

On the Fundamental Groups at Infinity of the Moduli Spaces of Compact Riemann Surfaces

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1. Introduction and Statement of the Result

In his seminal manuscript *Esquisse d'un programme* (1984; now available in [LS]), Grothendieck explains that the structure of the tower of all moduli spaces of curves is somehow governed by its “first two levels” (“deux premiers étages”), that is, the moduli spaces of dimensions 1 and 2. We refer to the *Esquisse* and to [L] for more context and details about this statement. Let us only mention that, speaking in terms of topology, Grothendieck was concerned more precisely with the orbifold fundamental groups of the moduli spaces of curves; he explains that the above “principle” is essentially equivalent to the fact that the orbifold fundamental group of any moduli space of dimension > 2 is equal to its fundamental group at infinity. We do not recall here the notion of orbifold fundamental group, which is due to Thurston in a topological context, because we will be concerned only with the ordinary topological fundamental group—that is, the fundamental groups of the moduli spaces of curves viewed as manifolds, forgetting about their orbifold structure. In terms of analytic or algebraic geometry, this amounts to viewing them as coarse and not as fine moduli spaces for curves.

Before we consider moduli spaces of curves in detail, let us make precise the notion of fundamental group at infinity in a topological context. Note that it is less easy (although feasible) to do it in terms of algebraic geometry, because a quasiprojective variety cannot usually be exhausted by an increasing sequence of projective subvarieties; nor is it easy to define tubular neighborhoods of closed subvarieties. So let M be a paracompact differentiable manifold and partially order the compact submanifolds (possibly with boundary) of M by inclusion. Their complements define an obvious inverse system: if $K \subset K'$, we simply consider the inclusion $M \setminus K' \subset M \setminus K$. We need a base point for our fundamental group and exploit the fact that a fundamental group need not be based at a point but in fact at any simply connected subset of the ambient manifold. Here a *base point at infinity*, simply denoted by $*$, is given by an open part $U \subset M$ such that, for any compact set K , there exists a compact set K' with $K \subset K'$ and $U \setminus K'$ nonempty and *simply* connected. Let π_1 denote, as usual, the topological fundamental group (functor).

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DEFINITION (The fundamental group at infinity). Let M be a paracompact differentiable manifold and assume there exists a base point at infinity $*$ for M , defined by an open set U . We define the topological fundamental group at infinity of M based at $*$ as:

$$\pi_1^\infty(M, *) = \varprojlim \pi_1(M \setminus K, U \setminus K),$$

where the inverse limit is over the cofinal family of compact subsets K of M such that $U \setminus K$ is simply connected, partially ordered by inclusion, and uses the natural induced maps on the fundamental groups.

From now on we will often lightheartedly ignore the base points in the notation, having done with the problem of base points at infinity as before. Let us very briefly recall a few notions from the theory of Teichmüller and moduli spaces of curves, essentially in order to fix notation. We refer to any standard textbook on the subject (e.g. [IT]) for the necessary background information. For simplicity we will mainly consider in this note the case of Riemann surfaces without marked or deleted points, postponing to the closing remarks some observations on the more general case. Yet in the intermediary steps, we will have to consider surfaces with marked points anyway, so let us introduce the more general objects right away. We will denote by $\mathcal{T}_{g,n}$ the Teichmüller space of compact Riemann surfaces of type (g, n) , that is, those obtained from surfaces of genus g by marking n points. One may also consider, taking a hyperbolic rather than conformal viewpoint, that the points are deleted, giving rise to surfaces with cusps. Let $\mathcal{M}_{g,n}$ be the fine moduli space of surfaces of type (g, n) obtained as the quotient of $\mathcal{T}_{g,n}$ by the (Teichmüller) modular group (alias mapping class group) $\Gamma_{g,n}$. More precisely, $\Gamma_{g,n}$ acts properly and discontinuously on $\mathcal{T}_{g,n}$ with quotient $\mathcal{M}_{g,n}$. Since the action is not free, the latter space naturally inherits an orbifold structure (a structure of stack in the algebraic context). One can also “forget” about the orbifold structure, thus getting a bona fide (in general normal but singular) manifold $M_{g,n}$, which is a coarse moduli space for the surfaces of type (g, n) . In all this the points or punctures are labeled and $\Gamma_{g,n}$ denotes the pure modular group, preserving the marked points individually. One can now allow permutations of these points and obtain the full group $\Gamma_{g,[n]}$, which is an extension of the permutation group S_n by the pure group $\Gamma_{g,n}$. We also get the corresponding spaces $\mathcal{M}_{g,[n]} = \mathcal{T}_{g,n}/\Gamma_{g,[n]}$ and $M_{g,[n]}$. Note that there is no notion of permuting the punctures at the level of the Teichmüller space and that this operation is not a mere ornament when one is interested in torsion elements, as we will be. The effect may be dramatic: typically $\Gamma_{0,n}$ is torsion-free, whereas $\Gamma_{0,[n]}$ is generated by its torsion elements. Finally we mention that we drop n from the notation when it is 0: we write Γ_g for $\Gamma_{g,0}$, and so on.

