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How Far Is an Ultraflat Sequence of Unimodular Polynomials from Being Conjugate-Reciprocal?

Tamás Erdélyi

1. Introduction

Let D be the open unit disk of the complex plane. Its boundary, the unit circle of the complex plane, is denoted by ∂D. Let

$$\mathcal{K}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \mathbb{C},\ |a_k| = 1 \right\}.$$

The class $\mathcal{K}_n$ is often called the collection of all *complex* unimodular polynomials of degree n. Let

$$\mathcal{L}_n := \left\{ p_n : p_n(z) = \sum_{k=0}^{n} a_k z^k,\ a_k \in \{-1, 1\} \right\}.$$

The class $\mathcal{L}_n$ is often called the collection of all *real* unimodular polynomials of degree n. By Parseval's formula,

$$\int_0^{2\pi} |P_n(e^{it})|^2\, dt = 2\pi(n+1)$$

for all $P_n \in \mathcal{K}_n$. Therefore

$$\min_{z \in \partial D} |P_n(z)| \le \sqrt{n+1} \le \max_{z \in \partial D} |P_n(z)|. \tag{1.1}$$

An old problem (or rather an old theme) is the following.

Problem 1.1 (Littlewood's flatness problem). How close can a unimodular polynomial $P_n \in \mathcal{K}_n$ or $P_n \in \mathcal{L}_n$ come to satisfying

$$|P_n(z)| = \sqrt{n+1}, \quad z \in \partial D\,? \tag{1.2}$$

Obviously (1.2) is impossible if $n \ge 1$. So one must look for less than (1.2), but then there are various ways of seeking such an "approximate situation". One way is the following. Littlewood [Li1] suggested that there might conceivably exist a sequence (P_n) of polynomials $P_n \in \mathcal{K}_n$ (possibly even $P_n \in \mathcal{L}_n$) such that

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$(n+1)^{-1/2}|P_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbb{R}$. We shall call such sequences of unimodular polynomials "ultraflat". More precisely, we give the following definitions.

DEFINITION 1.2. Given a positive number ε, we say that a polynomial $P_n \in \mathcal{K}_n$ is *ε-flat* if

$$(1-\varepsilon)\sqrt{n+1} \le |P_n(z)| \le (1+\varepsilon)\sqrt{n+1}, \quad z \in \partial D, \tag{1.3}$$

or equivalently

$$\max_{z\in\partial D}\big||P_n(z)| - \sqrt{n+1}\big| \le \varepsilon\sqrt{n+1}.$$

DEFINITION 1.3. Given a sequence (ε_{n_k}) of positive numbers tending to 0, we say that a sequence (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$ is *(ε_{n_k})-ultraflat* if

$$(1-\varepsilon_{n_k})\sqrt{n_k+1} \le |P_{n_k}(z)| \le (1+\varepsilon_{n_k})\sqrt{n_k+1}, \quad z \in \partial D, \tag{1.4}$$

or equivalently

$$\max_{z\in\partial D}\big||P_{n_k}(z)| - \sqrt{n_k+1}\big| \le \varepsilon_{n_k}\sqrt{n_k+1}.$$

The existence of an ultraflat sequence of unimodular polynomials seemed very unlikely in view of a 1957 conjecture of P. Erdős (Problem 22 in [Er]) asserting that, for all $P_n \in \mathcal{K}_n$ with $n \ge 1$,

$$\max_{z\in\partial D}|P_n(z)| \ge (1+\varepsilon)\sqrt{n+1}, \tag{1.5}$$

where $\varepsilon > 0$ is an absolute constant (independent of n). Yet, by refining a method of Körner [Kö], Kahane [Ka] proved that there exists a sequence (P_n) with $P_n \in \mathcal{K}_n$ that is (ε_n)-ultraflat, where

$$\varepsilon_n = O\big(n^{-1/17}\sqrt{\log n}\big). \tag{1.6}$$

Thus the Erdős conjecture (1.5) was disproved for the classes $\mathcal{K}_n$. For the more restricted class $\mathcal{L}_n$, the analogous Erdős conjecture is unsettled to this date. It is a common belief that the analogous Erdős conjecture for $\mathcal{L}_n$ is true and that consequently there is no ultraflat sequence of polynomials $P_n \in \mathcal{L}_n$.

An extension of Kahane's breakthrough is given in [Be]. For an account of some of the work done until the mid-1960s, see Littlewood's book [Li2] and [QS].

2. New Results

In this paper we study ultraflat sequences (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ in general, not necessarily those produced by Kahane in his paper [Ka]. With trivial modifications our results remain valid even if we study ultraflat sequences (P_{n_k}) of unimodular polynomials $P_{n_k} \in \mathcal{K}_{n_k}$. It is left to the reader to formulate these analogous results. We examine how far an ultraflat sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is from being conjugate reciprocal. Our main results are formulated by the following theorems. In each of Theorems 2.1–2.3 we assume that (ε_n) is a sequence of positive numbers tending to 0 and that the sequence (P_n) of unimodular polynomials $P_n \in \mathcal{K}_n$ is (ε_n)-ultraflat.

