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Triangular Hopf Algebras with the Chevalley Property

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1. Introduction

Triangular Hopf algebras were introduced by Drinfeld [Dr]. They are the Hopf algebras whose representations form a symmetric tensor category. In that sense, they are the class of Hopf algebras closest to group algebras. The structure of triangular Hopf algebras is far from trivial and yet is more tractable than that of general Hopf algebras, owing to their proximity to groups and Lie algebras. This makes triangular Hopf algebras an excellent testing ground for general Hopf algebraic ideas, methods, and conjectures.

A general classification of triangular Hopf algebras is not known yet. However, there are two classes that are relatively well understood. One of them is semisimple triangular Hopf algebras over **C**, for which a complete classification is given in [EG1; EG2]. The key theorem about such Hopf algebras states that each of them is obtained by twisting a group algebra of a finite group (see [EG1, Thm. 2.1]). The proof of this theorem is based on Deligne's theorem on Tannakian categories [D1].

Another important class of Hopf algebras is that of *pointed* ones. These are Hopf algebras whose simple co-modules are all 1-dimensional. Theorem 5.1 in [G] gives a classification of minimal triangular pointed Hopf algebras (we note that the additional assumption made in [G, Thm. 5.1] is, by our Theorem 6.1, superfluous).

Recall that a finite-dimensional algebra is called *basic* if all of its simple modules are 1-dimensional (i.e., if its dual is a pointed co-algebra). The same Theorem 5.1 of [G] gives a classification of minimal triangular basic Hopf algebras, since the dual of a minimal triangular Hopf algebra is again minimal triangular.

Basic and semisimple Hopf algebras share a common property—namely, the Jacobson radical $\mathrm{Rad}(H)$ of such a Hopf algebra H is a Hopf ideal and therefore the quotient $H/\mathrm{Rad}(H)$ (the semisimple part) is itself a Hopf algebra. The representation-theoretic formulation of this property is: The tensor product of two simple H-modules is semisimple. A remarkable classical theorem of Chevalley [C, p. 88] states that, over **C**, this property holds for the group algebra of any (not necessarily finite) group. So let us call this property of H the *Chevalley property*.

The Chevalley property certainly fails for many finite-dimensional Hopf algebras—for example, for Lusztig's [L] finite-dimensional quantum groups $\mathrm{U}_q(\mathbf{g})'$

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at roots of unity (also known as Frobenius–Lusztig kernels). However, we found that this property holds for all examples we know of finite-dimensional *triangular* Hopf algebras in characteristic 0. We felt, therefore, that it is natural to classify all finite-dimensional triangular Hopf algebras with the Chevalley property. This is what we do in this paper.

We start by classifying triangular Hopf algebras with R-matrix of rank ≤ 2. We show that such a Hopf algebra is a suitable modification of a co-commutative Hopf superalgebra (i.e. the group algebra of a supergroup). On the other hand, by a theorem of Kostant [Ko], a finite supergroup is a semidirect product of a finite group with an odd vector space on which this group acts.

Next we prove our main result: Any finite-dimensional triangular Hopf algebra with the Chevalley property is obtained by twisting a triangular Hopf algebra with R-matrix of rank ≤ 2. We also prove the converse result that any such Hopf algebra does have the Chevalley property. As a corollary, we prove that any finite-dimensional triangular Hopf algebra whose co-radical is a Hopf subalgebra (e.g. pointed) is obtained by twisting a triangular Hopf algebra with R-matrix of rank ≤ 2.

The paper is organized as follows. In Section 2 we give the definitions of Hopf superalgebras and twists for them. We also discuss co-commutative Hopf superalgebras and describe their classification (Kostant's theorem [Ko]).

In Section 3 we establish a correspondence between usual Hopf algebras and Hopf superalgebras, and we show how this correspondence extends to twists and to triangular Hopf algebras. In Section 4 we discuss the Chevalley property, and in Section 5 we prove our main result and discuss its consequences and some open questions.

In Section 6, using the main theorem, we show that a finite-dimensional co-triangular pointed Hopf algebra is generated by its grouplike and skewprimitive elements. Thus we confirm the conjecture that this is the case for any finite-dimensional pointed Hopf algebra over **C** [AS2] in the co-triangular case. This allows us to strengthen the main result of [G].

In Section 7 we prove that the categorical dimensions of objects in any abelian symmetric rigid category with finitely many irreducible objects are integers. In particular, this is the case for the representation category of a triangular Hopf algebra. This gives supporting evidence for a positive answer to the question we ask in Section 5: Is any finite-dimensional triangular Hopf algebra a twist of a modified supergroup algebra?

In the appendix we use the lifting method [AS1; AS2] to give other proofs of Theorem 5.2.1 and Corollary 6.3 as well as a generalization of Lemma 5.3.4.

We note that, similarly to the case of semisimple Hopf algebras, the proof of our main result is based on Deligne's theorem [D1]. In fact, we use Theorem 2.1 of [EG1] to prove the main result of this paper.

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2. Hopf Superalgebras

2.1. Supervector Spaces

The ground field in this paper will always be the field **C** of complex numbers.

We start by recalling the definition of the category of supervector spaces. A Hopf algebraic way to define this category is as follows.

Let u be the generator of the group $\mathbb{Z}_2$ of two elements, and set

$$R_u := \tfrac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u) \in \mathbf{C}[\mathbb{Z}_2] \otimes \mathbf{C}[\mathbb{Z}_2]. \tag{1}$$

Then $(\mathbf{C}[\mathbb{Z}_2], R_u)$ is a minimal triangular Hopf algebra.

Definition 2.1.1. The category of supervector spaces over **C** is the symmetric tensor category $\mathrm{Rep}(\mathbf{C}[\mathbb{Z}_2], R_u)$ of representations of the triangular Hopf algebra $(\mathbf{C}[\mathbb{Z}_2], R_u)$. This category will be denoted by SuperVect.

For $V \in$ SuperVect and $v \in V$, we say that v is even if $uv = v$ and odd if $uv = -v$. The set of even vectors in V is denoted by V_0 and the set of odd vectors by V_1, so $V = V_0 \oplus V_1$. We define the parity of a vector v to be $p(v) = 0$ if v is even and $p(v) = 1$ if v is odd (if v is neither odd nor even then $p(v)$ is not defined).

Thus, as an ordinary tensor category, SuperVect is equivalent to the category of representations of $\mathbb{Z}_2$, but the commutativity constraint is different from that of $\mathrm{Rep}(\mathbb{Z}_2)$ and equals $\beta := R_u P$, where P is the permutation of components. In other words, we have

$$\beta(v \otimes w) = (-1)^{p(v)p(w)} w \otimes v, \tag{2}$$

where both v and w are either even or odd.

2.2. Hopf Superalgebras

Recall that, in any symmetric tensor category, one can define an algebra (coalgebra, bi-algebra, Hopf algebra, triangular Hopf algebra, etc.) to be an object of this category equipped with the usual structure maps (morphisms in this category), subject to the same axioms as in the usual case. In particular, any of these algebraic structures in the category SuperVect is usually identified by the prefix "super". For example, we have the following definition.

Definition 2.2.1. A Hopf superalgebra is a Hopf algebra in SuperVect.

More specifically, a Hopf superalgebra $\mathcal{H}$ is an ordinary $\mathbb{Z}_2$-graded associative unital algebra with multiplication m, equipped with a co-associative map $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ (a morphism in SuperVect) that is multiplicative in the super-sense, and with a co-unit and antipode satisfying the standard axioms. Here "multiplicativity in the super-sense" means that Δ satisfies the relation

$$\Delta(ab) = \sum (-1)^{p(a_2)p(b_1)} a_1 b_1 \otimes a_2 b_2 \tag{3}$$

for all $a, b \in \mathcal{H}$ (where $\Delta(a) = \sum a_1 \otimes a_2$ and $\Delta(b) = \sum b_1 \otimes b_2$). This is because the tensor product of two algebras A, B in SuperVect is defined to be $A \otimes B$ as a vector space, with multiplication

$$(a \otimes b)(a' \otimes b') := (-1)^{p(a')p(b)} aa' \otimes bb'. \tag{4}$$

REMARK 2.2.2. Hopf superalgebras appear in [Ko] under the name of "graded Hopf algebras".

Similarly, a (quasi)triangular Hopf superalgebra $(\mathcal{H}, \mathcal{R})$ is a Hopf superalgebra with an R-matrix (an even element $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$) satisfying the usual axioms. As in the even case, an important role is played by the Drinfeld element u of $(\mathcal{H}, \mathcal{R})$:

$$u := m \circ \beta \circ (\mathrm{Id} \otimes S)(\mathcal{R}). \tag{5}$$

For instance, $(\mathcal{H}, \mathcal{R})$ is triangular if and only if u is a grouplike element of $\mathcal{H}$.

