ the ramification locus, and $C := \pi^*(B) - 2R$. We set $E := \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$ (so that $K_S = -3E + R$), $N := \deg\pi$, $d := (\deg B)/2$, $\delta := \deg B^*$, and $g := g(B) = g(B^*) = g(R)$. With a standard abuse of notation, we will not distinguish a divisor from the associated line bundle.

In order to prove the main theorem of this section, we follow now the arguments of Kulikov in our more general case. Although some of the proof of Kulikov works without correction, we have decided (for the convenience of the reader) to repeat also those proofs, with the exception of Proposition 3.8. We start with some numerical relations.

LEMMA 3.3.

(1) $d \in \mathbb{N}$;
(2) $E^2 = N$;
(3) $(E, R) = 2d$;
(4) $\delta = 4d + 2g - 2 - \sum_{i=1}^{r} s_i(n_i - 1)$.

*Proof*. By the Hurwitz formula we have

$$2 - 2g(E) = 2N - \deg B.$$

Thus $\deg B$ is even and so (1) is proved; (2) and (3) are trivial.

Using a generic projection onto a line yields

$$e(B) = 2de(\mathbb{P}^1) - \delta - \sum_{i=1}^{r}(s_i n_i - 1).$$

Thus

$$2 - 2g = e(R) = e(B) + \sum_{i=1}^{r}(s_i - 1) = 4d - \delta - \sum_{i=1}^{r} s_i(n_i - 1),$$

since $R$ is the normalization of $B$ and is obtained by separating locally the irreducible components of $B$. This completes the proof. □

LEMMA 3.4.

$$R^2 = 3d + g - 1.$$

*Proof*. By genus formula,

$$2g - 2 = (K_S + R, R) = (-3E + 2R, R) = -6d + 2R^2.$$ □

Since $\delta \geq 0$, by Lemma 3.4 and Lemma 3.3(4) we have the following.

COROLLARY 3.5.

$$\sum_{i=1}^{r} s_i(n_i - 1) \leq 2g - 2 + 4d < 2R^2 = 2(3d + g - 1).$$

By Hodge's index theorem ($E^2$ is positive by definition), we have

$$\begin{vmatrix} E^2 & (E, R) \\ (E, R) & R^2 \end{vmatrix} = N(3d + g - 1) - 4d^2 \leq 0.$$

This yields the following corollary.

COROLLARY 3.6.

$$N \leq \frac{4d^2}{3d + g - 1}.$$

We can now compute the invariants of $S$ as follows.

LEMMA 3.7.

$$K_S^2 = 9N - 9d + g - 1,$$

$$e(S) = 3N + \delta - 4d = 3N + 2g - 2 - \sum_{i=1}^{r} s_i(n_i - 1),$$

$$\chi(\mathcal{O}_S) = N + \frac{3g - 3 - 9d - \sum_{i=1}^{r} s_i(n_i - 1)}{12}.$$

*Proof*. Since $K_S = -3E + R$, we have $K_S^2 = 9N - 12d + R^2$. Using a generic pencil of lines in $\mathbb{P}^2$ and its preimage in $S$, we obtain

$$e(S) = 2e(E) - N + \delta,$$
$$e(E) = -(K_S + E, E) = 2N - 2d.$$

From Noether's formula it follows that

$$12\chi(\mathcal{O}_S) = K_S^2 + e(S) = 12N + 3g - 3 - 9d - \sum_{i=1}^{r} s_i(n_i - 1)$$

and we are done. □

Note that $\sum_{i=1}^{r} s_i(n_i - 1)$ must be divisible by 3.

Assume that there exist two nonequivalent smooth projective generic covers $(S_1, \pi_1)$ and $(S_2, \pi_2)$ with the same branch curve $B$. Write $N_i = \deg \pi_i$ and $\pi_i^*(B) = 2R_i + C_i$ for $i = 1, 2$. Let $X$ be the normalization of the fiber product $S_1 \times_{\mathbb{P}^2} S_2$. Denote by $g_i\colon X \to S_i$ and $\pi_{1,2}\colon X \to \mathbb{P}^2$ the corresponding natural morphisms, as summarized in the following diagram:

$$\begin{tikzcd} X \arrow[r, "g_1"] \arrow[d, "g_2"'] \arrow[dr, "\pi_{1,2}"] & S_1 \arrow[d, "\pi_1"] \\ S_2 \arrow[r, "\pi_2"] & \mathbb{P}^2. \end{tikzcd}$$

We have $\deg g_1 = N_2$ and $\deg g_2 = N_1$, so that $\deg \pi_{1,2} = N_1 N_2$.

The following result is proved in [Ku, Sec. 2, Prop. 2]. Although Kulikov assumes at the very beginning that $B$ has only nodes and cusps as singularities, the proof does not require this hypothesis.

Proposition 3.8. *If $(S_1, \pi_1)$ and $(S_2, \pi_2)$ are not equivalent, then $X$ is irreducible.*

Let $Y$ be the set of points $p \in X$ such that (a) $\pi_{1,2}(p) \subset \operatorname{Sing} B$ and (b) $\pi_1$ and $\pi_2$, restricted (respectively) to neighborhoods of $g_1(p)$ and $g_2(p)$, are normal generic covers with different branch loci.

Lemma 3.9. Sing $X \subset Y$.

*Proof.* If $g_1(p) \notin R_1$ or $g_2(p) \notin R_2$, then $p$ is clearly smooth.