If Q_n is a polynomial of degree n of the form

$$Q_n(z) = \sum_{k=0}^{n} a_k z^k, \quad a_k \in \mathbb{C},$$

then its conjugate polynomial is defined by

$$Q_n^*(z) := z^n \bar{Q}_n\left(\frac{1}{z}\right) := \sum_{k=0}^{n} \bar{a}_{n-k} z^k.$$

THEOREM 2.1. *We have*

$$\int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2 \, |dz| = 2\pi\left(\tfrac{1}{3} + \gamma_n\right)n^3,$$

where (γ_n) is a sequence of real numbers converging to 0.

THEOREM 2.2. *If the coefficients of P_n are denoted by $a_{k,n}$, that is, if*

$$P_n(z) = \sum_{k=0}^{n} a_{k,n} z^k, \quad k = 0, 1, \ldots, n, \;\; n = 1, 2, \ldots,$$

then

$$\sum_{k=0}^{n} k^2 |a_{k,n} - \bar{a}_{n-k,n}|^2 \geq \left(\tfrac{1}{3} + \delta_n\right)n^3,$$

where (δ_n) is a sequence of real numbers converging to 0.

THEOREM 2.3. *We have*

$$\int_{\partial D} |P_n(z) - P_n^*(z)|^2 \, |dz| \geq 2\pi\left(\tfrac{1}{3} + \gamma_n\right)n,$$

where (γ_n) is a sequence of real numbers converging to 0. *Using the notation of Theorem 2.2, in terms of the coefficients of P_n we have*

$$\sum_{k=0}^{n} |a_{k,n} - \bar{a}_{n-k,n}|^2 \geq \left(\tfrac{1}{3} + \delta_n\right)n,$$

where (δ_n) is a sequence of real numbers converging to 0.

REMARK 2.4. Theorem 2.3 tells us much more than the nonexistence of an ultraflat sequence of conjugate reciprocal unimodular polynomials. It measures how far such an ultraflat sequence is from being a sequence of conjugate reciprocal polynomials.

3. Lemmas

To prove the theorems in Section 2, we need two lemmas. The first one can be checked by a simple calculation.

LEMMA 3.1. *Let P_n be an arbitrary polynomial of degree n with complex coefficients having no zeros on the unit circle. Let*

$$f_n(z) := \frac{zP_n'(z)}{P_n(z)} \quad \textit{and} \quad f_n^*(z) := \frac{zP_n^{*\prime}(z)}{P_n^*(z)}.$$

Then

$$\overline{f_n(z)} + f_n^*(z) = n, \quad z \in \partial D.$$

Our next lemma may be found in [MMR, p. 676] and is due to Malik.

LEMMA 3.2. *Let P_n be an arbitrary polynomial of degree n with complex coefficients. We have*

$$\max_{z\in\partial D}(|P_n'(z)| + |P_n^{*\prime}(z)|) \le n \max_{z\in\partial D}|P_n(z)|.$$

LEMMA 3.3 (Bernstein's inequality in $L_2(\partial D)$). *If Q_n is a polynomial of degree at most n with complex coefficients, then*

$$\int_{\partial D} |Q_n'(z)|^2\, |dz| \le n^2 \int_{\partial D} |Q_n(z)|^2\, |dz|.$$

4. Proof of the Theorems

Proof of Theorem 2.1. Lemma 3.2 when combined with the ultraflatness of (P_n) implies that

$$|P_n'(z)| + |P_n^{*\prime}(z)| \le n \max_{z\in\partial D}|P_n(z)| \le (1+\varepsilon_n)(n+1)^{3/2}$$

for every $z \in \partial D$. Lemma 3.1 when combined with the ultraflatness of P_n implies

$$|P_n'(z)|\frac{1}{(1-\varepsilon_n)\sqrt{n+1}} + |P_n^{*\prime}(z)|\frac{1}{(1-\varepsilon_n)\sqrt{n+1}} \ge \frac{|P_n'(z)|}{|P_n(z)|} + \frac{|P_n^{*\prime}(z)|}{|P_n^*(z)|} \ge n,$$

that is,

$$|P_n'(z)| + |P_n^{*\prime}(z)| \ge (1-\varepsilon_n)n^{3/2}$$

for every $z \in \partial D$. We conclude that

$$(1-\varepsilon_n)^2 n^3 \le (|P_n'(z)| + |P_n^{*\prime}(z)|)^2 \le (1+\varepsilon_n)^2(n+1)^3, \quad z \in \partial D.$$

Multiplying out the expression in the middle and integrating on ∂D with respect to $|dz|$, we obtain

$$\begin{aligned} 2\pi(1-\varepsilon_n)^2 n^3 &\le \int_{\partial D} |P_n'(z)|^2\, |dz| + \int_{\partial D} |P_n^{*\prime}(z)|^2\, |dz| + 2\int_{\partial D} |P_n'(z)P_n^{*\prime}(z)|\, |dz| \\ &\le 2\pi(1+\varepsilon_n)^2 n^3. \end{aligned}$$