Let us now stick to the unmarked (or compact) case $n = 0$ for stating the results. As mentioned previously, marked surfaces reappear as intermediate objects in the course of the proof (Section 2) and are briefly commented on again in the closing section. By definition $\pi_1(\mathcal{M}_g) = \Gamma_g$, where one considers the orbifold fundamental group and assumes $g > 1$ in order to deal with hyperbolic surfaces.

One can actually also include $g = 0$ because \mathcal{M}_0 is a point (this is the Schönfliess theorem) and $\mathcal{M}_1 \simeq \mathcal{M}_{1,1}$. For the coarse moduli spaces, Maclachlan [M] proves that they are simply connected: $\pi_1(M_g) = \{1\}$ for $g > 1$ (in fact, one can again include the cases $g = 0, 1$). This deals with the ordinary topological fundamental group, considering M_g as a manifold. In view of [A], this amounts to saying that Γ_g is generated by its torsion elements. Note that both results hold in the algebraic context: if one considers \mathcal{M}_g as a Deligne–Mumford stack (see [DMu]) over \mathbf{C} (or over \mathbf{Q}), its fundamental group is the profinite completion of Γ_g ; whereas the coarse moduli space M_g , which is nothing else than \mathcal{M}_g viewed as a scheme, is simply connected. In [L] it is shown, confirming Grothendieck’s prediction, that $\pi_1^\infty(\mathcal{M}_g) \simeq \Gamma_g$ for $g \geq 2$. Note that the assertion is content-free for $g = 0$ and does not hold for $g = 1$ (even identifying \mathcal{M}_1 with $\mathcal{M}_{1,1}$). To put this result in context, it is best to state the result with marked points. Recall that $\mathcal{M}_{g,n}$ has dimension $d(g, n) = 3g - 3 + n$ (assume $2g - 2 + n > 0$, that is, the surfaces are hyperbolic). With this in mind, it is shown in [L] that $\pi_1^\infty(\mathcal{M}_{g,n}) \simeq \Gamma_{g,n}$ ($= \pi_1(\mathcal{M}_{g,n})$) if and only if $d(g, n) > 2$, which is indeed Grothendieck’s prediction. This result is also valid if one does not label the marked points (replace n by $[n]$ everywhere) and can be immediately transposed (as before) in the algebraic context.

In this note we prove, at least for surfaces without marked points, a statement that was left as a “plausible assertion” in [L]. In fact, we show the following.

THEOREM. $\pi_1^\infty(M_g) = \{1\}$ ($= \pi_1(M_g)$) for $g > 2$ and $\pi_1^\infty(M_2) \simeq \mathbf{Z}/5\mathbf{Z}$.

Section 2 is devoted to some geometrical properties that provide, much as in [L], the crucial ingredients for the proof of the theorem. We recall and make precise in passing some properties of the loci of curves with nontrivial automorphisms, which may have some independent interest. In Section 3 we give the proof of the theorem, including the determination of the fundamental group at infinity in the case of genus 2; at the end, we briefly comment on the cases with marked points.

