

Extension Operators for Locally Univalent Mappings

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1. Preliminaries

Let \mathbf{C}^n denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$.

Let $z' = (z_2, \dots, z_n)$ so that $z = (z_1, z')$. Let $B_r^n = \{z \in \mathbf{C}^n : \|z\| < r\}$ and let $B^n = B_1^n$. In the case of one variable, B_r^n is denoted by U_r and U_1 by U . If $G \subset \mathbf{C}^n$ is an open set, let $H(G)$ denote the set of holomorphic mappings from G into \mathbf{C}^n . If $f \in H(B_r^n)$, we say that f is *normalized* if $f(0) = 0$ and $Df(0) = I$. Let $S(B_r^n)$ be the set of normalized univalent mappings in $H(B_r^n)$. The sets of normalized convex (resp., starlike) mappings of B_r^n are denoted by $K(B_r^n)$ (resp., $S^*(B_r^n)$). When $n = 1$, the sets $S(U)$, $S^*(U)$, and $K(U)$ are denoted by S , S^* , and K , respectively. For vectors and matrices, A^* denotes the conjugate transpose of A .

We recall that a mapping $F: B^n \times [0, \infty) \rightarrow \mathbf{C}^n$ is called a *Loewner chain* if $F(\cdot, t)$ is univalent on B^n , $F(0, t) = 0$, $DF(0, t) = e^t I$ for $t \geq 0$, and

$$F(z, s) \prec F(z, t), \quad z \in B^n, \quad 0 \leq s \leq t < \infty,$$

where the symbol \prec means the usual subordination. We will consider the set $S^0(B^n)$ consisting of those mappings $F \in S(B^n)$ that can be imbedded in Loewner chains. It is well known that, in the case of several complex variables, $S^0(B^n)$ is a proper subset of $S(B^n)$ (see [K; GrHK]). If $F: B_r^n \rightarrow \mathbf{C}^n$ ($0 < r \leq 1$), we say that $F \in S^0(B_r^n)$ if $F_r \in S^0(B^n)$, where $F_r(z) = \frac{1}{r} F(rz)$ and $z \in B^n$.

A mapping $f \in H(B^n)$ with $f(0) = 0$ is called *starlike* if f is univalent on B^n and if $f(B^n)$ is a starlike domain with respect to zero.

It is known that starlikeness can be characterized in terms of Loewner chains: f is starlike on B^n iff $f(z, t) = e^t f(z)$ ($z \in B^n$, $t \geq 0$) is a Loewner chain. For the analytical characterization of starlikeness, see [S1; S2].

A key role in our discussion is played by the n -dimensional version of the Carathéodory set:

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$$\mathcal{M} = \{h \in H(B^n) : h(0) = 0, Dh(0) = I, \operatorname{Re}\langle h(z), z \rangle > 0, z \in B^n \setminus \{0\}\}.$$

Recently, three of the present authors have shown that \mathcal{M} is compact [GrHK].

In order to generate mappings in $S^0(B^n)$, we will make use of a modification of a criterion of Pfaltzgraff [Pf1]. In his initial result, Pfaltzgraff used the following additional assumption on $h(z, t)$, which now is not necessary: For each $T > 0$ and $r \in (0, 1)$, there exists a number $M = M(r, T)$ such that

$$\|h(z, t)\| \leq M(r, T), \quad \|z\| \leq r, \quad 0 \leq t \leq T.$$

LEMMA 1.1. *Let $f(z, t) = e^t z + \dots$ be a mapping from $B^n \times [0, \infty)$ into \mathbf{C}^n such that (a) $f(\cdot, t) \in H(B^n)$ for each $t \geq 0$ and (b) $f(z, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z \in B^n$.*

Let $h : B^n \times [0, \infty) \rightarrow \mathbf{C}^n$ satisfy the following conditions:

- (i) $h(\cdot, t) \in \mathcal{M}$, $t \geq 0$;
- (ii) for each $z \in B^n$, $h(z, t)$ is a measurable function of $t \in [0, \infty)$.

Suppose that

$$\frac{\partial f}{\partial t}(z, t) = Df(z, t)h(z, t) \quad \text{a.e. } t \geq 0$$

and for all $z \in B^n$, and suppose there exists a sequence $\{t_m\}$ ($t_m > 0$) increasing to ∞ such that

$$\lim_{m \rightarrow \infty} e^{-t_m} f(z, t_m) = G(z)$$

locally uniformly on B^n . Then $f(z, t)$ is a Loewner chain.

The Roper–Suffridge extension operator is defined for normalized locally univalent functions on U by

$$\Phi_n(f)(z) = F(z) = (f(z_1), \sqrt{f'(z_1)z'}), \quad (1.1)$$

where the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

Roper and Suffridge [RS] proved that if $f \in K$ then $\Phi_n(f) \in K(B^n)$, and in [GrK1] it was shown that if $f \in S^*$ then $\Phi_n(f) \in S^*(B^n)$.

In this paper we consider the operators

$$\Psi_{n,\alpha,\beta}(f)(z) = F_{\alpha,\beta}(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1} \right)^\alpha (f'(z_1))^\beta z' \right), \quad z \in B^n, \quad (1.2)$$

where $\alpha \geq 0$, $\beta \geq 0$, and f is a locally univalent function on U , normalized by $f(0) = f'(0) - 1 = 0$, and such that $f(z_1) \neq 0$ for $z_1 \in U \setminus \{0\}$. We choose the branches such that

$$\left(\frac{f(z_1)}{z_1} \right)^\alpha \Big|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1))^\beta \Big|_{z_1=0} = 1.$$

If $\alpha \in [0, 1]$ and $\beta = 0$ (resp., $\alpha = 0$ and $\beta \in [0, 1/2]$) then we obtain 1-parameter families of operators, which have been recently considered in [GrK2; GrKK]. Of course, when $\alpha = 0$ and $\beta = 1/2$ we obtain the Roper–Suffridge operator Φ_n .

We remark that all of the $\Psi_{n,\alpha,\beta}$ fall into the general class of operators of the form $\Psi_g(f)(z) = (f(z_1), z'_1g(z_1))$, where f is normalized locally univalent on U and g is a nonvanishing holomorphic function on U such that $g(0) = 1$. If f is univalent on U and if g is analytic and nonzero on U , then the mapping $\Psi_g(f)$ is clearly univalent on B^n . However, if we impose some geometric conditions on the extended mapping Ψ_g , our methods require that we have some connection between the functions f and g . In fact, for a given convex function f , it is difficult to find a function g such that $\Psi_g(f)$ is a convex mapping. As we shall see in Example 1.3, the only choice of $g(z_1)$ that makes the mapping $(z_1, z') \mapsto (z_1/(1 - z_1), z'_1g(z_1))$ a normalized convex mapping is $g(z_1) = 1/(1 - z_1)$.

In [PfS1, Ex. 1] it was shown that, if f is starlike on U , then $\Psi_{n,1,0}(f)$ is starlike on B^n ; in [GrKK] it was shown that $\Psi_{n,0,\beta}(f)$ is starlike whenever f is starlike and $\beta \in [0, 1/2]$. This suggests that one should examine the geometry associated with $\Psi_{n,\alpha,\beta}(f)$. We note that the operator $\Psi_{n,\alpha,\beta}$ has the property that the function $f(z_1) = z_1/(1 - z_1)$ is mapped to $(z_1/(1 - z_1), z'_1/(1 - z_1)^{\alpha+2\beta})$.