At a point $p$ such that $p_1 = g_1(p) \in R_1$ and $p_2 = g_2(p) \in R_2$, we can choose small neighborhoods $V_i(p_i) \subset S_i$ and $U(\pi_{1,2}(p)) \subset \mathbb{P}^2$ such that $\pi_i(V_i) = U$ and both $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are equivalent (up to possibly different base changes) to one of the following:

(i) if $n_i = 1$ or $\pi_{1,2}(p)$ is a smooth point of $B$,

$$f_{2,1}\colon \mathbb{C}^2 \to \mathbb{C}^2 \text{ defined by } (x, y) \mapsto (x^2, y);$$

(ii) if $n_i \geq 2$, the projection on the $(x, y)$-plane of the surface in $\mathbb{C}^4$,

$$\begin{cases} n_i w + z^{n_i} = (n_i + 1)x, \\ zw = y. \end{cases}$$

Suppose that the branch loci of $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are the same. We then have, in the first case, that $V_1 \times_U V_2$ has equations in $\mathbb{C}^4$,

$$\begin{cases} x_1^2 = x_2^2, \\ y_1 = y_2, \end{cases}$$

and the normalization of $V_1 \times_U V_2$ is the disjoint union of two smooth surfaces ($x_1 = x_2,\ y_1 = y_2$ and $x_1 = -x_2,\ y_1 = y_2$) in $\mathbb{C}^4$.

In the second case, $\tilde{V} = V_1 \times_U V_2$ is the surface in $\mathbb{C}^6$,

$$\begin{cases} n_i w_1 + z_1^{n_i} = n_i w_2 + z_2^{n_i} = (n_i + 1)x, \\ z_1 w_1 = z_2 w_2 = y, \end{cases}$$

which has two irreducible components: namely,

$$\tilde{V}_+ = \begin{cases} w_1 = w_2, \\ z_1 = z_2, \\ n_i w_1 + z_1^{n_i} = (n_i + 1)x, \\ z_1 w_1 = y, \end{cases}$$

which is isomorphic to $V_i$ via $g_i$; and

$$\tilde{V}_- = \begin{cases} n_i w_1 = z_2(z_2^{n_i - 1} + z_2^{n_i - 2} z_1 + \cdots + z_2 z_1^{n_i - 2} + z_1^{n_i - 1}), \\ n_i w_1 + z_1^{n_i} = n_i w_2 + z_2^{n_i} = (n_i + 1)x, \\ z_1 w_1 = z_2 w_2 = y, \end{cases}$$

which is expressed by $g_1$ (resp., $g_2$) as a normal generic cover of degree $N_2 - 1$ (resp., $N_1 - 1$) branched over $C_1$ (resp., $C_2$).

Both $\tilde{V}_+$ and $\tilde{V}_-$ are smooth and intersect in $g_1^{-1}(R_1) \cap g_2^{-1}(R_2)$. The normalization will be the disjoint union of these two smooth components. $\square$

Now suppose $p \in Y$ and let $V_1$ (resp., $V_2$) be the neighborhood of $g_1(p)$ (resp., $g_2(p)$) as in the definition of $Y$; the branch loci of $\pi_1|_{V_1}$ and $\pi_2|_{V_2}$ are different.

PROPOSITION 3.10. *$X$ has only rational double points as singularities.*

*More precisely, for every point $P \in Y$, if $p_i = \pi_{1,2}(P)$, then $P$ is a point of $X$ of type $A_{m_i - 1}$, and these are all the singular points of $X$.*

For instance, if $n_i = m_i = 1$ (the case of nodes), we get $A_0$—that is, a smooth point.

*Proof.* By Lemma 3.9, $\pi_{1,2}(P) = p_i$ for some $i$.

If $n_i = 1$, we can assume the two branch loci to be $\{x = 0\}$ and $\{x + y^{m_i} = 0\}$, which yields

$$\begin{cases} x = z_1^2, \\ z_2^2 = x + y^{m_i}; \end{cases}$$

that is, $z_2^2 = z_1^2 + y^{m_i}$, which is clearly a singularity of type $A_{m_i - 1}$ (if $m_i = 1$ implies $X$ is smooth at $P$). On the other hand, if $n_i \neq 1$ then $m_i = n_i + 1$ and $V_1 \times_U V_2$ is the surface in $\mathbb{C}^4$,

$$\begin{cases} z_1^{m_i} - m_i x z_1 = -(m_i - 1)y, \\ z_2^{m_i} - m_i \alpha x z_2 = -(m_i - 1)y, \end{cases}$$

with $\alpha^{s_i m_i} = 1$ but $\alpha^{m_i} \neq 1$, which is isomorphic to the surface in $\mathbb{C}^3$,

$$z_1^{m_i} - m_i x z_1 = z_2^{m_i} - m_i \alpha x z_2,$$

which has a double point at the origin.

The Hessian matrix ($m_i \geq 3$) at the origin is

$$\begin{bmatrix} 0 & -m_i & -m_i\alpha \\ -m_i & 0 & 0 \\ -m_i\alpha & 0 & 0 \end{bmatrix}$$

and has rank 2, hence $X$ has in $p$ a singularity of type $A_k$ for some $k \geq 2$. In order to compute $k$, set $z := m_i(z_1 - \alpha z_2)$. Then, in the coordinate system $(x, z, z_1)$, our equation can be written as $z_1^{m_i} = z(x + f(z, z_1))$, with $f(0, 0) = 0$; setting then $\bar{x} = x + f$, we find that near the origin the triple $(\bar{x}, z, z_1)$ is still a coordinate system in terms of which $V_1 \times_U V_2$ has equation $z_1^{m_i} = \bar{x}z$, that is, the standard expression for the singularity $A_{m_i-1}$. □

Note that, if $P$ is singular for $X$, then $g_1^{-1}(R_1) \cap g_2^{-1}(R_2) \cap (V_1 \times_U V_2) = P$.

In general, if $D_1$ and $D_2$ are two divisors in a normal surface, then we define $(D_1, D_2)$ ("the greatest common divisor") as the greatest divisor contained in both. By the local equations for the ramification divisor given in Remark 2.3, we notice that the "singular" points in Proposition 3.10 are isolated points for $g_1^{-1}(R_1) \cap g_2^{-1}(R_2)$.

REMARK 3.11. If $R = (g_1^{-1}(R_1), g_2^{-1}(R_2))$, then $R$ does not intersect Sing $X$ and, by the local considerations in the proof of Lemma 3.9, $R$ is smooth and $g_i|_R \colon R \to R_i$ is a (unramified) double cover.