As in the even case, the tensorands of the R-matrix of a (quasi)triangular Hopf superalgebra $\mathcal{H}$ generate a finite-dimensional sub-Hopf superalgebra $\mathcal{H}_m$, called the *minimal part of* $\mathcal{H}$ (the proof does not differ essentially from the proof of the analogous fact for Hopf algebras). A (quasi)triangular Hopf superalgebra is said to be minimal if it coincides with its minimal part. The dimension of the minimal part in the triangular case is the *rank* of the R-matrix.

2.3. Co-commutative Hopf Superalgebras

DEFINITION 2.3.1. We will say that a Hopf superalgebra $\mathcal{H}$ is *commutative* (resp., *co-commutative*) if $m = m \circ \beta$ (resp., $\Delta = \beta \circ \Delta$).

EXAMPLE 2.3.2 [Ko]. Let G be a group and $\mathbf{g}$ a Lie superalgebra with an action of G by automorphisms of Lie superalgebras. Let $\mathcal{H} := \mathbf{C}[G] \ltimes \mathrm{U}(\mathbf{g})$, where $\mathrm{U}(\mathbf{g})$ denotes the universal enveloping algebra of $\mathbf{g}$. Then $\mathcal{H}$ is a co-commutative Hopf superalgebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$ $(x \in \mathbf{g})$ and $\Delta(g) = g \otimes g$ $(g \in G)$. In this Hopf superalgebra, we have $S(g) = g^{-1}$, $S(x) = -x$, and in particular $S^2 = \mathrm{Id}$.

The Hopf superalgebra $\mathcal{H}$ is finite-dimensional if and only if G is finite, and $\mathbf{g}$ is finite-dimensional and purely odd (and hence commutative). Then $\mathcal{H} = \mathbf{C}[G] \ltimes \Lambda V$, where $V = \mathbf{g}$ is an odd vector space with a G-action. In this case, $\mathcal{H}^*$ is a commutative Hopf superalgebra.

REMARK 2.3.3. We note that, as in the even case, it is convenient to think about $\mathcal{H}$ and $\mathcal{H}^*$ in geometric terms. Consider, for instance, the finite-dimensional case. In this case, it is useful to think of the "affine algebraic supergroup" $\tilde{G} := G \ltimes V$. Then one can regard $\mathcal{H}$ as the group algebra $\mathbf{C}[\tilde{G}]$ of this supergroup and $\mathcal{H}^*$ as its function algebra $F(\tilde{G})$. Having this in mind, we will call the algebra $\mathcal{H}$ a *supergroup algebra*.

It turns out that, as in the even case, any co-commutative Hopf superalgebra is of the type described in Example 2.3.2. Namely, we have the following theorem.

THEOREM 2.3.4 [Ko, Thm. 3.3]. *Let $\mathcal{H}$ be a co-commutative Hopf superalgebra over $\mathbf{C}$. Then $\mathcal{H} = \mathbf{C}[\mathbf{G}(\mathcal{H})] \ltimes \mathrm{U}(\mathbf{P}(\mathcal{H}))$, where $\mathrm{U}(\mathbf{P}(\mathcal{H}))$ is the universal* *enveloping algebra of the Lie superalgebra of primitive elements of $\mathcal{H}$ and where $\mathbf{G}(\mathcal{H})$ is the group of grouplike elements of $\mathcal{H}$.*

In the finite-dimensional case we obtain a corollary.

COROLLARY 2.3.5. *Let $\mathcal{H}$ be a finite-dimensional co-commutative Hopf superalgebra over $\mathbf{C}$. Then $\mathcal{H} = \mathbf{C}[\mathbf{G}(\mathcal{H})] \ltimes \Lambda V$, where V is the space of primitive elements of $\mathcal{H}$ (regarded as an odd vector space) and $\mathbf{G}(\mathcal{H})$ is the finite group of grouplikes of $\mathcal{H}$. In other words, $\mathcal{H}$ is a supergroup algebra.*

We shall use this corollary and so (although it follows at once from Theorem 2.3.4) we will give its proof in Section 5 for the sake of completeness.

2.4. Twists

A twist for a Hopf algebra in any symmetric tensor category is defined in the same way as in the usual case (see [Dr]). However, for the reader's convenience, we will repeat this definition (for Hopf superalgebras).

Let $\mathcal{H}$ be a Hopf superalgebra. The multiplication, unit, co-multiplication, co-unit, and antipode in $\mathcal{H}$ will be denoted by m, 1, Δ, ε, and S, respectively.

DEFINITION 2.4.1. A twist for $\mathcal{H}$ is an invertible even element $\mathcal{J} \in \mathcal{H} \otimes \mathcal{H}$ that satisfies

$$(\Delta \otimes \mathrm{Id})(\mathcal{J})(\mathcal{J} \otimes 1) = (\mathrm{Id} \otimes \Delta)(\mathcal{J})(1 \otimes \mathcal{J}),$$
$$(\varepsilon \otimes \mathrm{Id})(\mathcal{J}) = (\mathrm{Id} \otimes \varepsilon)(\mathcal{J}) = 1, \tag{6}$$

where Id is the identity map of $\mathcal{H}$.

Given a twist $\mathcal{J}$ for $\mathcal{H}$, one can define a new Hopf superalgebra structure

$$(\mathcal{H}^{\mathcal{J}}, m, 1, \Delta^{\mathcal{J}}, \varepsilon, S^{\mathcal{J}})$$

on the algebra $(\mathcal{H}, m, 1)$ as follows. The co-product is determined by

$$\Delta^{\mathcal{J}}(a) = \mathcal{J}^{-1}\Delta(a)\mathcal{J} \quad \text{for any } a \in \mathcal{H}, \tag{7}$$

and the antipode is determined by

$$S^{\mathcal{J}}(a) = Q^{-1}S(a)Q \quad \text{for any } a \in \mathcal{H}, \tag{8}$$

where $Q := m \circ (S \otimes \mathrm{Id})(\mathcal{J})$.

If $\mathcal{H}$ is (quasi)triangular with the universal R-matrix $\mathcal{R}$, then so is $\mathcal{H}^{\mathcal{J}}$ with the universal R-matrix $\mathcal{R}^{\mathcal{J}} := \mathcal{J}_{21}^{-1}\mathcal{R}\mathcal{J}$.

3. Triangular Hopf Algebras with Drinfeld Element of Order ≤ 2

3.1. The Correspondence between Hopf Algebras and Superalgebras

We can now prove our first results, which will be essential in the next section. We start with a correspondence theorem between Hopf algebras and Hopf superalgebras.

THEOREM 3.1.1. *There is a one-to-one correspondence between*

1. *isomorphism classes of pairs* (H, u), *where* H *is an ordinary Hopf algebra and* u *is a grouplike element in* H *such that* $u^2 = 1$, *and*
2. *isomorphism classes of pairs* $(\mathcal{H}, g)$, *where* $\mathcal{H}$ *is a Hopf superalgebra and* g *is a grouplike element in* $\mathcal{H}$ *such that* $g^2 = 1$ *and* $gxg^{-1} = (-1)^{p(x)}x$ (*i.e.,* g *acts on* x *by its parity*),

such that the tensor categories of representations of H *and* $\mathcal{H}$ *are equivalent.*

Proof. Let (H, u) be an ordinary Hopf algebra with co-multiplication Δ, co-unit ε, antipode S, and a grouplike element u such that $u^2 = 1$. Let $\mathcal{H} = H$ regarded as a superalgebra, where the $\mathbb{Z}_2$-grading is given by the adjoint action of u. For $h \in H$ define Δ_0, Δ_1 by writing $\Delta(h) = \Delta_0(h) + \Delta_1(h)$, where $\Delta_0(h) \in H \otimes H_0$ and $\Delta_1(h) \in H \otimes H_1$. Define a map $\tilde{\Delta} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by $\tilde{\Delta}(h) := \Delta_0(h) - (-1)^{p(h)}(u \otimes 1)\Delta_1(h)$. Define $\tilde{S}(h) := u^{p(h)}S(h)$, $h \in H$. Then it is straightforward to verify that $(\mathcal{H}, \tilde{\Delta}, \varepsilon, \tilde{S})$ is a Hopf superalgebra.

The element u remains grouplike in the new Hopf superalgebra and acts by parity, so we can set $g := u$.

Conversely, suppose that $(\mathcal{H}, g)$ is a pair where $\mathcal{H}$ is a Hopf superalgebra with co-multiplication $\tilde{\Delta}$, co-unit ε, antipode $\tilde{S}$, and a grouplike element g (with $g^2 = 1$) acting by parity. For $h \in \mathcal{H}$ define $\tilde{\Delta}_0, \tilde{\Delta}_1$ by writing $\tilde{\Delta}(h) = \tilde{\Delta}_0(h) + \tilde{\Delta}_1(h)$, where $\tilde{\Delta}_0(h) \in \mathcal{H} \otimes \mathcal{H}_0$ and $\tilde{\Delta}_1(h) \in \mathcal{H} \otimes \mathcal{H}_1$. Let $H = \mathcal{H}$ as algebras, and define a map $\Delta : H \to H \otimes H$ by $\Delta(h) := \tilde{\Delta}_0(h) - (-1)^{p(h)}(g \otimes 1)\tilde{\Delta}_1(h)$. Define $S(h) := g^{p(h)}\tilde{S}(h)$, $h \in H$. Then it is straightforward to verify that $(H, \Delta, \varepsilon, S)$ is an ordinary Hopf algebra, and we can set $u := g$.