2. Some Geometry at Infinity on the Moduli Spaces of Curves

We consider hyperbolic surfaces of fixed type (g, n) (with $2g - 2 + n > 0$) and first recall some observations from [L]. Note that these purely geometric features do not depend on whether we view the moduli spaces as being fine or coarse. First, for $\varepsilon > 0$ we define the set $\mathcal{M}_{g,n}^\varepsilon \subset \mathcal{M}_{g,n}$ of points $[X] \in \mathcal{M}_{g,n}$ representing surfaces X such that it has at least one geodesic γ with length $l(\gamma) < \varepsilon$ in the Poincaré metric. For $\varepsilon \leq \varepsilon'$ there is an obvious inclusion $\mathcal{M}_{g,n}^\varepsilon \subset \mathcal{M}_{g,n}^{\varepsilon'}$ and thus a natural map between fundamental groups: $\pi_1(\mathcal{M}_{g,n}^\varepsilon) \rightarrow \pi_1(\mathcal{M}_{g,n}^{\varepsilon'})$. As mentioned previously, one can replace the fine moduli (\mathcal{M}) by the coarse moduli (M), which will be used shortly.

By [Mu] (see also [B2]), the complements of $\mathcal{M}_{g,n}^\varepsilon$ in $\mathcal{M}_{g,n}$ are compact and form a cofinal sequence, that is, they exhaust the space $\mathcal{M}_{g,n}$. So one has

$\pi_1^\infty(\mathcal{M}_{g,n}) = \lim_{\varepsilon \rightarrow 0} \pi_1(\mathcal{M}_{g,n}^\varepsilon)$. The same statement holds for coarse moduli spaces. Let $p: \mathcal{T}_{g,n} \rightarrow \mathcal{M}_{g,n}$ denote the canonical projection, and set $\mathcal{T}_{g,n}^\varepsilon = p^{-1}(\mathcal{M}_{g,n}^\varepsilon)$. One may show (see [L]) that:

- (i) $\mathcal{T}_{g,n}^\varepsilon$ is simply connected for any $\varepsilon > 0$; and
- (ii) there exists an absolute constant ε_0 such that, if $\varepsilon \leq \varepsilon' < \varepsilon_0$, then $\mathcal{T}_{g,n}^\varepsilon$ is a deformation retract of $\mathcal{T}_{g,n}^{\varepsilon'}$ and the retraction can be chosen to be $\Gamma_{g,n}$ -equivariant, thus producing in particular a diffeomorphism between $\mathcal{M}_{g,n}^\varepsilon$ and $\mathcal{M}_{g,n}^{\varepsilon'}$. From this one concludes that the sequence of groups $(\pi_1(\mathcal{M}_{g,n}^\varepsilon))_{\varepsilon > 0}$ stabilizes for $\varepsilon < \varepsilon_0$. In fact, one can pick $\varepsilon_0 = \frac{1}{3} \ln(1 + \sqrt{2})$. Again one may replace fine with coarse moduli spaces.

We now turn more specifically to the case of coarse moduli spaces, which are the subject of this note. By an elementary topological result [A], assertion (i) implies (see again [L] for details) that $\pi_1(\mathcal{M}_{g,n}^\varepsilon) = \Gamma_{g,n}/G^\varepsilon$, where G^ε is the normal subgroup of $\Gamma_{g,n}$ generated by the set T^ε (T stands for ‘‘torsion’’) of elements that have a nonempty fixed point set when acting on $\mathcal{T}_{g,n}^\varepsilon$. We thus have a decreasing sequence of subgroups $(G^\varepsilon)_{\varepsilon > 0}$. By (ii), this sequence is stationary and we let G be the limit, so that $G = G^\varepsilon$ for any $\varepsilon < \varepsilon_0$. The following statement records what we have so far.

PROPOSITION 1. $\pi_1^\infty(\mathcal{M}_{g,n}) = \Gamma_{g,n}/G$, where $G = G^\varepsilon$ for any $\varepsilon < \varepsilon_0$.