We obtain a number of extension results that are valid for $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$: if $f \in S$ then $\Psi_{n,\alpha,\beta}(f) \in S^0(B^n)$; if $f \in S^*$ then $\Psi_{n,\alpha,\beta}(f) \in S^*(B^n)$. Also, if f is a univalent function that satisfies known growth and distortion estimates, then $\Psi_{n,\alpha,\beta}(f)$ is a univalent mapping that satisfies a related growth estimate. It is interesting that the same set of parameter values arises in these different extension problems.

We will also prove that the operators $\Psi_{n,0,\beta}$ can be used to construct further examples of linear-invariant families that have minimum order $(n + 1)/2$ and that are not subsets of $K(B^n)$ for $n \geq 2$ (cf. [GrK2; PfS3]).

As already indicated, the preservation of convexity seems to be a very rigid property of the Roper–Suffridge operator. Only for $(\alpha, \beta) = (0, 1/2)$ does $\Psi_{n,\alpha,\beta}$ preserve convexity. It would be of interest to determine whether there is any perturbation of the Roper–Suffridge operator that preserves convexity.

We now give the example that shows that the only choice of $g(z_1)$ that makes the mapping $(z_1, z') \mapsto (z_1/(1 - z_1), z'_1g(z_1))$ a normalized convex mapping is $g(z_1) = 1/(1 - z_1)$. For this purpose, we need the following result.

Let $F : B^n \rightarrow \mathbf{C}^n$ be a normalized holomorphic univalent mapping of the ball B^n onto a convex domain Ω . Suppose that Ω is unbounded. Also let $L = L(u) = \{ru : r \geq 0\}$, where u is a unit vector in \mathbf{C}^n . Then we have the following lemma [MS, Lemma 2.1].

LEMMA 1.2. *If $v \in \Omega$ and $L(u) \subset \Omega$, then $v + L(u) \subset \Omega$.*

EXAMPLE 1.3. Let $n = 2$ and $F : B^2 \rightarrow \mathbf{C}^2$ be given by

$$F(z) = \left(\frac{z_1}{1 - z_1}, \frac{z_2g(z_1)}{1 - z_1} \right), \quad z = (z_1, z_2) \in B^2,$$

where g is a nonvanishing analytic function on U with $g(0) = 1$. We show that the mapping F is convex only for $g(z_1) \equiv 1$.

Observe that the line $L = \{(it, 0) : t \in \mathbf{R}\} \subset F(B^2)$. From Lemma 1.2 we deduce that, for every $W \in F(B^2)$, the line $W + L \subset F(B^2)$.

Let

$$u = \frac{z_1}{1 - z_1} \quad \text{and} \quad v = \frac{z_2 g(z_1)}{1 - z_1}.$$

Then $\operatorname{Re} u > -1/2$ and

$$|z_2|^2 < 1 - |z_1|^2 = 1 - \frac{|u|^2}{|1 + u|^2} = \frac{1 + 2 \operatorname{Re} u}{|1 + u|^2},$$

so the mapping F is of the form

$$(u, \rho e^{i\varphi} \sqrt{1 + 2 \operatorname{Re} u} h(u)), \quad \rho < 1, \quad h(u) = g\left(\frac{u}{1 + u}\right).$$

Note that, from the nature of the mapping F and Lemma 1.2, we have the following relations:

$$(u, v) \in F(B^2) \iff (u, |v|) \in F(B^2) \iff (u + it, |v|) \in F(B^2) \quad \forall t \in \mathbf{R}. \quad (1.3)$$

For each u , let

$$M(u) = \sup\{|v| : (u, v) \in F(B^2)\} = \sqrt{1 + 2 \operatorname{Re}(u)} |h(u)|.$$

Then (1.3) implies that $M(u)$ is independent of $\operatorname{Im} u$, so that $|h(u)|$ is constant on the lines $\operatorname{Re} u = \text{constant} > -1/2$.

By the Schwarz reflection principle, we may reflect the function h with the domain restricted to the right half-plane across the unit circle (it is the unit circle because $h(0) = 1$) by $h(-\bar{u}) = 1/\overline{h(u)}$. This extended function is entire because $h(u) \neq 0$. Write $u = \sigma + i\tau$ and observe that $h(u) = R e^{i\phi}$ where R is independent of τ . Using the Cauchy–Riemann equations, it is easy to see that ϕ is independent of σ and, in fact, $h(u) = e^{au}$ for some real a . Using the convexity of the mapping, it follows that the set (u, v) such that $u > -1/2$ and $0 < v < \sqrt{1 + 2u} e^{au} = k(u)$ is convex. By elementary calculus, since $k''(u) > 0$ for large u when $a \neq 0$, this set (u, v) cannot be convex unless $a = 0$.

We recall that a linear-invariant family (L.I.F.) is a family \mathcal{F} of locally univalent mappings $F : B^n \rightarrow \mathbf{C}^n$ such that, if $F \in \mathcal{F}$, then:

- (i) $F(0) = 0$ and $DF(0) = I$; and
- (ii) $\Lambda_\phi(F) \in \mathcal{F}$ for all $\phi \in \operatorname{Aut}(B^n)$, where $\operatorname{Aut}(B^n)$ is the set of holomorphic automorphisms of B^n and $\Lambda_\phi(F)$ is the Koebe transform of F given by

$$\Lambda_\phi(F)(z) = [D\phi(0)]^{-1} [DF(\phi(0))]^{-1} (F(\phi(z)) - F(\phi(0))), \quad z \in B^n$$

(see [Pf2]).

The order of the L.I.F. \mathcal{F} is defined by

$$\operatorname{ord} \mathcal{F} = \sup\{|\operatorname{trace}\{\frac{1}{2} D^2 f(0)(w, \cdot)\}| : f \in \mathcal{F}, \|w\| = 1\}$$

(see [BFG; Pf2]).

We determine the order of the L.I.F. generated by $\Psi_{n,0,\beta}(\mathcal{F})$, where \mathcal{F} is a L.I.F. on U of known order. We then consider the radius of convexity of $\Phi_n(\mathcal{F})$ and the radius of starlikeness of $\Psi_{n,\alpha,\beta}(\mathcal{F})$, where again \mathcal{F} is a L.I.F. on U of known order.

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2. Loewner Chains Associated with the Operator $\Psi_{n,\alpha,\beta}$

We begin this section with the following result.

THEOREM 2.1. *Assume that $f \in S$ and that $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ such that $\alpha + \beta \leq 1$. Then $F_{\alpha,\beta} = \Psi_{n,\alpha,\beta}(f) \in S^0(B^n)$.*

Proof. It suffices to give the proof when $n = 2$. Since $f \in S$, there exists a Loewner chain $f(z_1, t)$ such that $f(z_1) = f(z_1, 0)$ for all $z_1 \in U$. Let $F_{\alpha,\beta}(z, t)$ be defined by

$$F_{\alpha,\beta}(z, t) = \left(f(z_1, t), e^{(1-\alpha-\beta)t} z_2 \left(\frac{f(z_1, t)}{z_1} \right)^\alpha (f'(z_1, t))^\beta \right) \quad (2.1)$$

for $z = (z_1, z_2) \in B^2$ and $t \geq 0$. We shall show that $F_{\alpha,\beta}(z, t)$ is a Loewner chain.

Since $f(z_1, t)$ is a Loewner chain in U , it follows that (a) $f(z_1, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z_1 \in U$ and (b) for each $r \in (0, 1)$, there exists a positive constant $M_0 = M_0(r)$ such that

$$|f(z_1, t)| \leq M_0 e^t, \quad |z_1| \leq r, \quad t \geq 0.$$

Also there exists a function $p(z_1, t)$ that is holomorphic on U , measurable in $t \geq 0$, with $p(0, t) = 1$ and $\operatorname{Re} p(z_1, t) > 0$ for $z_1 \in U$ and $0 \leq t < \infty$, and such that

$$\frac{\partial f}{\partial t}(z_1, t) = z_1 f'(z_1, t) p(z_1, t) \quad \text{a.e. } t \geq 0 \quad (2.2)$$

and for all $z_1 \in U$.