It is obvious that the two assignments just constructed are inverse to each other. The equivalence of tensor categories is straightforward to verify. The theorem is proved. □

Theorem 3.1.1 implies the following. Let $\mathcal{H}$ be *any* Hopf superalgebra and let $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}$ be the semidirect product, where the generator g of $\mathbb{Z}_2$ acts on $\mathcal{H}$ by $gxg^{-1} = (-1)^{p(x)}x$. Then we can define an ordinary Hopf algebra $\bar{\mathcal{H}}$, which is the one corresponding to $(\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}, g)$ under the correspondence of Theorem 3.1.1.

The constructions of this section have the following explanation in terms of Radford's bi-product construction [R2]. Namely, $\mathcal{H}$ is a Hopf algebra in the Yetter–Drinfeld category of $\mathbf{C}[\mathbb{Z}_2]$, so Radford's bi-product construction yields a Hopf algebra structure on $\mathbf{C}[\mathbb{Z}_2] \otimes \mathcal{H}$; it is straightforward to see that this Hopf algebra is exactly $\bar{\mathcal{H}}$. Moreover, it is clear that, for any pair (H, u) as in Theorem 3.1.1, gu is central in $\bar{\mathcal{H}}$ and $H = \bar{\mathcal{H}}/(gu - 1)$.

Let us give an interesting corollary of Theorem 3.1.1, even though we will not use it.

COROLLARY 3.1.2. *Let* $\mathcal{H}$ *be a finite-dimensional Hopf superalgebra over* $\mathbf{C}$. *Then*:

1. $\mathcal{H}$ *is semisimple if and only if it is co-semisimple*;
2. *if* $\mathcal{H}$ *is semisimple then* $S^4 = \mathrm{Id}$; *and*

3. *if $\mathcal{H}$ is semisimple and $S^2 = \mathrm{Id}$, then $\mathcal{H}$ is purely even, that is, it is a usual semisimple Hopf algebra.*

Proof. 1. If $\mathcal{H}$ is semisimple then so is $\bar{\mathcal{H}}$, hence so is $(\bar{\mathcal{H}})^*$. But it is easy to show that $(\bar{\mathcal{H}})^*$ is isomorphic as an algebra to $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}^*$ (unlike the dual of $\mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}$, which is isomorphic to $\mathbf{C}[\mathbb{Z}_2] \otimes \mathcal{H}^*$). Therefore, this crossed product algebra is semisimple. It is well known (and easy to show) that this implies the semisimplicity of $\mathcal{H}^*$.

2. The Hopf algebra $\bar{\mathcal{H}}$ is semisimple, so we have $S^2 = \mathrm{Id}$ in it. Thus, in $\mathcal{H}$ we have $S^2 = \mathrm{Ad}(g)$ and so $S^4 = \mathrm{Ad}(g^2) = \mathrm{Id}$.

3. Since $S^2 = \mathrm{Ad}(g)$, g must be central. Thus, $\mathcal{H}$ is purely even. □

REMARK 3.1.3. The example of supergroup algebras shows that, for finite-dimensional Hopf superalgebras (unlike usual Hopf algebras), $S^2 = \mathrm{Id}$ does not imply semisimplicity or co-semisimplicity. In fact, Corollary 3.1.2(3) shows that, in a sense, the situation is exactly the opposite.

3.2. Correspondence of Twists

Let us say that a twist J for a Hopf algebra H with an involutive grouplike element g is *even* if it is invariant under $\mathrm{Ad}(g)$.

PROPOSITION 3.2.1. *Let $(\mathcal{H}, g)$ be a pair as in Theorem 3.1.1, and let H be the associated ordinary Hopf algebra. Let $\mathcal{J} \in \mathcal{H} \otimes \mathcal{H}$ be an even element. Write $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$, where $\mathcal{J}_0 \in \mathcal{H}_0 \otimes \mathcal{H}_0$ and $\mathcal{J}_1 \in \mathcal{H}_1 \otimes \mathcal{H}_1$. Define $J := \mathcal{J}_0 - (g \otimes 1)\mathcal{J}_1$. Then J is an even twist for H if and only if $\mathcal{J}$ is a twist for $\mathcal{H}$. Moreover, $\mathcal{H}^{\mathcal{J}}$ corresponds to H^J under the correspondence in Theorem 3.1.1. Thus, there is a one-to-one correspondence between even twists for H and twists for $\mathcal{H}$ that is given by $J \to \mathcal{J}$.*

Proof. Straightforward. □

3.3. The Correspondence between Triangular Hopf Algebras and Superalgebras

Let us now return to our main subject, which is triangular Hopf algebras and superalgebras. For triangular Hopf algebras whose Drinfeld element u is involutive, we will make the natural choice of the element u in Theorem 3.1.1—namely, we define it to be the Drinfeld element of H.

THEOREM 3.3.1. *The correspondence of Theorem 3.1.1 extends to a one-to-one correspondence between*

1. *isomorphism classes of ordinary triangular Hopf algebras H with Drinfeld element u such that $u^2 = 1$ and*
2. *isomorphism classes of pairs $(\mathcal{H}, g)$, where $\mathcal{H}$ is a triangular Hopf superalgebra with Drinfeld element 1 and g is an element of $\mathbf{G}(\mathcal{H})$ such that $g^2 = 1$ and $gxg^{-1} = (-1)^{p(x)}x$.*

Proof. Let (H, R) be a triangular Hopf algebra with $u^2 = 1$. Since $(S \otimes S)(R) = R$ and $S^2 = \mathrm{Ad}(u)$ [Dr], $u \otimes u$ and R commute. Hence we can write $R = R_0 + R_1$, where $R_0 \in H_0 \otimes H_0$ and $R_1 \in H_1 \otimes H_1$. Let $\mathcal{R} := (R_0 + (1 \otimes u)R_1)R_u$. Then $\mathcal{R}$ is even. Indeed, since $R_0 = 1/2(R + (u \otimes 1)R(u \otimes 1))$ and $R_1 = 1/2(R - (u \otimes 1)R(u \otimes 1))$, it follows that $u \otimes u$ and $\mathcal{R}$ commute.

It is now straightforward to show that $(\mathcal{H}, \mathcal{R})$ is triangular with Drinfeld element 1. Let us show, for instance, that $\mathcal{R}$ is unitary. We use the notation $a * b$ and X^{21} for multiplication and opposition in the tensor square of a superalgebra, and we use the notation ab and X^{op} for usual algebras. Then

$$\mathcal{R} * \mathcal{R}^{21} = (R_0 + (1 \otimes u)R_1)R_u * (R_0^{\mathrm{op}} - (u \otimes 1)R_1^{\mathrm{op}})R_u.$$

Since $R_u R_0 = R_0 R_u$ and $R_u R_1 = -(u \otimes u)R_1 R_u$, it follows that the RHS equals

$$(R_0 + (1 \otimes u)R_1) * (R_0^{\mathrm{op}} + (1 \otimes u)R_1^{\mathrm{op}}) = R_0 R_0^{\mathrm{op}} + R_1 R_1^{\mathrm{op}} + (1 \otimes u)(R_1 R_0^{\mathrm{op}} + R_0 R_1^{\mathrm{op}}).$$

But $R_0 R_0^{\mathrm{op}} + R_1 R_1^{\mathrm{op}} = 1$ and $(1 \otimes u)(R_1 R_0^{\mathrm{op}} + R_0 R_1^{\mathrm{op}}) = 0$, since $RR^{\mathrm{op}} = 1$, so we are done.

Conversely, suppose that $(\mathcal{H}, g)$ is a pair, where $\mathcal{H}$ is a triangular Hopf superalgebra with R-matrix $\mathcal{R}$ and Drinfeld element 1. Let $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$, where $\mathcal{R}_0$ has even components and $\mathcal{R}_1$ has odd components. Let $R := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g$. Then it is straightforward to show that (H, R) is triangular with Drinfeld element $u = g$. The theorem is proved. □

COROLLARY 3.3.2. *If $(\mathcal{H}, \mathcal{R})$ is a triangular Hopf superalgebra with Drinfeld element* 1, *then the Hopf algebra $\bar{\mathcal{H}}$ is also triangular with the R-matrix*

$$\bar{R} := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g, \tag{9}$$

where g is the grouplike element adjoined to $\mathcal{H}$ to obtain $\bar{\mathcal{H}}$. Moreover, $\mathcal{H}$ is minimal if and only if $\bar{\mathcal{H}}$ is minimal.

Proof. Clear. □

The following corollary, combined with Kostant's theorem, gives a classification of triangular Hopf algebras with R-matrix of rank ≤ 2 (i.e., of the form R_u as in (1), where u is grouplike of order ≤ 2).