In order to vindicate the description of G given in [L] (see Proposition 2, which follows), we bring in the relative Teichmüller and moduli spaces from [HM] and [GH]. Let $h \in \Gamma_{g,n}$ be a nontrivial torsion element; we denote by $\mathcal{T}_{g,n}(h)$ the fixed point set of h , or equivalently of the finite cyclic group $\langle h \rangle$ it generates, acting on $\mathcal{T}_{g,n}$. The elements of the modular group permute these relative Teichmüller spaces according to the familiar rule: $f(\mathcal{T}_{g,n}(h)) = \mathcal{T}_{g,n}(fhf^{-1})$ for $f \in \Gamma_{g,n}$. The stabilizer of $\mathcal{T}_{g,n}(h)$ is thus the normalizer of the cyclic group $\langle h \rangle$ in $\Gamma_{g,n}$, which we denote by $\Gamma_{g,n}(h)$. The quotient $\mathcal{T}_{g,n}(h)/\Gamma_{g,n}(h) = \tilde{\mathcal{M}}_{g,n}(h)$ is called the *relative moduli space*. Again there is a fine and a coarse version, and by a classical theorem of Cartan the underlying variety is normal. As a matter of fact, if $\mathcal{M}_{g,n}(h) = p(\mathcal{T}_{g,n}(h))$ denotes the image of the relative Teichmüller space in the moduli space, then one can show (see [GH]) that $\tilde{\mathcal{M}}_{g,n}(h)$ is precisely the normalization of $\mathcal{M}_{g,n}(h)$. In this setting the generating set T^ε introduced before Proposition 1 can be described as

$$T^\varepsilon = \{h \in \Gamma_{g,n}, \mathcal{M}_{g,n}(h) \cap \mathcal{M}_{g,n}^\varepsilon \neq \emptyset\}.$$

By a classical result of Nielsen, any element $h \in \Gamma_{g,n}$ of finite order can be realized as an automorphism of some Riemann surface X of type (g, n) . Let $X' = (X/\langle h \rangle)^*$ denote the surface obtained by puncturing the quotient $X/\langle h \rangle$ at the ramification points of the covering $X \rightarrow X/\langle h \rangle$, and let X' be of type (p, v) . Then (see [HM; GH]) one has $\mathcal{T}_{g,n}(h) \simeq \mathcal{T}_{p,v}$ and $\tilde{\mathcal{M}}_{g,n}(h)$ is a finite cover of $\mathcal{M}_{p,v}$. As in [L], we call a torsion element *maximal* if its fixed point set is just a point—that is, if $\mathcal{T}_{g,n}(h)$ has dimension zero, which is equivalent to saying that X' is a thrice-punctured sphere $((p, v) = (0, 3))$.

Now, since the order of an automorphism of a Riemann surface of given type is bounded, one may use the elementary theory of coverings of Riemann surfaces to find that there is only a finite number of surfaces having maximal automorphisms (i.e., whose underlying mapping classes are maximal). The lengths of their geodesics are bounded from below by some ε_{\min} and so, for ε small enough (indeed, for $\varepsilon < \varepsilon_{\min}$), the set T^ε does *not* contain any maximal element. The key point of this section is that this accounts for the set of torsion elements that are excluded from T^ε for ε small enough. In other words, our next goal is to prove the following.

PROPOSITION 2. *For ε sufficiently small, T^ε consists of the set of nonmaximal torsion elements. In particular, G is the group generated by these elements.*

In order to prove this we need two more geometric lemmas, leading to a geometric version (Proposition 3) of Proposition 2. For simplicity, we omit the subscript (g, n) from the notation until the end of this section.

LEMMA 1. *For any given h , the family of subvarieties $f(\mathcal{T}(h))_{f \in \Gamma}$ is locally finite. That is, for any point $t \in \mathcal{T}$, there is a neighborhood of t that meets only finitely many of these subvarieties.*

Proof. Since Γ acts properly discontinuously on \mathcal{T} , there is an open set U containing t such that U and $f(U)$ are disjoint except for finitely many $f \in \Gamma$, say h_1, \dots, h_r . Now for any $f, h \in \Gamma$ (h of finite order), $f(\mathcal{T}(h)) = \mathcal{T}(fhf^{-1})$; in other words, $f(\mathcal{T}(h))$ is just the fixed point set of fhf^{-1} . This implies that $U \cap f(\mathcal{T}(h)) \subset U \cap fhf^{-1}(U)$, from which we infer that if this intersection is nonempty then $fhf^{-1} = h_i$ for some $i \in (1, r)$, and hence $f(\mathcal{T}(h)) = \mathcal{T}(h_i)$. \square