Obviously $F_{\alpha,\beta}(\cdot, t) \in H(B^2)$, $F_{\alpha,\beta}(0, t) = 0$, and $DF_{\alpha,\beta}(0, t) = e^t I$; also, $F_{\alpha,\beta}(z, t)$ satisfies the absolute continuity hypothesis of Lemma 1.1. Using (2.1), we deduce that

$$\begin{aligned} \frac{\partial F_{\alpha,\beta}}{\partial t}(z, t) &= \left(\frac{\partial f}{\partial t}(z_1, t), z_2 e^{(1-\alpha-\beta)t} \left((1-\alpha-\beta) \left(\frac{f(z_1, t)}{z_1} \right)^\alpha (f'(z_1, t))^\beta \right. \right. \\ &\quad \left. \left. + \frac{\partial}{\partial t} \left(\left(\frac{f(z_1, t)}{z_1} \right)^\alpha (f'(z_1, t))^\beta \right) \right) \right). \end{aligned}$$

Because $f(z_1, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z_1 \in U$, we can deduce that, for almost all $t \geq 0$,

$$\begin{aligned} \frac{\partial}{\partial t} (f'(z_1, t))^\beta &= \beta (f'(z_1, t))^{\beta-1} \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial z_1}(z_1, t) \right) \\ &= \beta (f'(z_1, t))^{\beta-1} \frac{\partial}{\partial z_1} \left(\frac{\partial f}{\partial t}(z_1, t) \right) \\ &= \beta (f'(z_1, t))^{\beta-1} \frac{\partial}{\partial z_1} (z_1 f'(z_1, t) p(z_1, t)) \\ &= \beta (f'(z_1, t))^\beta \left[p(z_1, t) + z_1 \frac{f''(z_1, t)}{f'(z_1, t)} p(z_1, t) + z_1 p'(z_1, t) \right], \end{aligned}$$

making use of (2.2) and the fact that the order of differentiation may be changed.

Consequently, we obtain the relation

$$\begin{aligned} \frac{\partial F_{\alpha,\beta}}{\partial t}(z, t) &= \left(z_1 f'(z_1, t) p(z_1, t), z_2 e^{(1-\alpha-\beta)t} \left(\frac{f(z_1, t)}{z_1} \right)^\alpha (f'(z_1, t))^\beta \right. \\ &\quad \times \left[1 - \alpha - \beta + \alpha \frac{z_1 f'(z_1, t)}{f(z_1, t)} p(z_1, t) + \beta p(z_1, t) \right. \\ &\quad \left. \left. + \beta z_1 p'(z_1, t) + \beta \frac{z_1 f''(z_1, t)}{f'(z_1, t)} p(z_1, t) \right] \right) \end{aligned}$$

a.e. $t \geq 0$ and for all $z \in B^2$.

Moreover, straightforward computation now yields

$$\begin{aligned} [DF_{\alpha,\beta}(z, t)]^{-1} \frac{\partial F_{\alpha,\beta}}{\partial t}(z, t) \\ = (z_1 p(z_1, t), z_2 (1 - \alpha - \beta + (\alpha + \beta) p(z_1, t) + \beta z_1 p'(z_1, t))) \end{aligned}$$

a.e. $t \geq 0$ and for all $z \in B^2$. Let

$$h_{\alpha,\beta}(z, t) = (z_1 p(z_1, t), z_2 (1 - \alpha - \beta + (\alpha + \beta) p(z_1, t) + \beta z_1 p'(z_1, t))).$$

Then we have

$$\frac{\partial F_{\alpha,\beta}(z, t)}{\partial t} = DF_{\alpha,\beta}(z, t) h_{\alpha,\beta}(z, t) \quad \text{a.e. } t \geq 0$$

and for all $z \in B^2$. We next show that $h_{\alpha,\beta}(z, t)$ satisfies the requirements (i) and (ii) from Lemma 1.1.

Obviously, $h_{\alpha,\beta}(\cdot, t) \in H(B^2)$, $h_{\alpha,\beta}(0, t) = 0$, $Dh_{\alpha,\beta}(0, t) = I$, and

$$\begin{aligned} \operatorname{Re} \langle h_{\alpha,\beta}(z, t), z \rangle &= |z_1|^2 \operatorname{Re} p(z_1, t) + (1 - \alpha - \beta) |z_2|^2 \\ &\quad + (\alpha + \beta) |z_2|^2 \operatorname{Re} p(z_1, t) + \beta |z_2|^2 \operatorname{Re} (z_1 p'(z_1, t)) \end{aligned} \quad (2.3)$$

for $z \in B^2$ and $t \geq 0$.

It is clear that if $z = (z_1, 0)$ then

$$\operatorname{Re} \langle h_{\alpha,\beta}(z, t), z \rangle = |z_1|^2 \operatorname{Re} p(z_1, t) \geq 0;$$

hence it suffices to assume that $z = (z_1, z_2)$ with $z_2 \neq 0$. In view of the minimum principle for harmonic functions, it suffices to prove that

$$\operatorname{Re} \langle h_{\alpha,\beta}(z, t), z \rangle \geq 0, \quad |z_1|^2 + |z_2|^2 = 1, \quad z_2 \neq 0, \quad t \geq 0.$$

We have $p(0, t) = 1$ and $\operatorname{Re} p(z_1, t) > 0$ ($z_1 \in U$, $t \geq 0$), and it is well known that

$$|p'(z_1, t)| \leq \frac{2 \operatorname{Re} p(z_1, t)}{1 - |z_1|^2}$$

(see e.g. [P2]); from this we obtain

$$\operatorname{Re} (z_1 p'(z_1, t)) \geq -\frac{2|z_1|}{1 - |z_1|^2} \operatorname{Re} p(z_1, t), \quad |z_1| < 1, \quad t \geq 0.$$

Using this inequality together with (2.3) and the fact that $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, and $\alpha + \beta \leq 1$, we obtain

$$\begin{aligned} & \operatorname{Re}\langle h_{\alpha,\beta}(z, t), z \rangle \\ & \geq |z_1|^2 \operatorname{Re} p(z_1, t) + (1 - \alpha - \beta)(1 - |z_1|^2) \\ & \quad + (\alpha + \beta)(1 - |z_1|^2) \operatorname{Re} p(z_1, t) - 2\beta|z_1| \operatorname{Re} p(z_1, t) \\ & = (1 - \alpha - \beta)(1 - |z_1|^2) \\ & \quad + \operatorname{Re} p(z_1, t)[|z_1|^2(1 - \alpha - \beta) - 2\beta|z_1| + \alpha + \beta] \geq 0 \end{aligned} \quad (2.4)$$

for $z = (z_1, z_2)$, where $|z_1|^2 + |z_2|^2 = 1$, $z_2 \neq 0$, and $t \geq 0$.

In determining the nonnegative values of α and β for which the left-hand side of (2.4) is nonnegative on B^2 , there are three cases to consider:

- (i) $\alpha + \beta = 1$;
- (ii) $\alpha + \beta < 1$ and the quadratic polynomial

$$q(x) = (1 - \alpha - \beta)x^2 - 2\beta x + \alpha + \beta$$

assumes its minimum outside the interval $[0, 1]$;

- (iii) $\alpha + \beta < 1$ and $q(x)$ assumes its minimum inside the interval $[0, 1]$.