COROLLARY 3.3.3. *The correspondence of Theorem 3.3.1 restricts to a one-to-one correspondence between*

1. *isomorphism classes of ordinary triangular Hopf algebras with R-matrix of rank ≤ 2 and*
2. *isomorphism classes of pairs $(\mathcal{H}, g)$, where $\mathcal{H}$ is a co-commutative Hopf superalgebra and g is an element of $\mathbf{G}(\mathcal{H})$ such that $g^2 = 1$ and $gxg^{-1} = (-1)^{p(x)}x$.*

Proof. Let (H, R) be an ordinary triangular Hopf algebra with R-matrix of rank $\leq$ 2. In particular, the Drinfeld element u of H satisfies $u^2 = 1$ and $R = R_u$. Hence by Theorem 3.3.1, $(\mathcal{H}, \tilde{\Delta}, \mathcal{R})$ is a triangular Hopf superalgebra. Moreover, it is co-commutative because $\mathcal{R} = R_u R_u = 1$.

Conversely, for any $(\mathcal{H}, g)$, by Theorem 3.3.1 the pair (H, R_g) is an ordinary triangular Hopf algebra, and clearly the rank of R_g is ≤ 2. □

Corollaries 2.3.5 and 3.3.3 imply that finite-dimensional triangular Hopf algebras with R-matrix of rank ≤ 2 correspond to supergroup algebras. In view of this, we make the following definition.

DEFINITION 3.3.4. A finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2 is called a *modified supergroup algebra*.

3.4. Construction of Twists for Supergroup Algebras

PROPOSITION 3.4.1. *Let $\mathcal{H} = \mathbf{C}[G] \ltimes \Lambda V$ be a supergroup algebra. Let $r \in S^2V$. Then $\mathcal{J} := e^{r/2}$ is a twist for $\mathcal{H}$. Moreover, $((\Lambda V)^{\mathcal{J}}, \mathcal{J}_{21}^{-1}\mathcal{J})$ is minimal triangular if and only if r is nondegenerate.*

Proof. Straightforward. □

EXAMPLE 3.4.2. Let G be the group of order 2 with generator g. Let $V := \mathbf{C}$ be the nontrivial 1-dimensional representation of G, and write $\Lambda V = \mathrm{sp}\{1, x\}$. Then the associated ordinary triangular Hopf algebra to $(\mathcal{H}, g) := (\mathbf{C}[G] \ltimes \Lambda V, g)$ is Sweedler's [S] 4-dimensional Hopf algebra H with the triangular structure R_g. Namely, the algebra H is generated by a grouplike element g and a $1 : g$ skew primitive element x (i.e., $\Delta(x) = x \otimes 1 + g \otimes x$) satisfying the relations $g^2 = 1$, $x^2 = 0$, and $gx = -xg$. It is known [R2] that the set of triangular structures on H is parameterized by $\mathbf{C}$; namely, R is a triangular structure on H if and only if

$$R = R_\lambda := R_g - \frac{\lambda}{2}(x \otimes x - gx \otimes x + x \otimes gx + gx \otimes gx), \quad \lambda \in \mathbf{C}.$$

Clearly, (H, R_λ) is minimal if and only if $\lambda \neq 0$.

Let $r \in S^2V$ be defined by $r := \lambda x \otimes x$, $\lambda \in \mathbf{C}$. Set $\mathcal{J}_\lambda := e^{r/2} = 1 + \frac{1}{2}\lambda x \otimes x$; it is a twist for $\mathcal{H}$. Hence, $J_\lambda := 1 - \frac{1}{2}\lambda gx \otimes x$ is a twist for H. It is easy to check that $R_\lambda = (J_\lambda)_{21}^{-1} R_g J_\lambda$. Thus, $(H, R_\lambda) = (H, R_0)^{J_\lambda}$.

REMARK 3.4.3. In fact, Radford's classification of triangular structures on H can be easily deduced from Lemma 5.3.4 (see Section 5).

4. The Chevalley Property

Recall that in the introduction we made the following definition.

DEFINITION 4.1. A Hopf algebra H over $\mathbf{C}$ is said to have the *Chevalley property* if the tensor product of any two simple H-modules is semisimple. More generally, let us say that a tensor category has the Chevalley property if the tensor product of two simple objects is semisimple.

Let us give some equivalent formulations of the Chevalley property.

PROPOSITION 4.2. *Let H be a finite-dimensional Hopf algebra over* $\mathbf{C}$ *and let* $A := H^*$. *The following conditions are equivalent.*

1. *H has the Chevalley property.*
2. *The category of (right) A-co-modules has the Chevalley property.*
3. $\mathrm{Corad}(A)$ *is a Hopf subalgebra of* A.
4. $\mathrm{Rad}(H)$ *is a Hopf ideal and thus* $H/\mathrm{Rad}(H)$ *is a Hopf algebra.*
5. $S^2 = \mathrm{Id}$ *on* $H/\mathrm{Rad}(H)$ *or (equivalently) on* $\mathrm{Corad}(A)$.

Proof. (1. $\Leftrightarrow$ 2.) Clear, since the categories of left H-modules and right A-co-modules are equivalent.

(2. $\Rightarrow$ 3.) Recall the definition of a matrix coefficient of a co-module V over A. If $\rho\colon V \to V \otimes A$ is the co-action ($v \in V$, $\alpha \in V^*$), then

$$\phi^V_{v,\alpha} := (\alpha \otimes \mathrm{Id})\rho(v) \in A.$$

It is well known that:

(a) The co-radical of A is the linear span of the matrix coefficients of all simple A-co-modules.
(b) The product in A of two matrix coefficients is a matrix coefficient of the tensor product. Specifically,

$$\phi^V_{v,\alpha}\phi^W_{w,\beta} = \phi^{V\otimes W}_{v\otimes w,\alpha\otimes\beta}.$$

It follows at once from (a) and (b) that $\mathrm{Corad}(A)$ is a subalgebra of A. Since the co-radical is stable under the antipode, the claim follows.

(3. $\Leftrightarrow$ 4.) To say that $\mathrm{Rad}(H)$ is a Hopf ideal is equivalent to saying that $\mathrm{Corad}(H^*)$ is a Hopf algebra, since $\mathrm{Corad}(H^*) = (H/\mathrm{Rad}\, H)^*$.

(4. $\Rightarrow$ 1.) If V, W are simple H-modules then they factor through $H/\mathrm{Rad}(H)$. But $H/\mathrm{Rad}(H)$ is a Hopf algebra, so $V \otimes W$ also factors through $H/\mathrm{Rad}(H)$ and hence is semisimple.

(3. $\Rightarrow$ 5.) Clear, since a co-semisimple Hopf algebra is involutory.

(5. $\Rightarrow$ 3.) Consider the subalgebra B of A generated by $\mathrm{Corad}(A)$. This is a Hopf algebra, and $S^2 = \mathrm{Id}$ on it. Thus, B is co-semisimple and hence $B = \mathrm{Corad}(A)$ is a Hopf subalgebra of A. □

REMARK 4.3. The assumption that the base field has characteristic 0 is needed only in the proof of (5. $\Leftrightarrow$ 3.)

5. Classification of Triangular Hopf Algebras with the Chevalley Property

5.1. The Main Theorem

Our main result is the following theorem.

THEOREM 5.1.1. *Let H be a finite-dimensional triangular Hopf algebra over* $\mathbf{C}$. *Then the following are equivalent.*

1. *H is a twist of a finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2 (i.e., of a modified supergroup algebra).*
2. *H has the Chevalley property.*

The proof of this theorem is contained in Sections 5.2 and 5.3.

5.2. Local Finite-Dimensional Hopf Superalgebras Are Exterior Algebras

THEOREM 5.2.1. *Let $\mathcal{H}$ be a local finite-dimensional Hopf superalgebra (not necessarily supercommutative). Then $\mathcal{H} = \Lambda V^*$ for a finite-dimensional vector space V. In other words, $\mathcal{H}$ is the function algebra of an odd vector space V.*

REMARK 5.2.2. Note that, in the commutative case, Theorem 5.2.1 is a special case of Proposition 3.2 of [Ko].

Proof of Theorem 5.2.1. It is sufficient to show that $\mathcal{H}^* = \Lambda V$ for some vector space V, since $(\Lambda V)^* = \Lambda V^*$ as Hopf superalgebras. For this, it is sufficient to show that $\mathcal{H}^*$ is generated by primitive elements, since the sub-Hopf superalgebra in $\mathcal{H}^*$ generated by a basis of the space of primitive elements of $\mathcal{H}^*$ is clearly a *free* anti-commutative algebra on its generators.