Let $F\colon \tilde{X} \to X$ be the resolution of singularities of $X$, and let $\tilde{g}_i = g_i \circ F$ and $\tilde{\pi}_{1,2} = \pi_{1,2} \circ F$. We define $\tilde{R} := F^*(R)$, $\tilde{C}_1 := F^*((g_1^{-1}(R_1), g_2^{-1}(C_2)))$, and $\tilde{C}_2 := F^*((g_1^{-1}(C_1), g_2^{-1}(R_2)))$.

PROPOSITION 3.12.

(1) $(\tilde{R}, \tilde{C}_j) = \sum_{i=1}^r s_i(n_i - 1)$;
(2) $\tilde{R}^2 = 2(3d + g - 1) - \sum_{i=1}^r s_i(n_i - 1)$;
(3) $\tilde{C}_1^2 = (N_2 - 2)(3d + g - 1) - \sum_{i=1}^r s_i(n_i - 1)$;
(4) $\tilde{C}_2^2 = (N_1 - 2)(3d + g - 1) - \sum_{i=1}^r s_i(n_i - 1)$.

*Proof*. By Remark 3.11, $R$ does not intersect the singular points of $X$; hence we can compute the intersections of $\tilde{C}_1$ and $\tilde{R}$ in $X$. By definition, $\tilde{C}_1$ and $\tilde{R}$ intersect only at points of the preimage of $R_2 \cap C_2$, in particular over some singular point of $B$.

But we have already noticed that the only points $p \in R$ such that $\pi_{1,2}(p) \in$ Sing $B$ are those points such that $\pi_1$ near $g_1(p)$ and $\pi_2$ near $g_2(p)$ are branched over the same curve, as considered in the proof of Lemma 3.9. Let $p \in X$ be such a point. Because $\pi_{1,2}(p)$ is a singular point of $B$, there exists an $i$ such that $\pi_{1,2}(p) = p_i$.

If $n_i = 1$ then $\pi_1$ and $\pi_2$ are locally double covers, so $C_j$ does not contain $p$ and $p$ does not contribute to the intersection number. Otherwise, let $V_1$ (resp., $V_2$) be a small neighborhood of $g_1(p)$ (resp., $g_2(p)$) as in Lemma 3.9. Then, since $g_1|_R$ (and also $g_2|_R$) is an unramified double cover, there are exactly two points over $p_i$ contained both in $R$ and in the normalization of the fiber product of $V_1$ and $V_2$, say $P_{i+}$ and $P_{i-}$. These two points belong to the two components $\tilde{V}_+$ and $\tilde{V}_-$, respectively (see the proof of Lemma 3.9), but since $C_j$ does not intersect $\tilde{V}_+$, we may suppose that $p = P_{i-} \in \tilde{V}_-$.

Rewriting the equations for $\tilde{V}_-$ yields

$$\begin{aligned} w_1 &= \frac{z_2(z_1^{n_i} - z_2^{n_i})}{n_i(z_1 - z_2)}, \\ w_2 &= w_1 + \frac{z_1^{n_i} - z_2^{n_i}}{n_i}, \\ x &= \frac{n_i w_1 + z_1^{n_i}}{n_i + 1}, \\ y &= z_1 w_1. \end{aligned}$$

We remark that all the members of these equations are polynomials, and we can take $z_1, z_2$ as holomorphic coordinates for $\tilde{V}_-$.

Now, $R \cap \tilde{V}_-$ is cut by $w_1 = z_1^{n_i}$ and $w_2 = z_2^{n_i}$; that is,

$$\begin{cases} z_2 \dfrac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_1^{n_i}, \\ z_1^{n_i} - z_2^{n_i} + z_2 \dfrac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_2^{n_i}. \end{cases}$$

This implies $z_1^{n_i} = z_2^{n_i}$; that is, $z_1 = \lambda z_2$ with $\lambda^{n_i} = 1$. But if $\lambda \neq 1$ then the left members in our equations vanish and we obtain $z_1 = z_2 = 0$ (which is not a curve). From this it follows that $z_1 = z_2$ clearly solves our equations, so it is the local equation we were looking for.

A branch of $F(\tilde{C}_2) \cap \tilde{V}_-$ is given (cf. Remark 2.3) by the equations $\alpha w_1 = z_1^{n_i}$ and $w_2 = z_2^{n_i}$, where $\alpha = \left(\frac{n_i+\alpha}{n_i+1}\right)^{n_i+1}$ for $\alpha \neq 1$; that is,

$$\begin{cases} z_2 \dfrac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = \dfrac{n_i}{\alpha} z_1^{n_i}, \\ z_1^{n_i} - z_2^{n_i} + z_2 \dfrac{z_1^{n_i} - z_2^{n_i}}{z_1 - z_2} = n_i z_2^{n_i}. \end{cases}$$

This yields

$$z_1^{n_i} = \frac{\alpha(n_i+1)}{n_i+\alpha} z_2^{n_i} = \left(\frac{n_i+\alpha}{n_i+1}\right)^{n_i} z_2^{n_i},$$

so that $z_1 = \lambda \frac{n_i+\alpha}{n_i+1} z_2$ with $\lambda^{n_i} = 1$.

Moreover, if we set $t = \frac{n_i+\alpha}{n_i+1}$ (so that $t^{n_i+1} = (n_i+1)t - n_i$), then $\lambda$ must satisfy (by the first equation)

$$(\lambda t)^{n_i-1} + (\lambda t)^{n_i-2} + \cdots + 1 = \frac{n_i}{t}.$$

Hence

$$t^{n_i} - 1 = (\lambda t)^{n_i} - 1 = (\lambda t - 1)\frac{n_i}{t}$$

or

$$t^{n_i+1} = (\lambda n_i + 1)t - n_i;$$

that is,

$$(n_i+1)t = (\lambda n_i + 1)t.$$

Thus $\lambda = 1$, and $F(\tilde{C}_2) \cap \tilde{V}_-$ is the union of the $n_i - 1$ curves $z_1 = \frac{n_i+\alpha}{n_i+1} z_2$. Therefore, every component of $\tilde{C}_j$ intersects $\tilde{R}$ transversally, and we conclude that $(\tilde{R}, \tilde{C}_j) = \sum_{i=1}^r s_i(n_i - 1)$.