In case (i) we obtain $0 \leq \beta \leq 1/2$, $\alpha + \beta = 1$. In case (ii) we obtain $\alpha + 2\beta \geq 1$, $\alpha + \beta < 1$, $0 \leq \beta \leq 1/2$. In case (iii) we obtain $\alpha + 2\beta < 1$. These three possibilities together give the conditions on α and β in the statement of the theorem. It is also clear that the mapping $h_{\alpha,\beta}$ satisfies the measurability condition (ii) from Lemma 1.1.

Since $e^{-t}f(\cdot, t)$ is locally uniformly bounded on U for $t \geq 0$, there exists a sequence $\{t_m\}$ ($t_m > 0$), increasing to ∞ , such that

$$\lim_{m \rightarrow \infty} e^{-t_m} f(z_1, t_m) = g(z_1)$$

locally uniformly on U . Then, by Vitali's theorem, we have

$$\lim_{m \rightarrow \infty} e^{-t_m} F_{\alpha,\beta}(z, t_m) = \Psi_{n,\alpha,\beta}(g)(z)$$

locally uniformly on B^2 .

Taking into account Lemma 1.1, we deduce that $F_{\alpha,\beta}(z, t)$ is a Loewner chain and thus $F_{\alpha,\beta}(z) = F_{\alpha,\beta}(z, 0)$ belongs to $S^0(B^2)$. This completes the proof. \square

We mention that, for $\beta = 0$ and $\alpha \in [0, 1]$, the result of Theorem 2.1 has been obtained in [GrK2]. Also, when $\alpha = 0$ and $\beta \in [0, 1/2]$, the result was obtained in [GrKK]. Actually, in all cases it is possible to show that $F_{\alpha,\beta}(z)$ admits a parametric representation, which is a slightly stronger conclusion (see [GrHK]).

A direct application of Theorem 2.1 is the following corollary.

COROLLARY 2.2. *Let $f \in S^*$ and let $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ such that $\alpha + \beta \leq 1$. Then $F_{\alpha,\beta} = \Psi_{n,\alpha,\beta}(f) \in S^*(B^n)$.*

Proof. Since $f \in S^*$, we know that $f(z_1, t) = e^t f(z_1)$ is a Loewner chain. From the proof of Theorem 2.1, we deduce that $F_{\alpha, \beta}(z, t) = e^t F_{\alpha, \beta}(z)$ is also a Loewner chain. This completes the proof. \square

Certain cases of this result (i.e., $\alpha \in [0, 1]$ and $\beta = 0$; $\alpha = 0$ and $\beta \in [0, 1/2]$) have been studied in [GrK2; GrKK]. In particular, it is known that the Roper–Suffridge extension operator $\Phi_n = \Psi_{n, 0, 1/2}$ preserves starlikeness.

As for the preservation of convexity under the operator $\Psi_{n, \alpha, \beta}$, we know that $\Psi_{n, 0, 1/2}(K) \subseteq K(B^n)$, $\Psi_{n, 0, 0}(K) \not\subseteq K(B^n)$ (see [RS]) and also $\Psi_{n, 0, \beta}(K) \not\subseteq K(B^n)$ for $\beta \in [0, 1/2)$ (see [GrKK]). Moreover, we have the following.

THEOREM 2.3. *Let $\alpha \geq 0$ and $\beta \geq 0$. Also let $\Psi_{n, \alpha, \beta}$ be the operator defined by (1.2). Then $\Psi_{n, \alpha, \beta}(K) \subset K(B^n)$, for $n \geq 2$, if and only if $(\alpha, \beta) = (0, 1/2)$.*

Proof. We will use similar arguments to the proof of [RS, Thm. 2] to give a geometric proof. For this purpose, let $n = 2$ and let $f : U \rightarrow \mathbb{C}$ be defined by

$$f(z_1) = \frac{1}{2} \log\left(\frac{1+z_1}{1-z_1}\right).$$

Then f is convex, but the mapping

$$F_{\alpha, \beta}(z) = \left(\frac{1}{2} \log\left(\frac{1+z_1}{1-z_1}\right), z_2 \left(\frac{1}{2z_1} \log\left(\frac{1+z_1}{1-z_1}\right)\right)^\alpha \frac{1}{(1-z_1^2)^\beta}\right)$$

is only convex for $(\alpha, \beta) = (0, 1/2)$. To see this, let

$$u = \frac{1}{2} \log\left(\frac{1+z_1}{1-z_1}\right), \quad v = z_2 \left(\frac{1}{2z_1} \log\left(\frac{1+z_1}{1-z_1}\right)\right)^\alpha \frac{1}{(1-z_1^2)^\beta}.$$

If $F_{\alpha, \beta}(B^2)$ is convex, then so is its intersection with the plane $\text{Im } u = 0$, $\text{Im } v = 0$. This intersection contains the entire real u -axis and precisely the interval $(-1, 1)$ of the real v -axis. In order to show that convexity is not satisfied, it suffices to show that if $z_1 \rightarrow 1$ along the real axis then we are constrained to have $|v| \rightarrow 0$ or else it is possible to choose z_2 so that $|v| \rightarrow \infty$.

If $z_1 \rightarrow 1$ along the real axis then u is real, $u \rightarrow \infty$, and

$$|v|^2 = \frac{|z_2|^2}{|1-z_1^2|^{2\beta}} \left| \frac{1}{2z_1} \log\left(\frac{1+z_1}{1-z_1}\right) \right|^{2\alpha}.$$

Let $z_2 = \varepsilon > 0$ be small and let $z_1 = \sqrt{1-\varepsilon^2}$. Then it is elementary to show that if $(\alpha, \beta) \neq (0, 1/2)$ then $|v|^2 \rightarrow 0$ or $|v|^2 \rightarrow \infty$. This completes the proof. \square

EXAMPLE 2.4. (i) Let $F_{\alpha, \beta}$, u , and v be as in the proof of Theorem 2.3. Figures 1, 2, and 3 show $F_{\alpha, \beta}(B^2) \cap \{\text{Im } u = 0\} \cap \{\text{Im } v = 0\}$ when $(\alpha, \beta) = (0, 0.495)$, $(1, 0)$, and $(1/2, 1/2)$, respectively. The graphs are starlike but are not convex. See Theorem 2.1.

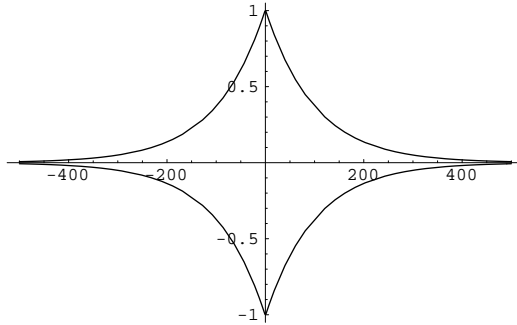


Figure 1 $\alpha = 0, \beta = 0.495$

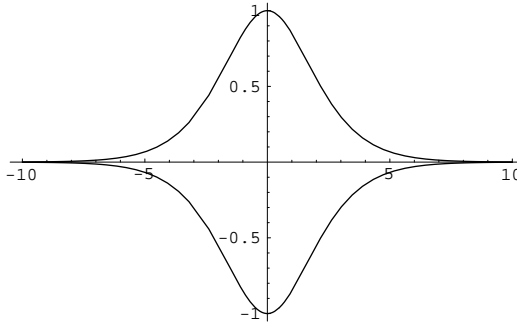


Figure 2 $\alpha = 1, \beta = 0$

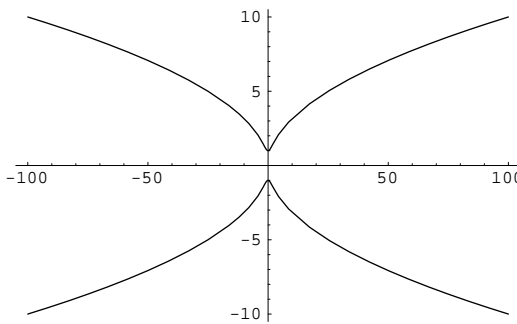


Figure 3 $\alpha = 1/2, \beta = 1/2$

(ii) Next, we give an example of a mapping $F_\beta = \Psi_{n,0,\beta}(f)$, where $f \in K$ and $\beta \in [0, 1/2]$, that satisfies a necessary condition for convexity; however, it is not convex for $\beta \neq 1/2$.