Let I be the kernel of the co-unit in $\mathcal{H}$. Then $I = \mathrm{Rad}(\mathcal{H})$ since $\mathcal{H}$ is local. So in particular there exists a positive integer N such that, for any $x_1, \ldots, x_N \in \mathcal{H}$, one has

$$(x_1 - \varepsilon(x_1)1) \cdots (x_N - \varepsilon(x_N)1) = 0.$$

Let $\delta_k \colon \mathcal{H}^* \to (\mathcal{H}^*)^{\otimes k}$ be the map dual to the map $\mathcal{H}^{\otimes k} \to \mathcal{H}$ defined by

$$x_1 \otimes \cdots \otimes x_k \mapsto (x_1 - \varepsilon(x_1)1) \cdots (x_k - \varepsilon(x_k)1)$$

(this map was introduced by Drinfeld in [Dr]). We see that we have a filtration of $\mathcal{H}^* : \mathcal{H}^* = \bigcup \mathcal{H}_k^*$, where $\mathcal{H}_k^*$ is the kernel of δ_k (the Nth term of this filtration is $\mathcal{H}^*$). In other words, $\mathcal{H}_k^*$ is the orthogonal complement of I^k.

Let $V \subseteq \mathcal{H}^*$ be the space of primitive elements, and let $\mathcal{B} := \Lambda V \subseteq \mathcal{H}^*$ be the corresponding Hopf supersubalgebra generated by them. We will prove by induction in k that $\mathcal{H}_k^*$ is contained in $\mathcal{B}$, which will complete the proof.

The base of induction is obvious (as $\delta_1(x) = x - \varepsilon(x)$, hence $\mathcal{H}_1^* = \mathbf{C}$). Suppose the statement is known for $k = n$, and let $a \in \mathcal{H}_{n+1}^*$. Then it is straightforward to verify that $j := \Delta(a) - a \otimes 1 - 1 \otimes a \in \mathcal{H}_n^* \otimes \mathcal{H}_n^*$. Hence, by the induction assumption, $j \in \mathcal{B} \otimes \mathcal{B}$. Thus, j is a symmetric (in the super-sense) 2-cocycle for the co-Hochschild complex of $\mathcal{B}$. But it is well known that a symmetric (in the super-sense) co-Hochschild 2-cocycle for the exterior algebra is a co-boundary. Thus, there exists $b \in \mathcal{B}$ with $j = \Delta(b) - b \otimes 1 - 1 \otimes b$. Hence $a - b$ is a primitive element and thus $a \in \mathcal{B}$. We are done. □

REMARK 5.2.3. In the appendix we give another proof of Theorem 5.2.1 using the lifting method of [AS2].

Theorem 5.2.1 will be used in Section 5.3, but it also allows one to give the following proof of Corollary 2.3.5.

Proof of Corollary 2.3.5. Let I be the ideal in $\mathcal{H}^*$ generated by all the odd elements. It is easy to see that this is a Hopf ideal. Consider the Hopf algebra $E := \mathcal{H}^*/I$ (the even part). This is an *ordinary* commutative Hopf algebra, so $E = F(G)$ for a suitable finite group G. Moreover, it is clear that every element of I is nilpotent, so $I = \mathrm{Rad}(\mathcal{H}^*)$. Thus, irreducible $\mathcal{H}^*$-modules are 1-dimensional and are parameterized by $g \in G$. Let us call them L_g. Also, we see that $G = \mathbf{G}(\mathcal{H})$.

Let P_g be the projective cover of the irreducible module L_g. Then $\mathcal{H}^* = \bigoplus_g P_g$, where the P_g are indecomposable two-sided ideals (the ideals are two-sided because the algebra is commutative in the super-sense). In particular, P_g are local algebras with 1-dimensional semisimple quotient. Also, we have a natural projection of algebras $\mathcal{H}^* \to P_g$ for all g; in particular, $\mathcal{H}^* \to P_1$.

Note that $\mathcal{H}$ acts on $\mathcal{H}^*$ on the left and right. In particular, so does the group G.

LEMMA. *The following hold*:

1. $g_1 P_g g_2 = P_{g_1 g g_2}$;
2. $\Delta(P_g) \subset \bigoplus_{g_1, g_2 : g_1 g_2 = g} P_{g_1} \otimes P_{g_2}$.

Proof. Straightforward. □

COROLLARY. *The ideal $\mathcal{I} := \bigoplus_{g \neq 1} P_g$ is a Hopf ideal, and thus $P_1 = \mathcal{H}^*/\mathcal{I}$ is a Hopf superalgebra.*

Thus, $P_1^* \subset \mathcal{H}$ is a sub-Hopf superalgebra with an action of G, and we have a factorization $\mathcal{H} = \mathbf{C}[G] \ltimes P_1^*$. The Hopf superalgebra P_1 is local, so $P_1^* = \Lambda V$ by Theorem 5.2.1. This concludes the proof of Corollary 2.3.5. □

REMARK 5.2.4. Here is the same proof, described in a more intuitive geometric language. Consider $\tilde{G} := \mathrm{Spec}(\mathcal{H}^*)$; this is an affine supergroup scheme. Let $G \subseteq \tilde{G}$ be the even part of $\tilde{G}$. Then G is a finite group scheme, so by a standard theorem it is a finite group. Let V be the connected component of the identity in $\tilde{G}$. Then the function algebra $\mathcal{O}(V)$ on V is a local finite-dimensional Hopf superalgebra. It follows by Theorem 5.2.1 that $\mathcal{O}(V) = \Lambda V^*$ for some finite-dimensional vector space V.

Thus, we have a split exact sequence of algebraic supergroups

$$1 \to V \to \tilde{G} \to G \to 1$$

(it is split because G is a subgroup of $\tilde{G}$ that is complementary to V). Hence $\tilde{G} = G \ltimes V$, as desired.

5.3. *Proof of the Main Theorem*

We start by giving a super-analog of Theorem 3.1 in [G].

LEMMA 5.3.1. *Let $\mathcal{H}$ be a minimal triangular pointed Hopf superalgebra. Then* $\mathrm{Rad}(\mathcal{H})$ *is a Hopf ideal, and* $\mathcal{H}/\mathrm{Rad}(\mathcal{H})$ *is minimal triangular.*

Proof. The proof is a tautological generalization of the proof of Theorem 3.1 in [G] to the super case.

First of all, it is clear that $\mathrm{Rad}(\mathcal{H})$ is a Hopf ideal, since its orthogonal complement (the co-radical of $\mathcal{H}^*$) is a sub-Hopf superalgebra (since $\mathcal{H}^*$ is isomorphic to $\mathcal{H}^{\mathrm{cop}}$ as a co-algebra and hence is pointed). Thus, it remains to show that the triangular structure on $\mathcal{H}$ descends to a minimal triangular structure on $\mathcal{H}/\mathrm{Rad}(\mathcal{H})$. For this, it suffices to prove that the composition of the Hopf superalgebra maps

$$\mathrm{Corad}(\mathcal{H}^{*\mathrm{cop}}) \hookrightarrow \mathcal{H}^{*\mathrm{cop}} \to \mathcal{H} \to \mathcal{H}/\mathrm{Rad}(\mathcal{H})$$

(where the middle map is given by the R-matrix) is an isomorphism. But this follows from the fact that, for any surjective co-algebra map $\eta\colon C_1 \to C_2$, the image of the co-radical of C_1 contains the co-radical of C_2 [M, Cor. 5.3.5]. One need only apply this statement to the map $\mathcal{H}^{*\mathrm{cop}} \to \mathcal{H}/\mathrm{Rad}(\mathcal{H})$. □

LEMMA 5.3.2. *Let $\mathcal{H}$ be a minimal triangular pointed Hopf superalgebra such that the R-matrix $\mathcal{R}$ of $\mathcal{H}$ is unipotent* (*i.e., $\mathcal{R} - 1 \otimes 1$ is 0 in $\mathcal{H}/\mathrm{Rad}(\mathcal{H}) \otimes \mathcal{H}/\mathrm{Rad}(\mathcal{H})$*). *Then $\mathcal{H} = \Lambda V$ as a Hopf superalgebra and $\mathcal{R} = e^r$, where $r \in S^2V$ is a nondegenerate symmetric* (*in the usual sense*) *bilinear form on V^*.*

Proof. By Lemma 5.3.1, $\mathrm{Rad}(\mathcal{H})$ is a Hopf ideal, and $\mathcal{H}/\mathrm{Rad}(\mathcal{H})$ is minimal triangular. But the R-matrix of $\mathcal{H}/\mathrm{Rad}(\mathcal{H})$ must be $1 \otimes 1$, so $\mathcal{H}/\mathrm{Rad}(\mathcal{H})$ is 1-dimensional. Hence $\mathcal{H}$ is local and so, by Theorem 5.2.1, $\mathcal{H} = \Lambda V$. If $\mathcal{R}$ is a triangular structure on $\mathcal{H}$ then it comes from an isomorphism $\Lambda V^* \to \Lambda V$ of Hopf superalgebras that is induced by a linear isomorphism $r\colon V^* \to V$. Thus $\mathcal{R} = e^r$, where r is regarded as an element of $V \otimes V$. Since $\mathcal{R}\mathcal{R}_{21} = 1$, we have $r + r^{21} = 0$ (where $r^{21} = -r^{\mathrm{op}}$ is the opposite of r in the super-sense), so $r \in S^2V$. □

REMARK 5.3.3. The classification of pointed finite-dimensional Hopf algebras with co-radical of dimension 2 is known [CD; N]. In the appendix we use the lifting method [AS1; AS2] to give an alternative proof. We shall need the following more precise version of this result in the triangular case.