Let $E_{\tilde{X}} = F^*\pi_{1,2}^*(\mathcal{O}_{\mathbb{P}^2}(1)) = F^*g_i^*(E_i)$. It is immediate to verify that

$$E_{\tilde{X}}^2 = N_1 N_2,$$

$$(E_{\tilde{X}}, \tilde{R}) = 4d,$$

$$(E_{\tilde{X}}, \tilde{C}_1) = 2d(N_1 - 2),$$

$$(E_{\tilde{X}}, \tilde{C}_2) = 2d(N_2 - 2).$$

Since the canonical divisor of $\tilde{X}$ is $F^*K_X$ ($X$ has only rational double points as singularities), we have

$$K_{\tilde{X}} = -3E_{\tilde{X}} + \tilde{R} + \tilde{C}_1 + \tilde{C}_2$$

and

$$(K_{\tilde{X}} + \tilde{R}, \tilde{R}) = e(\tilde{R}) = 4g - 4;$$

hence

$$\tilde{R}^2 = 6d + 2g - 2 - \sum_{i=1}^r s_i(n_i - 1).$$

Because $F^*g_1^*(R_1) = \tilde{R} + \tilde{C}_1$, we have

$$N_2 R_1^2 = (\tilde{R} + \tilde{C}_1, \tilde{R} + \tilde{C}_1),$$

from which we obtain

$$\begin{aligned}\tilde{C}_1^2 &= N_2(3d + g - 1) - 6d - 2g + 2 - \sum_{i=1}^r s_i(n_i - 1)\\ &= (N_2 - 2)(3d + g - 1) - \sum_{i=1}^r s_i(n_i - 1).\end{aligned}$$ □

We can now finally prove the next theorem.

THEOREM 3.13. *Let $B$ be the branch locus of a smooth projective generic cover $\pi : S \to \mathbb{P}^2$ having $r$ singular points of type $x^{n_i s_i} = y^{m_i s_i}$ with $n_i \leq m_i$, $(n_i, m_i) = 1$. Then, if*

$$\deg \pi > \frac{4(3d+g-1)}{2(3d+g-1) - \sum_{i=1}^r s_i(n_i-1)},$$

*where $2d = \deg B$ and $g = g(B)$ is its genus, then $\pi$ is unique.*

In Section 1 we wrote the statement using a different notation that we found better there.

*Proof*. By Corollary 3.5, $\tilde{R}^2 > 0$, so by the Hodge index theorem we have

$$\begin{vmatrix} \tilde{R}^2 & (\tilde{C}_1, \tilde{R}) \\ (\tilde{C}_1, \tilde{R}) & \tilde{C}_1^2 \end{vmatrix} = 2(N_2-2)(3d+g-1)^2 - N_2(3d+g-1)\sum_{i=1}^r s_i(n_i-1) \leq 0;$$

the same equation holds if we replace $\tilde{C}_1$ by $\tilde{C}_2$ and $N_2$ by $N_1$. Consequently, we obtain

$$N_j \leq \frac{4(3d+g-1)}{2(3d+g-1) - \sum_{i=1}^r s_i(n_i-1)}$$

for $j = 1, 2$. □

Following an idea of Nemirovski [Ne], we may prove the following.

THEOREM 3.14. *In the hypothesis of Theorem 3.13, if $\deg \pi \geq 12$ then $\pi$ is unique.*

*Proof*. If $S$ is not an irrational ruled surface of genus greater than or equal to 2, then it satisfies the Bogomolov–Miyaoka–Yau inequality

$$K_S^2 \leq 3e(S).$$

By Lemma 3.7,

$$K_S^2 = 9N - 9d + g - 1,$$

$$e(S) = 3N + 2g - 2 - \sum_{i=1}^r s_i(n_i-1);$$

hence

$$\sum_{i=1}^r s_i(n_i-1) \leq 3d + \frac{5}{3}(g-1).$$

With this inequality, we can estimate the quantity

$$\frac{4(3d+g-1)}{2(3d+g-1) - \sum_{i=1}^r s_i(n_i-1)} \leq \frac{12d + 4(g-1)}{3d + \frac{1}{3}(g-1)} = 4 + \frac{8(g-1)}{9d+g-1} < 12.$$

If $S$ is an irrational ruled surface then it satisfies

$$K_S^2 \leq 2e(S);$$

the same argument as before can now be used to show that

$$\sum_{i=1}^{r} s_i(n_i - 1) \leq \frac{-3N + 9d + 3(g-1)}{2} < \frac{3}{2}(3d + g - 1).$$

Thus we derive the stronger estimate

$$\frac{4(3d + g - 1)}{2(3d + g - 1) - \sum_{i=1}^{r} s_i(n_i - 1)} < 8. \qquad \square$$

As a last remark, note that one can rewrite (with the obvious changes) all the results in [Ku, Thms. 3–12].

## 4. A Family and a Counterexample

In this section we will describe an interesting family of projective generic covers branched over a curve $\bar{B}$ with singularities of type $x^n = y^m$ that will produce a counterexample to Chisini's conjecture if we drop the hypothesis that the ramification divisor is smooth.

Let $\bar{B} \subset \mathbb{P}^2$ be a plane curve of equation $\bar{g}(x, w) = \bar{f}(y, w)$, where $\bar{g}$ and $\bar{f}$ are homogeneous polynomials of degree $d$, of the form

$$\bar{g}(x, w) = \prod_{i=1}^{r} (x - \alpha_i w)^{n_i},$$

$$\bar{f}(y, w) = \prod_{j=1}^{s} (y - \beta_j w)^{m_j};$$

here $\alpha_1, \dots, \alpha_r$ and $\beta_1, \dots, \beta_s$ are mutually distinct. In a neighborhood $U_{i,j}$ of the point $P_{i,j} = (\alpha_i, \beta_j, 1)$, the curve $\bar{B}$ is analytically equivalent to $x^{n_i} = y^{m_j}$.