Let $n = 2$, $f(z_1) = z_1/(1 - z_1)$, and $\beta \in [0, 1/2]$. Then

$$\begin{aligned} F_\beta(z) &= \Psi_{2,0,\beta}(f)(z) = \left(\frac{z_1}{1 - z_1}, \frac{z_2}{(1 - z_1)^{2\beta}} \right) \\ &= z + (z_1^2, 2\beta z_1 z_2) + (z_1^3, \beta(2\beta + 1)z_1^2 z_2) + \cdots \\ &\quad + \left(z_1^{k+1}, \frac{2\beta(2\beta + 1) \cdots (2\beta + k - 1)}{k!} z_1^k z_2 \right) + \cdots \end{aligned}$$

for all $z = (z_1, z_2) \in B^2$.

Let $w = (w_1, w_2) \in \mathbf{C}^2$ with $\|w\| = 1$ and let $k \geq 2$. It is obvious that the k th multilinear Taylor coefficient $(1/k!)D^k F_\beta(0)$ of F_β satisfies

$$\begin{aligned} &\left\| \frac{1}{k!} D^k F_\beta(0)(w, \dots, w) \right\| \\ &= |w_1|^{k-1} \sqrt{|w_1|^2 + \left[\frac{2\beta(2\beta + 1) \cdots (2\beta + k - 2)}{(k - 1)!} \right]^2 |w_2|^2} \leq 1. \end{aligned}$$

Hence

$$\sup_{\|z\|=1} \left\| \frac{1}{k!} D^k F_\beta(0)(w, \dots, w) \right\| \leq 1, \quad k \geq 2. \quad (2.5)$$

Now, let $A_k: \prod_{j=1}^k \mathbf{C}^2 \rightarrow \mathbf{C}^2$ be a k -linear symmetric mapping. Using [Hö, Thm. 4], we have

$$\|A_k\| = \sup_{\substack{\|w^{(j)}\|=1 \\ 1 \leq j \leq k}} \|A_k(w^{(1)}, \dots, w^{(k)})\| = \sup_{\|w\|=1} \|A_k(w, \dots, w)\|.$$

Combining these equalities with (2.5), one obtains that

$$\left\| \frac{1}{k!} D^k F_\beta(0) \right\| \leq 1, \quad k \geq 2.$$

Thus all multilinear Taylor coefficients of F_β satisfy the necessary condition for convexity on the unit ball of \mathbf{C}^2 via [Pfs4, Thm. 5.1], but F_β is not convex on B^2 for $\beta \neq 1/2$ by Example 1.3.

The graph of $F_\beta(B^2) \cap \{\operatorname{Re} u = 0\} \cap \{\operatorname{Im} v = 0\}$ when $(\alpha, \beta) = (0, 0.49)$ is shown in Figure 4.

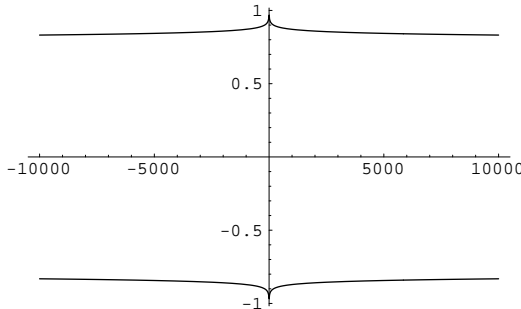


Figure 4 $\alpha = 0, \beta = 0.49$

3. Growth and Distortion Theorems for Families of the Form $\Psi_{n,\alpha,\beta}(\mathcal{F})$

In this section we show that the operator $\Psi_{n,\alpha,\beta}$ ($\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$) preserves growth results (cf. [Gr; GrK2; GrKK]).

THEOREM 3.1. *Suppose \mathcal{F} is a subset of S such that all $f \in \mathcal{F}$ satisfy*

$$\begin{aligned} \varphi(r) &\leq |f(z_1)| \leq \phi(r), & |z_1| &= r, \\ \varphi'(r) &\leq |f'(z_1)| \leq \phi'(r), & |z_1| &= r, \end{aligned} \tag{3.1}$$

where

$$\varphi, \phi \text{ are twice differentiable on } [0, 1), \tag{3.2}$$

$$\varphi(0) = \varphi'(0) - 1 = 0, \quad \varphi'(r) \geq 0, \quad \varphi''(r) \leq 0, \tag{3.3}$$

$$\phi(0) = \phi'(0) - 1 = 0, \quad \phi'(r) \geq 0, \quad \phi''(r) \geq 0. \tag{3.4}$$

If $F_{\alpha,\beta} = \Psi_{n,\alpha,\beta}(f)$ ($\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, $\alpha + \beta \leq 1$), then

$$\varphi(r) \leq \|F_{\alpha,\beta}(z)\| \leq \phi(r), \quad \|z\| = r. \tag{3.5}$$

Furthermore, if for some $f \in \mathcal{F}$ the lower (resp., upper) estimate in (3.1) is sharp at $z_1 \in U$, then the lower (resp., upper) estimate in (3.5) is sharp for $\Psi_{n,\alpha,\beta}(f)$ at $(z_1, 0, \dots, 0)$.

To prove this theorem, we must use the following result.

LEMMA 3.2. *Suppose φ and ϕ are functions that satisfy conditions (3.2)–(3.4) of Theorem 3.1, and suppose $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$. Then, for fixed $r \in [0, 1)$,*

the minimum of $(\varphi(t))^2 + (r^2 - t^2)(\varphi(t)/t)^{2\alpha}(\varphi'(t))^{2\beta}$ for $t \in [0, r]$ occurs when $t = r$;

the maximum of $(\phi(t))^2 + (r^2 - t^2)(\phi(t)/t)^{2\alpha}(\phi'(t))^{2\beta}$ for $t \in [0, r]$ occurs when $t = r$.

Proof. For the case $\alpha \in [0, 1/2]$, $\beta \in [0, 1/2]$, the sign of the first-order derivative of these functions on $(0, r]$ is easily determined using the relations

$$\varphi(t) \leq t, \quad \left(\frac{\varphi(t)}{t}\right)^{2\alpha-1} \geq 1, \quad (\varphi'(t))^{2\beta-1} \geq 1 \quad \text{for } t \in (0, r] \text{ and } \alpha \in [0, 1/2]$$

and

$$\phi(t) \geq t, \quad \left(\frac{\phi(t)}{t}\right)^{2\alpha-1} \leq 1, \quad (\phi'(t))^{2\beta-1} \leq 1 \quad \text{for } t \in (0, r] \text{ and } \alpha \in [0, 1/2].$$

For the case $\alpha \geq 1/2$, we use the fact that $\varphi'(t)t \leq \varphi(t)$ and

$$\begin{aligned} 2\varphi\varphi' - 2t(\varphi/t)^{2\alpha}(\varphi')^{2\beta} &\leq 2\varphi\varphi' - 2t(\varphi/t)^{2\alpha}(\varphi')^{2(1-\alpha)} \\ &= 2\varphi(\varphi')^{2(1-\alpha)}((\varphi')^{2\alpha-1} - (\varphi/t)^{2\alpha-1}) \leq 0, \end{aligned}$$

together with a similar result for ϕ with the inequalities reversed. □

Proof of Theorem 3.1. Let $\|z\| = r$. Using the result of Lemma 3.2, it is easy to deduce the lower and the upper bounds for

$$\begin{aligned} |f(z_1)|^2 + \|z'\|^{2\alpha} \left| \frac{f(z_1)}{z_1} \right|^{2\alpha} |f'(z_1)|^{2\beta} \\ = |f(z_1)|^2 + (r^2 - |z_1|^2) \left| \frac{f(z_1)}{z_1} \right|^{2\alpha} |f'(z_1)|^{2\beta}. \quad \square \end{aligned}$$

A direct consequence of Theorem 3.1 is the following growth result.

