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Mappings of Finite Distortion: Condition N

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1. Introduction

Suppose that f is a continuous mapping from a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ into $\mathbb{R}^n$. We consider the following Lusin condition N: If $E \subset \Omega$ with $\mathcal{L}^n(E) = 0$, then $\mathcal{L}^n(f(E)) = 0$. Physically, this condition requires that there be no creation of matter under the deformation f of the n-dimensional body Ω. This is a natural requirement, since the N property with differentiability a.e. is sufficient for validity of various change-of-variable formulas, including the area formula, and the condition N holds for a homeomorphism f if and only if f maps measurable sets to measurable sets.

If the coordinate functions of f belong to the Sobolev class $W^{1,1}_{\text{loc}}(\Omega)$ and $|Df| \in L^p(\Omega)$ for some $p > n$, then f satisfies the Lusin condition N (Marcus and Mizel, [14]). Recently we verified in [10] that this also holds when $|Df|$ belongs to the Lorentz space $\mathrm{L}^{n,1}(\Omega)$ and that this analytic assumption is essentially sharp even if the determinant of Df is nonnegative a.e. For a homeomorphism, less regularity is needed: it suffices to assume that $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$; this is due to Reshetnyak [19]. On the other hand, there exists a homeomorphism that does not satisfy the condition N and so $|Df|$ belongs to $L^p(\Omega)$ for each $p < n$; see the examples by Ponomarev [17; 18]. Some further results on the Lusin condition are listed in the survey paper [13].

We will need the concept of topological degree. We say that a continuous mapping f is *sense-preserving* if the topological degree with respect to any subdomain $G \subset\subset \Omega$ is strictly positive: $\deg(f, G, y) > 0$ for all $y \in f(G) \setminus f(\partial G)$. In this paper we show that, for a sense-preserving mapping, the sharp regularity assumption in the rearrangement-invariant scale to rule out the failure of the condition N is that

$$\lim_{\varepsilon \to 0+} \varepsilon \int_{\Omega} |Df|^{n-\varepsilon} = 0. \tag{1.1}$$

Theorem A. *Suppose that $f : \Omega \to \mathbb{R}^n$ is sense-preserving and that* (1.1) *holds. Then f satisfies condition N. On the other hand, there is a homeomorphism f from the closed unit cube Q_0 onto Q_0 such that*

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$$\sup_{0<\varepsilon\leq n-1} \varepsilon \int_\Omega |Df|^{n-\varepsilon} < \infty, \tag{1.2}$$

f does not satisfy condition N, and f restricted to the boundary of the unit cube is the identity mapping.

Let us define $L^{n)}(\Omega)$ as the collection of all the measurable functions u with

$$\|u\|_{n)} = \sup_{0<\varepsilon\leq n-1} \left(\varepsilon \int_\Omega |u(x)|^{n-\varepsilon}\, dx \right)^{1/(n-\varepsilon)} < \infty.$$

Then $L^{n)}(\Omega)$ is a Banach space and

$$L_b^{n)}(\Omega) = \left\{ u \in L^{n)}(\Omega) : \lim_{\varepsilon\to 0^+} \varepsilon \int_\Omega |u|^{n-\varepsilon}\, dx = 0 \right\}$$

is a closed subspace. These function spaces were introduced by Iwaniec and Sbordone [9]. The motivation for the subindex b in the definition of the latter space comes from the fact that $L_b^{n)}(\Omega)$ is the closure of bounded functions in $L^{n)}(\Omega)$; see [5], where the notation is slightly different from ours. It is immediate that $L_b^{n)}(\Omega) \subset L^{n)}(\Omega) \subset \bigcap_{p<n} L^p(\Omega)$ and that each measurable u with

$$\int_\Omega \frac{|u|^n}{\log(e+|u|)}\, dx < \infty$$

belongs to $L_b^{n)}(\Omega)$.

There are recent results related to Theorem A. Müller and Spector [15] prove the condition N for a Sobolev mapping that satisfies an invertibility assumption under the conditions that (a) the Jacobian determinant is strictly positive a.e. and (b) either the image of the domain has finite perimeter or the weak Jacobian, defined as a distribution using integration by parts, is represented by an appropriate measure. In our situation the weak Jacobian of the mapping f coincides with the pointwise Jacobian by a result of Greco [5] and thus no additional assumptions are needed. Yet another result in the same direction can be found in the work of Šverák [20]. Here again it is assumed that the Jacobian of the mapping is strictly positive almost everywhere. Thus our results are not covered by these earlier works.