Our (open) assumption is that the singular points of $\bar{B}$ are contained in the union of lines $\bar{g}(x, w) = 0$ or (if you prefer) in the set of the $P_{ij}$. By a classical result (see [De; F1]), if $\bar{B}$ is a nodal curve then $\pi_1(\mathbb{P}^2 \setminus \bar{B})$ is abelian; since $S_d$ has no center if $d \geq 3$, it follows that, if $\pi_1(\mathbb{P}^2 \setminus \bar{B})$ is abelian, then there is no projective generic cover of degree $d \geq 3$ whose branch locus is $\bar{B}$. Thus we will suppose that not all $n_i \leq 2$ and not all $m_j \leq 2$.

Note that $p = (0, 1, 0)$ does not belong to $\bar{B}$; therefore, in computing $\pi_1(\mathbb{P}^2 \setminus \bar{B})$, we can use the projection from $p$ onto the $x$-axis. More precisely, $\bar{B}$ intersects transversally the line at infinity $w = 0$ in the $d$ smooth points $(1, \xi, 0)$ with $\xi^d = 1$; then the line at infinity is not tangent to $\bar{B}$. This allows us to compute the fundamental group of the complement of $\bar{B}$ by computing the fundamental group of the complement of the affine curve $B$ in the chart $w \neq 0$, as we will do in Proposition 4.6.

Set $g(x) = \bar{g}(x, 1)$ and $f(y) = \bar{f}(y, 1)$, so that $B = \{g(x) = f(y)\}$. In order to compute the fundamental group of the complement of $B$, we can (by a deformation argument as in [O]) make the following assumptions without loss of generality:

(1) $\alpha_i, \beta_j \in \mathbb{R}$ for all $i, j$;
(2) $\alpha_1 < \alpha_2 < \cdots < \alpha_r$ and $\beta_1 < \beta_2 < \cdots < \beta_s$;
(3) if $\gamma_1, \ldots, \gamma_{s-1}$ are the roots of $f'$ such that $f(\gamma_i) \neq 0$, then the critical values for $f$, $f_1 = f(\gamma_1), \ldots, f_{s-1} = f(\gamma_{s-1})$ are mutually distinct;
(4) for a suitable $\varepsilon_0 > 0$, $|g(x)| < \min_i |f_i|$ for all $x \in (\alpha_1 - \varepsilon_0, \alpha_r + \varepsilon_0)$.

Let us point out (in order to justify assumption (3)) that the roots of $f'$ are those $\beta_j$ for which $m_j \geq 2$ (with multiplicity $m_j - 1$) and those roots of a polynomial of degree $s-1$ that has, by assumption (1), $s-1$ distinct real roots $\gamma_1, \ldots, \gamma_{s-1}$ such that $\beta_i < \gamma_i < \beta_{i+1}$.

The critical points of the projection from $p$ onto the $x$-axis are given by the intersection of $B$ with the union of horizontal lines $\{f'(y) = 0\}$. Then the critical values are (some of) the $\alpha_i$ (corresponding to points $P_{i,j}$) and the $d(s-1)$ distinct points $\delta_{j,h}$ for $h = 1, \ldots, d$ and $j = 1, \ldots, s-1$, where $g(\delta_{j,h}) = f_j$ (smooth points with vertical tangent). By assumption (4), no $\delta_{j,h}$ is contained in the interval $[\alpha_1, \alpha_r]$.

Choose $\varepsilon > 0$ small enough so that, for all $j$ (resp., for all $i$) and for every $t$ such that $0 < |t - \beta_j| \leq \varepsilon$ (resp., $0 < |t - \alpha_i| \leq \varepsilon$), we have that $f^{-1}(t)$ (resp., $g^{-1}(t)$) is given by $m_j$ (resp., $n_i$) distinct points. We denote by $b_{j,1}, \ldots, b_{j,m_j}$ (resp., $a_{i,1}, \ldots, a_{i,n_i}$) the points in $f^{-1}(\varepsilon)$ (resp., $g^{-1}(\varepsilon)$) ordered by their argument.

We fix now a free basis for $\Pi = \pi_1(\{y = 0\} \setminus \{\alpha_i, \delta_{j,k}\}, a_{1,1})$ in terms of which we will describe the braid monodromy of the projection. Let $C_\varepsilon(z_0) \subset \mathbb{C}$ be the circle of center $z_0$ and radius $\varepsilon$. We define by $c_i$ the closed path supported on the connected component of $g^{-1}(C_\varepsilon(0))$ near $\alpha_i$, with starting point the unique real point bigger than $\alpha_i$ and with counterclockwise orientation; we define by $c_i^+$ the "subpath" contained in the positive half-plane (imaginary part greater than 0) and by $c_i^-$ the "subpath" in the negative half-plane. Let $l_i$, $i = 1, \ldots, r-1$, be the (positively oriented) path contained in the real line connecting $c_i \cap \mathbb{R}$ and $c_{i+1} \cap \mathbb{R}$ but not containing any of the $\alpha_j$ (see Figure 3). Let $\omega$ be the small path supported on $c_1$ connecting $a_{1,1}$ with the base point of $c_1$ (in the clockwise direction).