COROLLARY 3.3. *Let $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$. If $f \in S$, then*

$$\frac{r}{(1+r)^2} \leq \|\Psi_{n,\alpha,\beta}(f)(z)\| \leq \frac{r}{(1-r)^2}, \quad \|z\| = r.$$

If $f \in K$, then

$$\frac{r}{1+r} \leq \|\Psi_{n,\alpha,\beta}(f)(z)\| \leq \frac{r}{1-r}, \quad \|z\| = r.$$

These estimates are sharp.

We next present the following covering result for the set $\Psi_{n,\alpha,\beta}(\mathcal{F})$.

THEOREM 3.4. *Let $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$. Also, let the set $\mathcal{F} \subset S$ and φ, ϕ satisfy the hypothesis of Theorem 3.1. Then, for all $f \in \mathcal{F}$, the image of $\Psi_{n,\alpha,\beta}(f)$ contains the ball B_ρ^n , where $\rho = \lim_{\rho \rightarrow 1^-} \varphi(\rho)$.*

Proof. Since $\varphi(r) - r$ is decreasing on $[0, 1)$, it follows that φ is bounded on $[0, 1)$. Also, φ is increasing on $[0, 1)$; hence the limit ρ exists. \square

Finally, we give the following distortion results for the set $\Psi_{n,\alpha,\beta}(\mathcal{F})$.

THEOREM 3.5. *Let $\alpha \geq 0$ and $\beta \geq 0$, and let the functions φ, ϕ satisfy the hypothesis of Theorem 3.1. Also let $\mathcal{F} \subset S$. Then, for all $f \in \mathcal{F}$ and $\|z\| = r$,*

$$\left(\frac{\varphi(r)}{r} \right)^{(n-1)\alpha} (\varphi'(r))^{1+(n-1)\beta} \leq |J_{F_{\alpha,\beta}}(z)| \leq \left(\frac{\phi(r)}{r} \right)^{(n-1)\alpha} (\phi'(r))^{1+(n-1)\beta},$$

where $J_{F_{\alpha,\beta}}(z) = \det DF_{\alpha,\beta}(z)$.

Proof. Using similar arguments to the proof of Theorem 3.1, we can obtain the minimum of

$$\left(\frac{\varphi(t)}{t} \right)^{(n-1)\alpha} (\varphi'(t))^{1+(n-1)\beta} \quad \text{for } t \in [0, r]$$

and the maximum of

$$\left(\frac{\phi(t)}{t} \right)^{(n-1)\alpha} (\phi'(t))^{1+(n-1)\beta} \quad \text{for } t \in [0, r]. \quad \square$$

COROLLARY 3.6. *Let $\alpha \geq 0$ and $\beta \geq 0$.*

If $f \in S$, then

$$\frac{(1-r)^{1+(n-1)\beta}}{(1+r)^{3+(2\alpha+3\beta)(n-1)}} \leq |J_{F_{\alpha,\beta}}(z)| \leq \frac{(1+r)^{1+(n-1)\beta}}{(1-r)^{3+(2\alpha+3\beta)(n-1)}}, \quad \|z\| = r.$$

If $f \in K$, then

$$\frac{1}{(1+r)^{2+(\alpha+2\beta)(n-1)}} \leq |J_{F_{\alpha,\beta}}(z)| \leq \frac{1}{(1-r)^{2+(\alpha+2\beta)(n-1)}}, \quad \|z\| = r.$$

These estimates are sharp.

4. Linear-Invariant Families Generated by the Operator $\Psi_{n,0,\beta}$

In this section we use the operator $\Psi_{n,0,\beta}$ to generate L.I.F.s on B^n and to study the order of these linear-invariant families. For other results concerning L.I.F.s in several complex variables, see for example [G; Pf2; PfS2; PfS3; PfS4].

Let $\mathcal{L}S_n$ denote the set of normalized locally univalent mappings on the unit ball B^n of \mathbf{C}^n . Let $\mathcal{F} \subset \mathcal{L}S_1$. Also let $\beta \in [0, 1/2]$ and $\Psi_{n,0,\beta}(\mathcal{F})$ be the corresponding set in \mathbf{C}^n ; that is,

$$\Psi_{n,0,\beta}(\mathcal{F}) = \{F_\beta(z) = (f(z_1), (f'(z_1))^\beta z') : f \in \mathcal{F}\}.$$

Let $\Lambda[\Psi_{n,0,\beta}(\mathcal{F})]$ denote the L.I.F. generated by $\Psi_{n,0,\beta}(\mathcal{F})$, as follows:

$$\Lambda[\Psi_{n,0,\beta}(\mathcal{F})] = \{\Lambda_\phi(F_\beta) : F_\beta \in \Psi_{n,0,\beta}(\mathcal{F}), \phi \in \text{Aut}(B^n)\}.$$

Note that even if the set \mathcal{F} is a L.I.F., it is not clear that the set $\Psi_{n,0,\beta}(\mathcal{F})$ is a L.I.F.

Let \mathcal{U} denote the set of unitary transformations in \mathbf{C}^n . The automorphisms of B^n (up to multiplication by unitary transformations) are the mappings

$$\varphi(z; a) = \varphi_a(z) = T_a\left(\frac{a-z}{1-a^*z}\right), \quad z \in B^n,$$

where

$$T_a = \frac{1}{\|a\|^2} \{(1-s_a)aa^* + s_a\|a\|^2 I\}$$

and

$$s_a = \sqrt{1 - \|a\|^2}.$$

In other words,

$$\text{Aut}(B^n) = \{V\varphi_a : a \in B^n, V \in \mathcal{U}\}.$$

The following lemmas will be useful in this work. The proof of Lemma 4.1 is contained in the first part of the proof of [PfS4, Thm. 3.3], but Lemma 4.2 is new.

LEMMA 4.1. *Let $\mathcal{F} \subset \mathcal{L}S_n$ and let $\Lambda[\mathcal{F}]$ be the L.I.F. generated by \mathcal{F} on B^n . Let $a \in U$ and $b \in B^{n-1}$. Then*

$\text{ord } \Lambda[\mathcal{F}]$

$$= \sup\{|\text{trace}\{\frac{1}{2}D^2\Lambda_{\varphi_b}\Lambda_{\varphi_a}(F)(0)(\gamma, \cdot)\}| : |a| < 1, \|b\| < 1, \|\gamma\| = 1, F \in \mathcal{F}\},$$

where $\varphi_a := \varphi_{ae_1}$ and $\varphi_b := \varphi_{(0,b)}$.