LEMMA 5.3.4. *Let H be a minimal triangular pointed Hopf algebra whose co-radical is $\mathbf{C}[\mathbb{Z}_2] = \mathrm{sp}\{1, u\}$, where u is the Drinfeld element of H. Then $H = \overline{(\Lambda V)^{\mathcal{J}}}$ with the triangular structure of Corollary 3.3.2, where $\mathcal{J} = e^{r/2}$ for $r \in S^2V$ a nondegenerate element. In particular, H is a twist of a modified supergroup algebra.*

Proof. Let $\mathcal{H}$ be the associated triangular Hopf superalgebra to H as described in Theorem 3.3.1. Then the R-matrix of $\mathcal{H}$ is unipotent because it turns into $1 \otimes 1$ after killing the radical.

Let $\mathcal{H}_m$ be the minimal part of $\mathcal{H}$. By Lemma 5.3.2, $\mathcal{H}_m = \Lambda V$ and $\mathcal{R} = e^r$, $r \in S^2V$. Hence, if $\mathcal{J} := e^{r/2}$ then $\mathcal{H}^{\mathcal{J}^{-1}}$ has R-matrix equal to $1 \otimes 1$. Thus, $\mathcal{H}^{\mathcal{J}^{-1}}$ is co-commutative and so, by Corollary 2.3.5, it equals $\mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$. Therefore, $\mathcal{H} = \mathbf{C}[\mathbb{Z}_2] \ltimes (\Lambda V)^{\mathcal{J}}$, and the result follows from Proposition 3.2.1. □

We shall need the following lemma.

LEMMA 5.3.5. *Let $B \subseteq A$ be finite-dimensional associative unital algebras. Then any simple B-module is a constituent* (*in the Jordan–Holder series*) *of some simple A-module.*

Proof. Since A when considered as a B-module contains B as a B-module, it follows that any simple B-module is a constituent of A.

Decompose A (in the Grothendieck group of A) into simple A-modules: $A = \sum V_i$. Further decomposing as B-modules, we obtain $V_i = \sum W_{ij}$ and hence $A = \sum_i \sum_j W_{ij}$. Now, by the Jordan–Holder theorem, since A (as a B-module) contains all simple B-modules, any simple B-module X is in $\{W_{ij}\}$. Thus, X is a constituent of some V_i, as desired. □

PROPOSITION 5.3.6. *Any minimal triangular Hopf algebra H with the Chevalley property is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2.*

Proof. By Proposition 4.2, the co-radical H_0 of H is a Hopf subalgebra, since $H \simeq H^{*\mathrm{cop}}$ is minimal triangular. Consider the Hopf algebra map $\varphi \colon H_0 \to H^{*\mathrm{cop}}/\operatorname{Rad}(H^{*\mathrm{cop}})$ given by the composition of the maps

$$H_0 \hookrightarrow H \simeq H^{*\mathrm{cop}} \to H^{*\mathrm{cop}}/\operatorname{Rad}(H^{*\mathrm{cop}}),$$

where the second map is given by the R-matrix. We claim that ϕ is an isomorphism. Indeed, H_0 and $H^{*\mathrm{cop}}/\operatorname{Rad}(H^{*\mathrm{cop}})$ have the same dimension, since $\operatorname{Rad}(H^{*\mathrm{cop}}) = (H_0)^{\perp}$, and ϕ is injective, since H_0 is semisimple by [LR]. Let $\pi : H \to H_0$ be the associated projection.

We see, arguing exactly as in [G, Thm. 3.1], that H_0 is also minimal triangular, say with R-matrix R_0. Now, by [EG1, Thm. 2.1], we can find a twist J in $H_0 \otimes H_0$ such that $(H_0)^J$ is isomorphic to a group algebra and has R-matrix $(R_0)^J$ of rank ≤ 2. Notice that here we are relying on Deligne's theorem, as mentioned in the introduction.

Let us now consider J as an element of $H_0 \otimes H_0$ and the twisted Hopf algebra H^J, which is again triangular. The projection $\pi : H^J \to (H_0)^J$ is still a Hopf algebra map and sends R^J to $(R_0)^J$. It induces a projection $(H^J)_m \to \mathbf{C}[\mathbb{Z}_2]$ whose kernel K_m is contained in the kernel of π. Because any simple $(H^J)_m$-module is contained as a constituent in a simple H-module (see Lemma 5.3.5), $K_m = \operatorname{Rad}((H^J)_m)$. Hence, $(H^J)_m$ is minimal triangular and $(H^J)_m/\operatorname{Rad}((H^J)_m) = (\mathbf{C}[\mathbb{Z}_2], R_u)$. It follows, again by minimality, that $(H^J)_m$ is also pointed with co-radical isomorphic to $\mathbf{C}[\mathbb{Z}_2]$. Therefore, by Lemma 5.3.4, $(H^J)_m$ (and hence H^J) can be further twisted into a triangular Hopf algebra with R-matrix of rank ≤ 2, as desired. □

Now we can prove the main theorem.

Proof of Theorem 5.1.1. (2. ⇒ 1.) By Proposition 4.2, $H/\operatorname{Rad}(H)$ is a semisimple Hopf algebra. Let H_m be the minimal part of H, and let H'_m be the image of H_m in $H/\operatorname{Rad}(H)$. Then H'_m is a semisimple Hopf algebra.

Consider the kernel K of the projection $H_m \to H'_m$. Then $K = \operatorname{Rad}(H) \cap H_m$. This means that any element $k \in K$ is zero in any simple H-module. This implies that k acts by zero in any simple H_m-module, since by Lemma 5.3.5 we have that any simple H_m-module occurs as a constituent of some simple H-module. Thus, K is contained in $\operatorname{Rad}(H_m)$. On the other hand, H_m/K is semisimple, so $K = \operatorname{Rad}(H_m)$. This shows that $\operatorname{Rad}(H_m)$ is a Hopf ideal. Therefore, H_m is minimal triangular satisfying the conditions of Proposition 5.3.6. By Proposition 5.3.6, H_m is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2. Hence H is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2 (by the same twist), as desired.

(1. $\Rightarrow$ 2.) By assumption, $\mathrm{Rep}(H)$ is equivalent to $\mathrm{Rep}(\tilde{G})$ for some supergroup $\tilde{G}$ (as a tensor category without braiding). But we know that supergroup algebras have the Chevalley property because, modulo their radicals, they are group algebras. This concludes the proof of the main theorem. □

REMARK 5.3.7. Notice that it follows from the proof of the main theorem that any triangular Hopf algebra with the Chevalley property can be obtained by twisting a triangular Hopf algebra with R-matrix of rank ≤ 2 by an *even* twist.

DEFINITION 5.3.8. If a triangular Hopf algebra H over $\mathbf{C}$ satisfies condition 1 or 2 of Theorem 5.1.1, then we will say that H is *of supergroup type.*

5.4. Corollaries of the Main Theorem

COROLLARY 5.4.1. *A finite-dimensional triangular Hopf algebra H is of supergroup type if and only if its minimal part H_m is also.*

Proof. If H is of supergroup type then $\mathrm{Rad}(H)$ is a Hopf ideal. Thus, as in the (2. $\Rightarrow$ 1.) proof of Theorem 5.1.1, we conclude that $\mathrm{Rad}(H_m)$ is a Hopf ideal, that is, H_m is of supergroup type.

Conversely, if H_m is of supergroup type then H_m is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2. Hence H is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2 (by the same twist), so H is of supergroup type. □

COROLLARY 5.4.2. *A finite-dimensional triangular Hopf algebra whose co-radical is a Hopf subalgebra is of supergroup type. In particular, this is the case for finite-dimensional triangular pointed Hopf algebras.*

Proof. This follows from Corollary 5.4.1. □

COROLLARY 5.4.3. *Any finite-dimensional triangular basic Hopf algebra is of supergroup type.*

Proof. A basic Hopf algebra automatically has the Chevalley property, since all its irreducible modules are 1-dimensional. Hence, the result follows from the main theorem. □

5.5. Questions

The previous results motivate the following question.

QUESTION 5.5.1. Does any finite-dimensional triangular Hopf algebra over $\mathbf{C}$ have the Chevalley property (i.e., is any such algebra of supergroup type)? Is it true under the assumption that $S^4 = \mathrm{Id}$ or at least under the assumption that $u^2 = 1$?