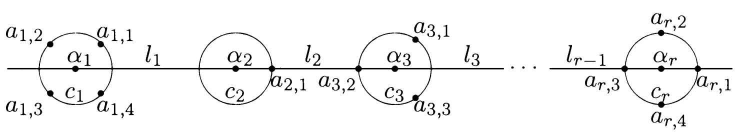


**Figure 3** $n_1 = n_r = 4$, $n_2 = 1$, $n_3 = 3$

Consider the paths $\rho_1, \ldots, \rho_r$ based at $a_{1,1}$ defined by

$$\rho_i = \big(\omega l_1 (c_2^+)^{-1} \cdots l_{i-2} (c_{i-1}^+)^{-1} l_{i-1}\big) c_i \big(\omega l_1 (c_2^+)^{-1} \cdots l_{i-2} (c_{i-1}^+)^{-1} l_{i-1}\big)^{-1},$$

where $\rho_1 = \omega c_1 \omega^{-1}$. These are paths around the $\alpha_i$ (see Figure 4). To complete the free basis of $\Pi$, we need some paths around the $\delta_{j,k}$.

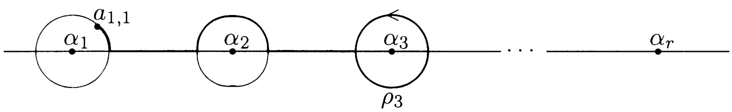


**Figure 4**

Consider the (real) critical values for $f$ and $f_i$ defined previously. Let $\omega_i$ be a loop around $f_i$ that is based at $\varepsilon$ and that is contained in the union of the paths $C_\varepsilon(f_i)$ and the real line constructed by the following algorithm: Follow the real line in the direction of $f_i$ until you meet the first $C_\varepsilon(f_j)$; if $j \neq i$, follow $C_\varepsilon(f_j)$ clockwise until you meet again the real line, then follow the real line again until a new $C_\varepsilon(f_j)$ is encountered and then repeat the algorithm; if $i = j$, follow counterclockwise the whole $C_\varepsilon(f_i)$ and come back to $\varepsilon$ from the way you arrived (and end the algorithm). Here are two examples (see Figure 5), where we have defined $C_\varepsilon^+$ and $C_\varepsilon^-$ in the natural way as we did for the $c_i$:

$$\omega_1 = TC_\varepsilon(f_1)T^{-1}T = [\varepsilon, f_4 - \varepsilon]C_\varepsilon^+(f_4)^{-1}[f_4 + \varepsilon, f_1 - \varepsilon],$$

$$\omega_3 = T'C_\varepsilon(f_3)T'^{-1}T' = C_\varepsilon^-(0)^{-1}[-\varepsilon, f_2 + \varepsilon]C_\varepsilon^-(f_2)^{-1}[f_2 - \varepsilon, f_3 + \varepsilon].$$

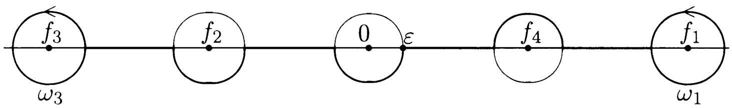


**Figure 5**

For every fixed pair $i, h$, we can uniquely lift $\omega_j$ to a (closed) path $\tilde{\Delta}_{j;i,h}$, based at $a_{i,h}$, such that $g(\tilde{\Delta}_{j;i,h}) = f(\omega_j)$; this is, in fact, a loop around some $\delta_{j,\bar{h}}$. Finally we define $\Delta_{j;i,h} \in \Pi$ to be the path based in $a_{1,1}$ obtained via conjugating $\tilde{\Delta}_{j;i,h}$ with a path connecting $a_{1,1}$ and $a_{i,h}$ that is obtained by following the orientation of each real interval and the reverse orientation of each circle. The paths $\rho_i$ and $\Delta_{j;i,h}$ clearly give a free basis for $\Pi$.

Now we can compute $\pi_1(\mathbb{C}^2 \setminus B)$ (and $\pi_1(\mathbb{P}^2 \setminus \bar{B})$). We can take, as generators of $\pi_1(\mathbb{C}^2 \setminus B)$ (and of $\pi_1(\mathbb{P}^2 \setminus \bar{B})$), a geometric basis $\mu_{j,k}$ (for $j = 1, \ldots, s$ and $k = 1, \ldots, m_j$) of $\pi_1(\{x = a_{1,1}\} \setminus B) \cong F_d$ in such a way that $\mu_{j,1}, \ldots, \mu_{j,m_j}$ are (conjugated to) the "standard generators" of $\pi_1(U_{1,j} \setminus B)$ (cf. [MP]), as shown in Figure 6.

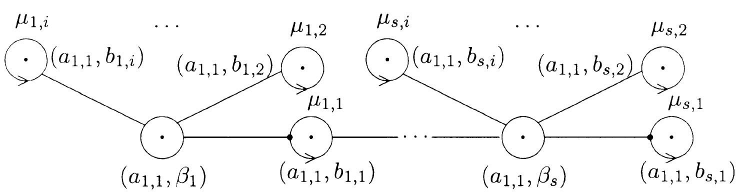


**Figure 6**

We recall now the following definition and theorem from [O].

DEFINITION 4.1 [O].

$$G_{m,n} := \langle g_1, \dots, g_m \mid g_k^{-1}(g_1 \cdots g_n) g_{k+n} (g_1 \cdots g_n)^{-1} \ \forall k = 1, \dots, m \rangle,$$

where the indices in the relators are taken to be cyclical mod $m$.

THEOREM 4.2 [O].

$$\pi_1(\mathbb{C}^2 \setminus \{x^m = y^n\}) \cong G_{m,n}.$$

PROPOSITION 4.3. *If $B = \{f(x) = g(y)\}$ then*

$$\pi_1(\mathbb{C}^2 \setminus B) \cong G_{m,n},$$

*where $n = (n_1, \dots, n_r)$ and $m = (m_1, \dots, m_s)$ are the greatest common divisors.*

*Proof.* Every path in $\Pi$ induces a braid (acting on $p^{-1}(a_{1,1})$) that is its *braid monodromy*; we compute the relations in $\pi_1(\mathbb{C}^2 \setminus B)$ by the braid monodromy of the generators of $\Pi$, following the method introduced in [Mo]. In order to express the braid monodromy of a path, we use the standard generators of the braid group on $d$ strands given by the positive half-twists $\sigma_i$ $(1 \le i \le d-1)$, exchanging the $i$th and the $(i+1)$th strands counterclockwise (the reader unfamiliar with the braid group can find precise definitions and more in, e.g., [Bi]).