Proof. We first observe that $\{\varphi_a \circ \varphi_b : a \in U, b \in B^{n-1}\}$ is a family of automorphisms ψ of B^n such that $\psi(0) = (a, \sqrt{1-|a|^2}b)$. Since this includes all of B^n , we conclude that the collection of all automorphisms consists of the composition of all unitary mappings with members of this special family. Since the trace is invariant under similarity, it follows that it is sufficient to consider automorphisms of the type just described. We know that $\Lambda_{\varphi_b} \Lambda_{\varphi_a} = \Lambda_{\varphi_a \circ \varphi_b}$, since a and b vary in U and B^{n-1} (respectively), so the lemma now follows. \square

LEMMA 4.2. *Assume $f: U \rightarrow \mathbf{C}$ is locally univalent, $g: U \rightarrow \mathbf{C}$ is holomorphic, and $f(0) = 0$ and $f'(0) = 1 = g(0)$. Define $F: B^n \rightarrow \mathbf{C}^n$ by $F(z) = (f(z_1), g(z_1)z')$, where $z = (z_1, z')$. With $G(z) = \Lambda_{\varphi_b}(F)(z)$, we have*

$$\begin{aligned} \sup\{|\text{trace}\{D^2G(0)(\gamma, \cdot)\}| : \|b\| < 1, \|\gamma\| = 1\} \\ = \max\{n + 1, \sup\{|\text{trace}\{D^2F(0)(\gamma, \cdot)\}| : \|\gamma\| = 1\}\}. \end{aligned}$$

Proof. Without loss of generality, we may assume the coordinates are chosen so that $b = xe_2$, where $0 \leq x < 1$. We write $z = (z_1, z_2, v)$, where $v \in B^{n-2}$ and $\|z\| < 1$. Of course, if $n = 2$ then v will not appear. Hence

$$\varphi_b(z) = \frac{\sqrt{1-x^2}}{1-xz_2}(-z_1, 0, -v) + \frac{x-z_2}{1-xz_2}e_2 := \sum_{j=1}^n \varphi^{(j)}(z)e_j.$$

Since

$$DF(\varphi_b(0))D\varphi_b(0) = \begin{pmatrix} -\sqrt{1-x^2} & 0 & 0 \\ -x\sqrt{1-x^2}g'(0) & -(1-x^2) & 0 \\ 0 & 0 & -\sqrt{1-x^2}I \end{pmatrix},$$

it follows that

$$G(z) = V(F(\varphi_b(z)) - xe_2),$$

where

$$V = \begin{pmatrix} -\frac{1}{\sqrt{1-x^2}} & 0 & 0 \\ \frac{xg'(0)}{1-x^2} & -\frac{1}{1-x^2} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{1-x^2}}I \end{pmatrix}.$$

Because of the form of G , the trace of $D^2G(0)(\gamma, \cdot)$ is

$$\begin{aligned} \frac{\partial^2 G_1}{\partial z_1^2}(0)\gamma_1 + \frac{\partial^2 G_1}{\partial z_1 \partial z_2}(0)\gamma_2 + \frac{\partial^2 G_2}{\partial z_1 \partial z_2}(0)\gamma_1 + \frac{\partial^2 G_2}{\partial z_2^2}(0)\gamma_2 \\ + \sum_{k=3}^n \left(\frac{\partial^2 G_k}{\partial z_1 \partial z_k}(0)\gamma_1 + \frac{\partial^2 G_k}{\partial z_2 \partial z_k}(0)\gamma_2 \right); \end{aligned}$$

thus the entries on the diagonal of $D^2G(0)(\gamma, \cdot)$ are

$$-\sqrt{1-x^2}f''(0)\gamma_1+x\gamma_2, -\sqrt{1-x^2}g'(0)\gamma_1+2x\gamma_2, \\ -\sqrt{1-x^2}g'(0)\gamma_1+x\gamma_2, \dots, -\sqrt{1-x^2}g'(0)\gamma_1+x\gamma_2.$$

The trace is therefore

$$-\sqrt{1-x^2}(f''(0)+(n-1)g'(0))\gamma_1+(n+1)x\gamma_2.$$

By elementary calculus, the supremum of this quantity over $0 \leq x < 1$, $\|\gamma\| = 1$, is

$$\max\{n+1, |f''(0)+(n-1)g'(0)|\}.$$

Since $\text{trace}\{D^2F(0)(\gamma, \cdot)\} = (f''(0)+(n-1)g'(0))\gamma_1$, the lemma follows. \square

We are now able to prove the following result.

THEOREM 4.3. *Let \mathcal{F} be a L.I.F. on U such that $\text{ord } \mathcal{F} = \delta < \infty$, and let $\beta \in [0, 1/2]$. Then $\text{ord } \Lambda[\Psi_{n,0,\beta}(\mathcal{F})] = \eta$, where*

$$\eta = (1+(n-1)\beta)\delta + \frac{(n-1)(1-2\beta)}{2}.$$

Proof. Let $f \in \mathcal{F}$ and set $G = \Psi_{n,0,\beta}(f)$. Using Lemmas 4.1 and 4.2, it follows that

$$\text{ord } \Lambda[\Psi_{n,0,\beta}(\mathcal{F})] \\ = \sup \left\{ \left| \frac{\text{trace}\{D^2\Lambda_{\varphi_a}(G)(0)(\gamma, \cdot)\}}{2} \right| : a \in U, \|\gamma\| = 1, f \in \mathcal{F} \right\}.$$

Write

$$f(z_1; a) = \frac{f\left(\frac{a-z_1}{1-\bar{a}z_1}\right) - f(a)}{-(1-|a|^2)f'(a)}.$$

Then

$$\Lambda_{\varphi_a}(G)(z) = \left(f(z_1; a), \left(\frac{f'\left(\frac{a-z_1}{1-\bar{a}z_1}\right)}{f'(a)} \right)^\beta \frac{1}{1-\bar{a}z_1} z' \right).$$

Now the diagonal of $D^2\Lambda_{\varphi_a}(G)(0)(\gamma, \cdot)$ has

$$\left(-\frac{(1-|a|^2)f''(a)}{f'(a)} + 2\bar{a} \right) \gamma_1$$

as its first entry and

$$\left(-\beta \frac{(1-|a|^2)f''(a)}{f'(a)} + \bar{a} \right) \gamma_1$$

in the remaining positions. The trace is therefore

$$\left(\frac{-(1-|a|^2)f''(a)}{f'(a)} + 2\bar{a} \right) \gamma_1(1+(n-1)\beta) + (1-2\beta)(n-1)\bar{a}\gamma_1.$$

Now, we may replace f by a function $g \in \mathcal{F}$ so that

$$g''(0) = \frac{-(1 - |a|^2)f''(a)}{f'(a)} + 2\bar{a}$$

(i.e., $g(z_1) = f(z_1; a)$). Thus, we want to find

$$\sup_{g \in \mathcal{F}, |a| < 1} \left| \frac{g''(0)}{2} (1 + (n-1)\beta) + (1-2\beta) \frac{n-1}{2} \bar{a} \right|.$$

This is clearly

$$\begin{aligned} \sup_{g \in \mathcal{F}, |a| < 1} \left(\frac{|g''(0)|}{2} (1 + (n-1)\beta) + (1-2\beta) \frac{n-1}{2} |a| \right) \\ = (1 + (n-1)\beta)\delta + \frac{(n-1)(1-2\beta)}{2}. \end{aligned}$$

This completes the proof. \square

We now give some interesting particular cases of Theorem 4.3. The following result was obtained by Pfaltzgraff [Pf2; Pf3] and by Liczberski and Starkov [LSt].