Note that

$$\begin{aligned} \int_{\partial D} |P_n'(z)|^2\, |dz| = \int_{\partial D} |P_n^{*\prime}(z)|^2\, |dz| &= 2\pi \sum_{k=1}^{n} k^2 \\ &= 2\pi\frac{n(n+1)(2n+1)}{6} \sim \frac{2\pi}{3}n^3. \end{aligned} \tag{2.1}$$

Hence

$$\int_{\partial D} |P_n'(z)||P_n^{*\prime}(z)|\,|dz| = 2\pi\left(\tfrac{1}{6}+\delta_n\right)n^3$$

with constants δ_n converging to 0. Integrating the equation

$$(|P_n'(z)| - |P_n^{*\prime}(z)|)^2 = |P_n'(z)|^2 + |P_n^{*\prime}(z)|^2 - 2|P_n'(z)P_n^{*\prime}(z)|$$

and using observation (2.1), we obtain the theorem. □

Proof of Theorem 2.2. Parseval's formula and the triangle inequality give

$$\begin{aligned} 2\pi\sum_{k=0}^{n} k^2|a_{k,n} - \bar{a}_{n-k,n}|^2 &= \int_{\partial D} |P_n'(z) - P_n^{*\prime}(z)|^2\,|dz| \\ &\geq \int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2\,|dz|, \end{aligned}$$

and the theorem then follows from Theorem 2.1. □

Proof of Theorem 2.3. Applying Theorem 2.1, the triangle inequality, and the Bernstein inequality in L_2 for $P_n - P_n^*$ (see Lemma 3.3), we obtain

$$\begin{aligned} 2\pi\left(\frac{1}{3}+\gamma_n\right)n^3 &= \int_{\partial D} (|P_n'(z)| - |P_n^{*\prime}(z)|)^2\,|dz| \leq \int_{\partial D} |P_n'(z) - P_n^{*\prime}(z)|^2\,|dz| \\ &\leq n^2 \int_{\partial D} |P_n(z) - P_n^*(z)|^2\,|dz|, \end{aligned}$$

where (γ_n) is a sequence of real numbers converging to 0. Now the first part of the theorem follows after dividing by n^2. To see the second part, we proceed as in the proof of Theorem 2.2 by using Parseval's formula. □

Last-Minute Addition

The author seems to be able to prove the following.

THEOREM (Saffari's orthogonality conjecture). *Assume that (P_n) is an ultraflat sequence of unimodular polynomials $P_n \in \mathcal{K}_n$. Let*

$$P_n(z) := \sum_{k=0}^{n} a_{k,n} z^k.$$

Then

$$\sum_{k=0}^{n} a_{k,n} a_{n-k,n} = o(n).$$

Here, as usual, $o(n)$ denotes a quantity for which $\lim_{n\to\infty} o(n)/n = 0$.

The proof of this may appear in a later publication.

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References

[Be] J. Beck, *"Flat" polynomials on the unit circle—Note on a problem of Littlewood,* Bull. London Math. Soc. 23 (1991), 269–277.

[BE] P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities,* Springer-Verlag, New York, 1995.

[DL] R. A. DeVore and G. G. Lorentz, *Constructive approximation,* Springer-Verlag, Berlin, 1993.

[Er] P. Erdős, *Some unsolved problems,* Michigan Math. J. 4 (1957), 291–300.

[Ka] J. P. Kahane, *Sur les polynomes a coefficient unimodulaires,* Bull. London Math. Soc. 12 (1980), 321–342.

[Kö] T. Körner, *On a polynomial of Byrnes,* Bull. London Math. Soc. 12 (1980), 219–224.

[Li1] J. E. Littlewood, *On polynomials* $\sum \pm z^m$, $\sum \exp(\alpha_m i) z^m$, $z = e^{i\theta}$, J. London Math. Soc. (2) 41 (1966), 367–376.

[Li2] ———, *Some problems in real and complex analysis,* Heath & Co., Lexington, MA, 1968.

[MMR] G. V. Milovanović, D. S. Mitrinović, and Th. M. Rassias, *Topics in polynomials: Extremal problems, inequalities, zeros,* World Scientific, River's Edge, NJ, 1994.

[QS] H. Queffelec and B. Saffari, *On Bernstein's inequality and Kahane's ultraflat polynomials,* J. Fourier Anal. Appl. 2 (1996), 519–592.

[Sa] B. Saffari, *The phase behavior of ultraflat unimodular polynomials,* Probabilistic and stochastic methods in analysis, with applications (Il Ciocco, 1991), pp. 555–572, Kluwer, Dordrecht, 1992.

Department of Mathematics
Texas A&M University
College Station, TX 77843

terdelyi@math.tamu.edu