In order to compute the braid monodromy of the generators we chose for $\Pi$, we consider the points $a_{i,h}$ lying on a line following the lexicographical order in their indices; that is, $a_{i,h} > a_{i',h'}$ if and only if $i > i'$, or $h > h'$ if $i = i'$. The braid monodromy of $\rho_1$ is

$$\tilde{\sigma}_1^{n_1} \cdots \tilde{\sigma}_s^{n_1},$$

where $\tilde{\sigma}_j = \sigma_{m_0+\cdots+m_j-1} \cdots \sigma_{m_0+\cdots+m_{j-1}+1}$ $(m_0 = 0)$. It gives us the relations

$$\mu_{j,k} = T_{j;1,n_1} \mu_{j,k+n_1} T_{j;1,n_1}^{-1}$$

for all $j, k$, where $T_{j;1,l} = \mu_{j,1}\mu_{j,2} \cdots \mu_{j,l}$ and the second index of the $\mu_{j,k}$ is taken to be cyclical (mod $m_j$).

Since (by condition (4)) lifting the path $l_i$ gives the identity braid for all $i$, it follows that the braid monodromy of $\rho_i$ is similar—that is,

$$\tilde{\sigma}_1^{n_i} \cdots \tilde{\sigma}_s^{n_i},$$

inducing in $\pi_1(\mathbb{C}^2 \setminus B)$ (and in $\pi_1(\mathbb{P}^2 \setminus \bar{B})$) the relations

$$\mu_{j,k} = T_{j;1,n_i} \mu_{j,k+n_i} T_{j;1,n_i}^{-1}$$

for all $i, j, k$. It is easy to see (cf. [MP, Prop. 1.1]) that these relations are equivalent to

$$\mu_{j,k} = T_{j;1,n} \mu_{j,k+n} T_{j;1,n}^{-1}$$

for all $j, k$, where $n = (n_1, \dots, n_r)$.

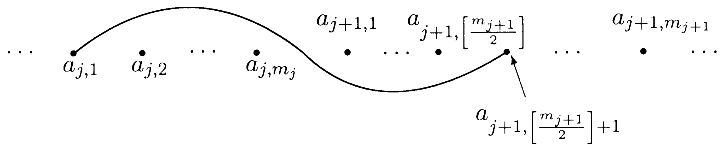


**Figure 7**

The monodromy of $\Delta_{j;1,1}$ is retrieved from the braid $f^{-1}(\omega_j)$ and gives the positive half-twist shown in Figure 7. That is,

$$T^{-1}\sigma_{m_0+\cdots+m_j}T \tag{4.4}$$

with

$$T = (\sigma_{m_0+\cdots+m_j+1}\cdots\sigma_{m_0+\cdots+m_{[(j+1)/2]}})(\tilde{\sigma}_j);$$

this gives the relation

$$\mu_{j,1} = (\mu_{j+1,1}\cdots\mu_{j+1,[m_{j+1}/2]})\mu_{j+1,[m_{j+1}/2]+1}(\mu_{j+1,1}\cdots\mu_{j+1,[m_{j+1}/2]})^{-1}.$$

This relation is best understood in terms of the "minimal standard generators" (cf. [MP])

$$\gamma_{j,k} = (\mu_{j,1}\cdots\mu_{j,k-1})\mu_{j,k}^{-1}(\mu_{j,1}\cdots\mu_{j,k-1})^{-1};$$

this relation becomes the simpler

$$\gamma_{j,1} = \gamma_{j+1,[m_{j+1}/2]+1}.$$

The braid monodromies of the other $\Delta_{j;i,h}$ are a conjugate of (4.4) by a multiple of $\tilde{\sigma}_j\tilde{\sigma}_{j+1}$ and give the relations

$$\gamma_{j,k} = \gamma_{j+1,[m_{j+1}/2]+k}$$

for all $j, k$.

These are cancellation relations because we can express each $\mu_{j,k}$ in terms of the $\mu_{1,k}$; moreover, they yield the relations

$$\mu_{1,k} = \mu_{1,k+m_j}$$

for all $j, k$. Therefore, if $m := (m_1, \dots, m_s)$, then the paths $\mu_1 = \mu_{1,1}, \dots, \mu_m = \mu_{1,m}$ generate $\pi_1(\mathbb{C}^2 \setminus B)$, and between them we have only the relations

$$\mu_k = T_{1,n}\mu_{k+n}T_{1,n}^{-1},$$

where $T_{1,n} = \mu_1\cdots\mu_n$ with cyclical indices mod $m$. □

Remark 4.5. By Theorem 4.2, $\pi_1(U_{i,j} \setminus B) \cong G_{n_i,m_j}$ and $\mu_1, \dots, \mu_{m_j}$ are (conjugated to) the standard generators for this group. In particular, the map $\pi_1(U_{i,j} \setminus B) \to \pi_1(\mathbb{C}^2 \setminus B)$ induced by the inclusion coincides with the map $(f_{n_i/n,m_j/m})_*$ introduced immediately after Remark 2.3.

This implies that, if $B$ is the branch curve of a normal generic cover with monodromy $\mu\colon G_{m,n} \to \mathcal{S}_d$, then the graph representing the local monodromy at $P_{i,j}$ is the pullback by $(f_{n_i/n,m_j/m})$ of the graph representing the global monodromy $\mu$.

PROPOSITION 4.6.

$$\pi_1(\mathbb{P}^2 \setminus \bar{B}) \cong G_{m,n} \big/ \langle (\mu_1 \cdots \mu_m)^{d/m} \rangle,$$

*where* $n = (n_1, \dots, n_r)$ *and* $m = (m_1, \dots, m_s)$.