LEMMA 2. *For any $h \in \Gamma$ of finite order, $\mathcal{M}(h)$ is a closed subvariety of \mathcal{M} .*

Proof. We have to show that $p^{-1}(\mathcal{M}(h)) = \bigcup_{f \in \Gamma} f(\mathcal{T}(h))$ is closed in \mathcal{T} ($p: \mathcal{T} \rightarrow \mathcal{M}$ the canonical projection). Let $t \in \mathcal{T}$ be in the closure of $p^{-1}(\mathcal{M}(h))$ and let $(t_m)_m$ be a sequence of points in $p^{-1}(\mathcal{M}(h))$ converging to t . Applying Lemma 1 and passing to a subsequence, we may assume that all the t_m belong to the same subvariety $f(\mathcal{T}(h)) = \mathcal{T}(fhf^{-1})$. But this last subvariety is well known to be closed, which implies that actually $t \in f(\mathcal{T}(h)) \subset p^{-1}(\mathcal{M}(h))$ as required. \square

We are now prepared to deduce our next proposition.

PROPOSITION 3. *If h is not a maximal torsion element in Γ , then $\mathcal{M}(h) \cap \mathcal{M}^\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.*

Proof. The sets $\mathcal{M} \setminus \mathcal{M}^\varepsilon$ are compact, as noted earlier (see [Mu] or [B1; B2]). If the proposition did not hold then $\mathcal{M}(h)$ would be a compact subvariety of \mathcal{M} , by Lemma 2. But the normalization $\hat{\mathcal{M}}(h)$ of \mathcal{M} is a finite cover of the moduli

space $\mathcal{M}_{p,v}$, where (p, v) is the type of the quotient surface (described previously) and this moduli space is not compact—except if $(p, v) = (0, 3)$, that is, if h is maximal. \square

The foregoing result, which can be rephrased by saying that $\mathcal{M}(h)$ “extends to infinity” except if it has dimension 0, is the content of statement (i) in [L, p. 152]. It immediately implies the validity of Proposition 2; hence, using Propositions 1 and 2, in order to determine the fundamental groups at infinity of the coarse moduli spaces we need only determine the subgroups of the modular groups generated by the nonmaximal torsion elements.

3. Nonmaximal Torsion Elements in the Teichmüller Modular Groups

We turn to the proof of the theorem stated in Section 1. The case $g > 2$ is now easy, using results in [McP]. Recall that we are reduced to showing that, for $g > 2$, the modular group Γ_g is generated by nonmaximal torsion elements (this is statement (ii) in [L, p. 152]). In [McP] it is shown that Γ_g is generated by involutions (elements of order 2) that fix two points if g is even and four points if g is odd. By the Riemann–Hurwitz formula, the genera of the respective quotients are thus $g/2$ and $(g - 1)/2$. This is enough to ensure that these generators are not maximal, which finishes the proof of the fact that the coarse moduli space M_g is simply connected for $g > 2$.

We now turn to the case $g = 2$, more precisely to showing that $\pi_1(M_2) = \mathbf{Z}/5\mathbf{Z}$. The key point is that one can realize M_2 as the quotient of \mathbf{C}^3 by the action of the cyclic group $C_5 = \mathbf{Z}/5\mathbf{Z}$ acting via a (nontrivial) fifth root of unity ζ by $\zeta \cdot (z_1, z_2, z_3) = (\zeta z_1, \zeta^2 z_2, \zeta^3 z_3)$ (see [Ig, p. 638]). This representation comes, of course, from the facts that (a) any genus-2 curve is hyperelliptic and (b) the coordinates z_i play a role that is analogous to the classical j (or λ) modular function in the genus-1 case.

The origin in \mathbf{C}^3 is the only fixed point of the C_5 -action (corresponding to the curve X_0 with equation $y^2 = x^5 - 1$) that has a cyclic automorphism group of order 10 generated by τ such that $\tau(x, y) = (\zeta x, -y)$. Both τ and τ^2 are maximal torsion elements of Γ_2 (see [S]). We now have the following lemma.