Remark 5.5.2. Recall from [G] that it is not known whether any finite-dimensional triangular Hopf algebra over **C** has the property $u^2 = 1$ or at least $S^4 = \mathrm{Id}$. It is also not known if $S^4 = \mathrm{Id}$ implies $u^2 = 1$ for triangular Hopf algebras. However, it is clear that, for finite-dimensional triangular Hopf algebras H of supergroup type, $u^2 = 1$ (and hence $S^4 = \mathrm{Id}$). Indeed, since $S^2 = \mathrm{Id}$ on the semisimple part of H, it follows that u acts by a scalar in any irreducible representation of H. In fact, since $\mathrm{tr}(u) = \mathrm{tr}(u^{-1})$, we have that $u = 1$ or $u = -1$ on any irreducible representation of H, and hence $u^2 = 1$ on any irreducible representation of H. Thus, u^2 is unipotent. But it is of finite order (since it is a grouplike element), so it is equal to 1 as desired.

Remark 5.5.3. Note that the answer to Question 5.5.1 is negative in the infinite-dimensional case. Namely, although the answer is positive in the co-commutative case (by [C]), it is negative already for triangular Hopf algebras with R-matrix of rank 2, which correspond to co-commutative Hopf superalgebras. Indeed, let us take the co-commutative Hopf superalgebra $\mathcal{H} := \mathrm{U}(\mathrm{gl}(n|n))$ (for the definition of the Lie superalgebra $\mathrm{gl}(n|n)$, see [K, p. 29]). The associated triangular Hopf algebra $\bar{\mathcal{H}}$ does not have the Chevalley property, since it is well known that the Chevalley theorem fails for Lie superalgebras (e.g., for $\mathrm{gl}(n|n)$); more precisely, already the product of the vector and co-vector representations for this Lie superalgebra is not semisimple.

Remark 5.5.4. It follows from Corollary 5.4.1 that a positive answer to Question 5.5.1 in the minimal case would imply the general positive answer.

Here is a generalization of Question 5.5.1.

Question 5.5.5. Does any **C**-linear abelian symmetric rigid tensor category, with $\mathrm{End}(\mathbf{1}) = \mathbf{C}$ and finitely many simple objects, have the Chevalley property?

Here is an even more ambitious question.

Question 5.5.6. Is such a category equivalent to the category of representations of a finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2? In particular, is it equivalent to the category of representations of a supergroup as a category without braiding? Are these statements valid at least for categories with Chevalley property? For semisimple categories?

6. Finite-Dimensional Co-triangular Pointed Hopf Algebras Are Generated by Grouplikes and Skewprimitives

There is a conjecture (see [AS2]) that any finite-dimensional pointed Hopf algebra over **C** is generated by grouplike and skewprimitive elements. Here we confirm it in the co-triangular case.

Theorem 6.1. *A finite-dimensional co-triangular pointed Hopf algebra H over* **C** *is generated by grouplike and skewprimitive elements.*

In order to prove the theorem, we will need the following lemma.

LEMMA 6.2. *Let H be a finite-dimensional pointed Hopf algebra or superalgebra. Then the following are equivalent.*

1. *H is generated by grouplike and skewprimitive elements.*
2. *There exists a faithful H^*-module that is a direct sum of tensor products of H^*-modules of dimension* 2.

Proof. Irreducible H^*-modules are 1-dimensional, so a 2-dimensional representation has the form

$$a \mapsto \begin{pmatrix} p(a) & r(a) \\ 0 & q(a) \end{pmatrix}, \quad a \in H^*,$$

where p, q are characters (i.e., they belong to $\mathbf{G}(H)$) and r is a $q : p$ skewprimitive. Conversely, such a 2-dimensional representation exists for any skewprimitive element. Matrix elements of tensor products of representations of H^* are products of the matrix elements of these representations (as elements of H). This implies the lemma. □

Now we are ready to give the following.

Proof of Theorem 6.1. We need to show that H^* has a faithful representation that is a direct sum of products of 2-dimensional representations. By [G], the Drinfeld element u of H^* satisfies $u^2 = 1$. Hence, by Theorem 3.1.1, we can replace H^* with the corresponding Hopf superalgebra $\widetilde{H^*}$ (since this does not change the representation category as a tensor category). But H^* is basic triangular, which means (by Corollary 5.4.3) that $\widetilde{H^*}$ is twist-equivalent to a supergroup algebra B. Thus, by Lemma 6.2, it suffices to show that B^* (the dual of B) is generated by grouplikes and skewprimitives.

But $B = \mathbf{C}[G] \ltimes \Lambda V$, where G is abelian. Thus, V is decomposed in the direct sum of eigenspaces for G. Let $v_1, \ldots, v_n$ be a basis of V such that $gv_i g^{-1} = \chi_i(g)v_i$, where the χ_i are some characters of G. Using this presentation of B, it is easy to compute its dual B^* and show that it is generated as an algebra by $G^\vee$ (the character group) and $\chi_i : 1$ skew primitive elements ξ_i, $i = 1, \ldots, n$. We are done. □

COROLLARY 6.3. *Theorem 5.1 of* [G] *gives the classification of all minimal triangular pointed Hopf algebras.*

Proof. Since minimal triangular pointed Hopf algebras are also co-triangular, by Theorem 6.1 they are generated by grouplikes and skewprimitives (which answers a question from [G]). On the other hand, [G, Thm. 5.1] gives a classification of minimal triangular Hopf algebras that are generated by grouplikes and skewprimitives. □

REMARK 6.4. Lemma 6.2 implies that, if H_1, H_2 are finite-dimensional pointed Hopf algebras and if the co-multiplication of H_1^* is obtained by conjugating that of H_2^* by an invertible element (not necessarily a twist), then H_1 is generated by grouplike and skewprimitive elements if and only if H_2 is.

7. Categorical Dimensions in Symmetric Categories with Finitely Many Irreducibles Are Integers

In this paper we classified finite-dimensional triangular Hopf algebras with the Chevalley property. In conclusion, let us give one result that is valid in the greater generality of any finite-dimensional triangular Hopf algebra—and even for any symmetric rigid category with finitely many irreducible objects.

Let $\mathcal{C}$ be a $\mathbf{C}$-linear abelian symmetric rigid category with $\mathbf{1}$ as its unit object, and suppose that $\mathrm{End}(\mathbf{1}) = \mathbf{C}$. Recall that there is a natural notion of dimension in $\mathcal{C}$ that generalizes the ordinary dimension of an object in Vect. Let β denote the commutativity constraint in $\mathcal{C}$ and, for an object V, let ev_V and coev_V denote the associated evaluation and coevaluation morphisms, respectively.

Definition 7.1 [DM]. The categorical dimension $\dim_c(V) \in \mathbf{C}$ of $V \in \mathrm{Ob}(\mathcal{C})$ is the morphism

$$\dim_c(V)\colon \mathbf{1} \xrightarrow{\mathrm{ev}_V} V \otimes V^* \xrightarrow{\beta_{V,V^*}} V^* \otimes V \xrightarrow{\mathrm{coev}_V} \mathbf{1}. \tag{10}$$

The main result of this section is the following.

Theorem 7.2. *In any $\mathbf{C}$-linear abelian symmetric rigid tensor category $\mathcal{C}$ with finitely many irreducible objects, the categorical dimensions of objects are integers.*

Proof. First note that the categorical dimension of any object V of $\mathcal{C}$ is an algebraic integer. Indeed, let $V_1, \dots, V_n$ be the irreducible objects of $\mathcal{C}$. Then $\{V_1, \dots, V_n\}$ is a basis of the Grothendieck ring of $\mathcal{C}$. Write $V \otimes V_i = \sum_j N_{ij}(V)V_j$ in the Grothendieck ring. Then $N_{ij}(V)$ is a matrix with integer entries, and $\dim_c(V)$ is an eigenvalue of this matrix. Thus, $\dim_c(V)$ is an algebraic integer.

Now, if $\dim_c(V) = d$ then it is easy to show (see e.g. [D1]) that

$$\dim_c(S^k V) = d(d+1)\cdots(d+k-1)/k!$$

and

$$\dim_c(\Lambda^k V) = d(d-1)\cdots(d-k+1)/k!;$$

hence these dimensions are also algebraic integers. Therefore, the theorem follows from our next lemma.