*Proof*. To compute $\pi_1(\mathbb{P}^2 \setminus B)$, we use the standard remark that the kernel of the surjective map

$$\pi_1(\mathbb{C}^2 \setminus B) \to \pi_1(\mathbb{P}^2 \setminus \bar{B}) \to 0$$

is infinite cyclic and is generated by a loop $L$ around the line at infinity. In our case this loop is

$$L = (\mu_{1,1} \cdots \mu_{1,m_1})(\mu_{2,1} \cdots \mu_{2,m_2}) \cdots (\mu_{s,1} \cdots \mu_{s,m_s})$$

or, in terms of the generators $\mu_i$,

$$L = (\mu_1 \cdots \mu_m)^{m_1/m} \cdots (\mu_1 \cdots \mu_m)^{m_s/m} = (\mu_1 \cdots \mu_m)^{d/m}. \qquad \square$$

Assume now that $\bar{B}$ is irreducible, that is, $(n, m) = 1$. In this case, the monodromy of the cover lifts to a generic (geometric loops map to transpositions) homomorphism $\mu\colon G_{n,m} \to \mathcal{S}_d$ for which $\mu(T_{1,n}^{d/n}) = 1$. By the classification of generic homomorphisms in Theorem 2.8, the monodromy graph is (exchanging $n$ and $m$, if necessary) a polygon.

We know that in this case there exist $h, k, a, b$ such that $n = a(h+k)$ and $m = bkh$ with $(h, k) = 1$. Now we can introduce our family: we define

$$\bar{g}_l(x, w) = (x - w)(x - 2w) \cdots (x - lw),$$

$$\bar{f}_l(y, w) = (y - w)(y - 2w) \cdots (y - lw),$$

and, given $h, k$ coprime, we consider the generic cover of degree $h + k$ branched over

$$\bar{g}_{h+k}(x, w)^{hk} = \bar{f}_{hk}(y, w)^{h+k}$$

with monodromy graph a polygon with $m$ edges, valence 1, and increment $h$.

Here all the singularities have the same form ($x^{hk} = y^{h+k}$) and so, by Remark 4.5, all the local monodromy graphs must coincide with the global one. In order to ensure the existence of the cover we need only check that the monodromy of $(\mu_1 \cdots \mu_{h+k})^{hk}$ is trivial, which was clear at the start because it belongs to the center of the (local) fundamental group (in fact, the order of the monodromy of $\mu_1 \cdots \mu_{h+k}$ is exactly $hk$).

Moreover, by Corollary 2.10, having all the singular points $a = b = 1$ means that the surface we defined is smooth, whence the cover is smooth if and only if $h = 1$; in this case one can easily check that the cover is given by the projection on the plane $z = 0$ from the point $(0, 0, 1, 0)$ of the surface

$$z^{k+1} - (k+1)z\bar{f}_k(x, w) + k\bar{g}_{k+1}(y, w) = 0.$$

We can finally state the counterexample we were looking for, as follows.

Proposition 4.7. *Let $B$ be the projective plane curve of degree 30 given by the equation*

$$\bar{g}_5(x, w)^6 = \bar{f}_6(y, w)^5.$$

*Then there are two generic covers, $\pi' \colon S' \to \mathbb{P}^2$ and $\pi'' \colon S'' \to \mathbb{P}^2$, where:*

(1) *$S'$ and $S''$ are smooth;*
(2) $\deg \pi' = 6$;
(3) $\deg \pi'' = 5$;
(4) *the ramification divisor of $\pi'$ is smooth;*
(5) *the ramification divisor of $\pi''$ has (exactly) 30 ordinary cusps as singularities.*

*Proof.* The cover $\pi'$ (resp., $\pi''$) is the cover of the family we just constructed for $h = 1$ and $k = 5$ (resp., $h = 2$ and $k = 3$). We check quickly the five properties: (1) holds for every surface in our family. Properties (2) and (3) follow because the degree is the number of vertices of the graph (i.e., $h + k$). Finally, (4) and (5) follow directly from Remark 2.3. □

This is a counterexample to Chisini's conjecture if we drop the assumption that the ramification divisor is nonsingular. This family does not produce counterexamples in higher degrees; in fact, the pair (5, 6) is the only one that can be expressed as sum and product of two coprime integers in two different ways.

Indeed, suppose we have $h+k = h'k'$ and $hk = h'+k'$ with $(h, k) = (h', k') = 1$ and (say) $h < k$, $h' < k'$, and $h + k < hk$. From $h' + k' > h'k'$ we have that $h' = 1$ and $k' = hk - 1$. But now $k(h - 1) = h + 1 = h - 1 + 2$ and it must be that $(h - 1)|2$; hence $h = 2$ and $k = 3$, which gives $k' = 5$.

In order to find counterexamples to a Chisini–Kulikov–Nemirovski-type result in arbitrarily large degrees, we must consider a slightly different family, as follows.

Proposition 4.8. *Let $t \in \mathbb{N}$ with $t \geq 2$ and let $B$ be the projective plane curve given by the equation*

$$\bar{g}_{4t+1}(x, w)^{2t(2t+1)} = \bar{f}_{2t(2t+1)}(y, w)^{4t+1}.$$

*Then there are two generic covers, $\pi' \colon S' \to \mathbb{P}^2$ and $\pi'' \colon S'' \to \mathbb{P}^2$, with $S'$, $S''$ smooth, of degrees $4t + 2$ and $4t + 1$ (respectively), and each a singular ramification divisor.*

In fact, the case $t = 1$ is exactly the case of Proposition 4.7, so the statement still holds except for the singularities of the ramification divisor.

*Proof.* The cover of degree $4t+1$ is the cover in our family for $h = 2t$, $k = 2t+1$. The cover of degree $4t + 2$ is simply the cover constructed in the same way as we did for our family, starting from the monodromy graph given by the polygon with $4t + 2$ vertices, valence $t$, and increment 1.

The smoothness comes from Corollary 2.10 by observing that (locally) we have $h = b = 1$. The other verifications are exactly as in the previous case and so we leave them to the reader. □

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