LEMMA 3. *Let $\tilde{G} \subset \Gamma_2$ be the subgroup generated by the set of (maximal or not) torsion elements that are not conjugate to either τ or τ^2 . Then $\tilde{G} = G$.*

Proof. We have that $G \subset \tilde{G}$ by definition. In the converse direction we first observe (see [S]) that, apart from X_0 , there are exactly two other curves with maximal automorphisms. The first one is the curve X_1 with equation $y^2 = x^6 - 1$ and automorphism group of order 24, generated by two elements γ and σ of order 6 and 4, respectively. Denoting by ζ_6 a primitive sixth root of unity, we can take $\gamma(x, y) = (\zeta_6 x, y)$ and $\sigma(x, y) = (x^{-1}, ix^{-3}y)$. The quotient of X_1 by σ is a sphere and there are four ramification values, so σ is not maximal. By the same token, γ is

found to be maximal, but $\gamma = \gamma^4\gamma^3$ and one finds that both γ^4 and γ^3 are not maximal; therefore, γ and hence the full automorphism group of X_1 is contained in G .

The other curve is X_2 with equation $y^2 = x(x^4 - 1)$. It has automorphism group of order 48 generated by the three elements ρ (order 8), β (order 3), and α (order 2) with $\rho = \alpha\beta$ (see [K] or [C]). One can again give explicit equations (see [C] for a detailed study of the automorphism groups of curves of genus 2), but for our purpose it is enough to observe that, because of their respective orders, α and β cannot be maximal. This implies that the full automorphism group of X_2 is contained in G , completing the proof of the lemma. □

In order to prove the assertion in the theorem for the case of genus 2, let us introduce the punctured space M_2^* , which is M_2 with the point corresponding to the curve X_0 removed (i.e., the origin in \mathbf{C}^3 in the representation of Igusa [Ig] recalled previously). Thus we have that $M_2^* \simeq (\mathbf{C}^3 \setminus \{0\})/C_5$ and also $M_2^* \simeq \mathcal{T}_2^*/\Gamma_2$, where of course \mathcal{T}_2^* denotes the Teichmüller space of genus 2 with the fiber $p^{-1}(X_0)$ deleted. Let us use these two representations to compute the fundamental group of M_2^* in two ways. First, by standard covering theory applied to Igusa's representation, we find that $\pi_1(M_2^*) = C_5$. On the other hand, since $p^{-1}(X_0)$ is simply connected, the result of [A] implies that $\pi_1(M_2^*) = \Gamma_2/\tilde{G}$. Since $\tilde{G} = G$ (by Lemma 2) we find indeed that $\Gamma_2/G = C_5$, thus finishing the proof of the assertion of the theorem: $\pi_1^\infty(M_2) \simeq \pi_1(M_2^*) = \mathbf{Z}/5\mathbf{Z}$. □

We note that the proof actually shows that $\pi_1^\infty(M_2)$ is generated by the mapping class τ^2 . It also shows that any set of generators for Γ_2 consisting of elements of finite order must contain a conjugate of τ^2 , which is a little more precise than saying that Γ_2 cannot be generated by nonmaximal torsion elements.

We close with a short comment on the case of surfaces with marked points. Let (g, n) be a given type and let $M_{g,[n]}$ be the coarse moduli space of surfaces of that type, where the points are *not* individually labeled, so giving rise to the full group $\Gamma_{g,[n]} (= \pi_1(\mathcal{M}_{g,[n]}))$.

The following result seems to hold: $\pi_1^\infty(M_{g,[n]}) = \{1\}$ for $g > 2$. This would follow just as the assertion of the theorem for $g > 2$ from the closing remark in [Pa], which states that $\Gamma_{g,[n]}$ ($g > 2$) is generated by involutions. These cannot be maximal. In fact, if h is such an involution for a Riemann surface S , then the quotient $S/\langle h \rangle$ either has genus > 0 or (if it has genus $= 0$) S is hyperelliptic and h is the hyperelliptic involution. In the latter case, h fixes $2g + 2 > 3$ points (with g the genus of S) and is thus again not maximal.

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