Lemma. *Suppose d is an algebraic integer such that $d(d+1)\cdots(d+k-1)/k!$ and $d(d-1)\cdots(d-k+1)/k!$ are algebraic integers for all k. Then d is an integer.*

Proof. Let Q be the minimal monic polynomial of d over $\mathbb{Z}$. Since $d(d-1)\cdots(d-k+1)/k!$ is an algebraic integer, so are $d'(d'-1)\cdots(d'-k+1)/k!$, where d' is any algebraic conjugate of d. Taking the product over all conjugates, we get that

$$N(d)N(d-1)\cdots N(d-k+1)/(k!)^n$$

is an integer, where n is the degree of Q. But $N(d-x) = (-1)^n Q(x)$. Hence we have that $Q(0)Q(1)\cdots Q(k-1)/(k!)^n$ is an integer. Similarly, from the identity for $S^k V$ it follows that $Q(0)Q(-1)\cdots Q(1-k)/(k!)^n$ is an integer. Now, without loss of generality, we can assume that $Q(x) = x^n + ax^{n-1} + \cdots$, where $a \leq 0$ (otherwise, replace $Q(x)$ by $Q(-x)$; we can do it because our condition is symmetric under this change). Then for large k we have $Q(k-1) < k^n$, so the sequence $b_k := Q(0)Q(1)\cdots Q(k-1)/k!^n$ is decreasing in absolute value or zero starting from some place. But a sequence of integers cannot be strictly decreasing in absolute value forever. Hence $b_k = 0$ for some k and so Q has an integer root. This means that d is an integer (i.e., Q is linear), since Q must be irreducible over the rationals. This concludes the proof of the lemma and hence of the theorem. □

COROLLARY 7.3. *For any triangular Hopf algebra H (not necessarily finite-dimensional), the categorical dimensions of its finite-dimensional representations are integers.*

Proof. It is enough to consider the minimal part H_m of H that is finite-dimensional, since $\dim_c(V) = \mathrm{tr}(u|_V)$ for any module V (where u is the Drinfeld element of H) and $u \in H_m$. Hence the result follows from Theorem 7.2. □

REMARK 7.4. Theorem 7.2 is false without the finiteness conditions. In fact, in this case any complex number can be a dimension, as is demonstrated in examples constructed by Deligne [D2, pp. 324–325]. Also, it is well known that the theorem is false for ribbon, nonsymmetric categories (e.g., for fusion categories of semisimple representations of finite-dimensional quantum groups at roots of unity, where dimensions can be irrational algebraic integers).

REMARK 7.5. Note that Theorem 7.2 can be regarded as a piece of supporting evidence for a positive answer to Question 5.5.6.

REMARK 7.6. In any rigid braided tensor category with finitely many irreducible objects, we can define the Frobenius–Perron dimension of an object V, $\mathrm{FPdim}(V)$, to be the largest positive eigenvalue of the matrix of multiplication by V in the Grothendieck ring. This dimension is well-defined by the Frobenius–Perron theorem and has the usual additivity and multiplicativity properties. For example, for the category of representations of a quasi-Hopf algebra, it is just the usual dimension of the underlying vector space. If the answer to Question 5.5.6 is positive then $\mathrm{FPdim}(V)$ for symmetric categories is always an integer that is equal to $\dim_c(V)$ modulo 2. It would be interesting to check this, at least in the case of modules over triangular Hopf algebras, when the integrality of FPdim is automatic (so only the mod 2 congruence has to be checked).

8. Appendix: On Pointed Hopf Algebras

In this appendix we use the lifting method [AS1; AS2] to give alternate proofs of Theorem 5.2.1 and Corollary 6.3 and a generalization of Lemma 5.3.4.

By *a braided Hopf algebra* we shall mean a Hopf algebra in the braided tensor category of Yetter–Drinfeld modules over a group algebra $\mathbf{C}[\Gamma]$, where Γ is a finite abelian group. For example, one can endow the exterior algebra ΛV with the structure of a braided Hopf algebra as follows. Let $x_1, \dots, x_N$ be a basis of V and let there be given $g_1, \dots, g_N \in \Gamma$ and $\chi_1, \dots, \chi_N \in \Gamma^\vee$ such that

$$\chi_i(g_j) = -1, \quad 1 \le i, j \le N.$$

Then V is a Yetter–Drinfeld module over $\mathbf{C}[\Gamma]$, where the action and co-action of Γ on x_i are given by χ_i and g_i (respectively). This action and co-action extend to ΛV and turn ΛV into a braided Hopf algebra.

LEMMA 8.1. *Let $R = \bigoplus_{n \ge 0} R(n)$ be a graded braided Hopf algebra such that $R(0) = \mathbf{C}$, $R(1) \simeq V$ as a Yetter–Drinfeld module (with the preceding assumptions), and R is generated by $R(1)$. Then R is isomorphic to ΛV as a graded braided Hopf algebra.*

Proof. It is known and easy to see that ΛV satisfies all the hypotheses that R does, plus that the primitive elements are concentrated in degree 1: $\mathbf{P}(\Lambda V) = \Lambda V(1) = V$ (see e.g. [AS1, Sec. 3]). In other words, ΛV is the Nichols algebra of V, and there exists a surjective homomorphism of graded braided Hopf algebras $R \to \Lambda V$ that is the identity in degree 1 (see e.g. [AS2, Lemma 5.5]). On the other hand, it is clear that ΛV can be presented by generators $x_1, \dots, x_N$ with relations

$$x_i x_j + x_j x_i = 0, \quad 1 \le i, j \le N. \tag{11}$$

So, in particular, $x_i^2 = 0$ for all i. It is thus enough to show that equations (11) also hold in R, with an evident abuse of notation. But $x_i x_j + x_j x_i$ is a primitive element of R, whose action is given by the character $\chi_i \chi_j$ and whose co-action is given by $g_i g_j$. Since $\chi_i \chi_j(g_i g_j) = 1$ and R is finite-dimensional, $x_i x_j + x_j x_i = 0$ in R by [AS1, Lemma 3.1]. □

Let H be a finite-dimensional pointed Hopf algebra such that $\mathbf{G}(H)$ is isomorphic to Γ. We recall that the lifting method [AS1; AS2] attaches several invariants to H:

(a) the graded Hopf algebra gr H associated to the co-radical filtration of H;
(b) a graded braided Hopf algebra R, the co-invariants of the homogeneous projection from gr H to $\mathbf{C}[\Gamma]$;
(c) a Yetter–Drinfeld module $W := R(1)$ over $\mathbf{C}[\Gamma]$, called the infinitesimal braided vector space of H.

We may conclude immediately from Lemma 8.1 as follows.

COROLLARY 8.2. *Let H be a finite-dimensional pointed Hopf algebra such that $\mathbf{G}(H)$ is isomorphic to Γ. Assume that the infinitesimal braiding of H is isomorphic to V as before. Then H is generated by grouplike and skewprimitive elements.*

REMARK 8.3. Notice that Corollary 8.2 allows one to give an alternative proof of Corollary 6.3. This is because Lemmas 5.3 and 5.4 in [G] imply that the infinitesimal braiding of any minimal triangular pointed Hopf algebra is isomorphic to a V as described previously.

Assume now that $\Gamma = \mathbb{Z}_2$. There is only one possible choice for V as before—namely, $g_1 = \cdots = g_N = u$ and $\chi_1 = \cdots = \chi_N =$ the sign. This gives the Hopf superalgebra as explained in Section 5. Let now H be a finite-dimensional pointed Hopf algebra such that $\mathbf{G}(H)$ is isomorphic to $\mathbb{Z}_2$. Then, for some natural number N, the infinitesimal braiding of H is isomorphic to V as before by [AS1, Lemma 3.1] again. The lifting method gives a very direct proof of the following well-known result.

THEOREM 8.4 [N, Thm. 4.2.1; CD]. *If H is a finite-dimensional pointed Hopf algebra with $\mathbf{G}(H)$ isomorphic to $\mathbb{Z}_2$, then $H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$.*

Proof. By the foregoing remarks and Corollary 8.2, we know that gr $H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$ for some V. The fact that $H \simeq$ gr H ("there are no liftings", in the jargon of the lifting method) is a particular case of the main result [AS1, Thm. 5.5]. □

We can now give another proof of Theorem 5.2.1.

It is enough to show that $\mathcal{H}^* = \Lambda V$ for some V as before, since $(\Lambda V)^* = \Lambda V^*$ as Hopf superalgebras. By the hypothesis, the co-radical of $\mathcal{H}^*$ is trivial: Corad$(\mathcal{H}^*) = \mathbf{C}1$. We can consider the bi-product $H := \mathbf{C}[\mathbb{Z}_2] \ltimes \mathcal{H}^*$; that is, $H = \bar{\mathcal{H}}$ in our notation. We claim that H is a finite-dimensional pointed Hopf algebra with $\mathbf{G}(H)$ isomorphic to $\mathbb{Z}_2$. Indeed, the filtration

$$\mathbf{C}[\mathbb{Z}_2] \subset \mathbf{C}[\mathbb{Z}_2] \ltimes (\mathcal{H}^*)_1 \subset \cdots \subset \mathbf{C}[\mathbb{Z}_2] \ltimes (\mathcal{H}^*)_j \subset \cdots$$

is a co-algebra filtration, where $1 \subset (\mathcal{H}^*)_1 \subset \cdots \subset (\mathcal{H}^*)_j \subset \cdots$ is the co-radical filtration of $\mathcal{H}^*$. Hence $\mathbf{C}[\mathbb{Z}_2]$ contains the co-radical of H, and the other inclusion is evident.

It follows then from Theorem 8.4 that $H \simeq \mathbf{C}[\mathbb{Z}_2] \ltimes \Lambda V$. By [AS2, Lemma 6.2], $\mathcal{H}^* \simeq \Lambda V$ as braided Hopf algebras—that is, as Hopf superalgebras. □

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