COROLLARY 4.4. *Let \mathcal{F} be a L.I.F. on U such that $\text{ord } \mathcal{F} = \delta < \infty$. Also, let Φ_n be the Roper–Suffridge extension operator defined by (1.1). Then $\text{ord } \Lambda[\Phi_n(\mathcal{F})] = \delta(n+1)/2$.*

COROLLARY 4.5. *Let $\beta \in [0, 1/2]$. Then*

$$\text{ord } \Lambda[\Psi_{n,0,\beta}(K)] = \frac{n+1}{2}.$$

Proof. It suffices to apply the result of Theorem 4.3 and then to use the fact that $\text{ord } K = 1$. \square

REMARK 4.6. It is well known that, if \mathcal{F} is a L.I.F. on the unit disc, then $\text{ord } \mathcal{F} = 1$ (the minimum order) if and only if $\mathcal{F} \subset K$ (see [P1, p. 134]). However, Corollary 4.5 suggests that in several complex variables this result does not remain true. Indeed, $\text{ord } \Lambda[\Psi_{n,0,\beta}(K)] = (n+1)/2$ whereas, for $\beta \neq 1/2$ and $n \geq 2$, $\Lambda[\Psi_{n,0,\beta}(K)] \not\subset K(B^n)$. For a similar conclusion, see [GrK2; PfS2; PfS3].

5. Radius of Univalence

Let \mathcal{F} be a nonempty subset of $S(B^n)$. Let

$$r^*(\mathcal{F}) = \sup\{r : f \text{ is starlike on } B_r^n \text{ and } f \in \mathcal{F}\}$$

and

$$r_c(\mathcal{F}) = \sup\{r : f \text{ is convex on } B_r^n \text{ and } f \in \mathcal{F}\}$$

denote the radius of starlikeness and the radius of convexity (respectively) of \mathcal{F} . In [GrKK] the authors obtained the exact values of $r^*(\Psi_{n,0,\beta}(S))$ and $r^*(\Psi_{n,0,\beta}(K))$ when $\beta \in [0, 1/2]$. In this section we shall give some simple and interesting consequences of Theorem 2.1 that are related to the radius of univalence of some subsets of $S(B^n)$. Throughout this section we consider only L.I.F.s of finite order.

REMARK 5.1. We note that for some $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$ and for some $r \in (0, 1)$, if $\Psi_{n,\alpha,\beta}(f) \in S(B_r^n)$ then $f \in S(U_r)$, too. Also, if $\Psi_{n,\alpha,\beta}(f) \in S^*(B_r^n)(K(B_r^n))$ then $f \in S^*(U_r)(K(U_r))$, too. Moreover, if $f \in S(U_r)$ then $\Psi_{n,\alpha,\beta}(f) \in S^0(B_r^n)$ for $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, and $\alpha + \beta \leq 1$, because the equality

$$\Psi_{n,\alpha,\beta}(f_r)(z) = \frac{1}{r} \Psi_{n,\alpha,\beta}(f)(rz)$$

holds on B^n .

THEOREM 5.2. Let \mathcal{F} be a L.I.F. on U such that $\text{ord } \mathcal{F} = \gamma$. Then $r_c(\Phi_n(\mathcal{F})) = \gamma - \sqrt{\gamma^2 - 1}$. In particular, $r_c(\Phi_n(S)) = r_c(\Phi_n(S^*)) = 2 - \sqrt{3}$.

Proof. Let $F \in \Phi_n(\mathcal{F})$. Then $F = \Phi_n(f)$ for some $f \in \mathcal{F}$. Taking into account [P1, Satz 2.5], one deduces that $f \in K(U_\rho)$ with $\rho = \gamma - \sqrt{\gamma^2 - 1}$, and this number is the radius of convexity for the set \mathcal{F} . Hence

$$\text{Re} \left[1 + \frac{z_1 f''(z_1)}{f'(z_1)} \right] > 0, \quad |z_1| < \rho,$$

and this quantity may be negative if $|z_1| > \rho$. Next, using [RS, Thm. 1], we conclude that $\Phi_n(f_\rho) \in K(B_{\rho_1}^n)$; by Remark 5.1, we deduce that F may fail to be convex in any ball $B_{\rho_1}^n$ with $\rho_1 > \rho$. Therefore, $r_c(\Phi_n(\mathcal{F})) = \gamma - \sqrt{\gamma^2 - 1}$. This completes the proof. \square

Note that, in dimension $n > 1$, there is in general no such connection between the order of a L.I.F. \mathcal{M} of B^n and its radius of convexity (see [PfS3; PfS4]). On the other hand, in [GrKK] the authors proved that, in \mathbf{C}^n ($n \geq 2$), the radius of convexity of $S^*(B^n)$ is strictly less than $2 - \sqrt{3}$.

THEOREM 5.3. Let \mathcal{F} be a L.I.F. on U such that $\text{ord } \mathcal{F} = \gamma$. Also let $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ be such that $\alpha + \beta \leq 1$. Then $\Psi_{n,\alpha,\beta}(\mathcal{F}) \subseteq S^*(B_\rho^n)$, where $\rho = 1/\gamma$.

Proof. Let $F_{\alpha,\beta} \in \Psi_{n,\alpha,\beta}(\mathcal{F})$. Then $F_{\alpha,\beta} = \Psi_{n,\alpha,\beta}(f)$ for some $f \in \mathcal{F}$. Because $\text{ord } \mathcal{F} = \gamma$, we may deduce from [P1, Folgerung 2.5] that $f \in S^*(U_\rho)$ for $\rho = 1/\gamma$. Using Corollary 2.2, we conclude that $\Psi_{n,\alpha,\beta}(f) \in S^*(B_\rho^n)$. This completes the proof. \square

Another radius problem is presented in the following.

THEOREM 5.4. $r^*(\Psi_{n,\alpha,\beta}(S)) = \tanh(\pi/4)$ for all $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$.

Proof. Let $F \in \Psi_{n,\alpha,\beta}(S)$. Then $F_{\alpha,\beta} = \Psi_{n,\alpha,\beta}(f)$ for some $f \in S$. It follows that $f \in S^*(U_\rho)$, where $\rho = \tanh(\pi/4)$, and this number is the radius of starlikeness for the set S (see [P2]). Again using Corollary 2.2 and Remark 5.1, we deduce that $F_{\alpha,\beta} \in S^*(B_\rho^n)$ and also that $F_{\alpha,\beta}$ may fail to be starlike in any ball $B_{\rho_1}^n$ with $\rho_1 > \rho$. This completes the proof. \square

Note that Theorem 5.4 generalizes a previous result obtained in [GrKK].

Our final result relates r_1 , the radius of univalence of the set $\Psi_{n,\alpha,\beta}(\mathcal{F})$ with \mathcal{F} a L.I.F. on U , with r_0 , the radius of nonvanishing of the set \mathcal{F} . Let

$$r_0 = \sup\{r : f(\zeta) \neq 0, |\zeta| < r, f \in \mathcal{F}\}$$

and

$$r_1 = \sup\{r : F_{\alpha,\beta} \text{ is univalent on } B_r^n, F_{\alpha,\beta} \in \Psi_{n,\alpha,\beta}(\mathcal{F})\}.$$

THEOREM 5.5. *Let $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$ with $\alpha + \beta \leq 1$. Also let \mathcal{F} be a L.I.F. on U . Then $r_1 = r_0 / (1 + \sqrt{1 - r_0^2})$.*

Proof. Taking into account [P1, Lemma 2.4], we deduce that each $f \in \mathcal{F}$ is univalent on U_{r_1} with $r_1 = r_0 / (1 + \sqrt{1 - r_0^2})$; in fact, this number is the radius of univalence of the set \mathcal{F} . By Theorem 2.1 and Remark 5.1, we deduce that $\Psi_{n,\alpha,\beta}(f) \in S^0(B_{r_1}^n)$ and that $\Psi_{n,\alpha,\beta}(f)$ may fail to be univalent in any ball $B_{r_2}^n$ with $r_2 > r_1$. Therefore, r_1 is the radius of univalence of $\Psi_{n,\alpha,\beta}(\mathcal{F})$. This completes the proof. \square

We note that a similar result, in the general context of linear-invariant families on the unit ball of \mathbf{C}^n , was obtained in [PFS4].

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