Let us now move on to mappings of finite distortion. We say that a Sobolev mapping $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ has *finite distortion* if there is a measurable function $K = K(x) \geq 1$, finite almost everywhere, such that

$$|Df(x)|^n \leq K(x) J_f(x) \text{ a.e.} \tag{1.3}$$

Here $J_f(x) = J(x, f) = \det Df(x)$ is the Jacobian determinant of f. We call (1.3) the *distortion inequality* for f. Notice that, unless we put extra conditions on K, we require only that $J_f(x) \geq 0$ a.e. and that the differential Df vanish a.e. in the zero set of the Jacobian determinant J_f. Gol'dstein and Vodopyanov [4] proved that Sobolev mappings of finite distortion with $|Df| \in L^n(\Omega)$ satisfy the Lusin condition N. We are interested here in mappings of finite distortion with lower integrability of the gradient. For the basic properties of such mappings, see [8] and

[11]. Our results in [11] together with Theorem A and its proof yield the following corollary.

COROLLARY B. *Let f be a mapping of finite distortion that satisfies* (1.1). *Then f satisfies condition N. On the other hand, there is a homeomorphism f of finite distortion from the closed unit cube Q_0 onto Q_0 such that* (1.2) *holds and f does not satisfy condition N.*

As a consequence of Theorem A, we also deduce that Sobolev mappings whose dilatations are exponentially integrable satisfy condition N. This result in the planar case is essentially due to David [1]. More generally, we have the following result.

COROLLARY C. *Suppose that $f \in W^{1,1}(\Omega, \mathbb{R}^n)$, $J_f \in L^1(\Omega)$, and*

$$|Df(x)|^n \le K(x) J_f(x)$$

a.e. $x \in \Omega$, where $\exp(\lambda K) \in L^1(\Omega)$ for some $\lambda > 0$. Then f satisfies condition N. On the other hand, if $\Psi\colon (0,\infty) \to (0,\infty)$ is a strictly increasing and continuous function such that

$$\int_1^\infty \frac{\Psi'(t)}{t}\, dt < \infty, \tag{1.4}$$

then there is a homeomorphism f of finite distortion from the closed unit cube Q_0 onto Q_0 such that $J_f \in L^1(Q_0)$ with

$$\int_{Q_0} \exp(\Psi(K(x)))\, dx < \infty$$

and such that f does not satisfy condition N.

The conclusion of the first part of Corollary C was previously known only in even dimensions—under the assumption that $\lambda > \lambda(n) > 0$. For this result see the paper [7] by Iwaniec, Koskela, and Martin, where the condition N was obtained as a consequence of nontrivial regularity results for mappings of exponentially integrable distortion. Notice that the exponential integrability of K in Corollary C cannot be substantially relaxed because (1.4) holds, for example, for the function $\Psi(t) = t/\log^2(1+t)$.

Our proof of Theorem A goes as follows. The topological degree is related to the weak Jacobian by a degree formula. On the other hand, by a result of Greco [5], the weak Jacobian coincides with the determinant of Df under the assumptions on f. We are then able to estimate the measure of $f(E)$ by an integral of the determinant of Df. The example showing the sharpness of (1.1) is a natural homeomorphism that maps a regular Cantor set of measure zero onto a Cantor set of positive measure. The construction is similar to that of Ponomarev's [18]. Extra care is needed, however, as we also use this very same mapping for Corollaries B and C and hence must estimate the distortion of our homeomorphism.

Note added in December 2000: It has been very recently noticed that the divergence of the integral in (1.4) is sufficient (modulo minor technical assumptions) for condition N in the setting of Corollary C. See [12] for details.

2. Degree Formula

If A is a real $n \times n$ matrix, we denote the cofactor matrix of A by $\operatorname{cof} A$. Then the entries of $\operatorname{cof} A$ are $b_{ij} = (-1)^{i+j} \det A_{ij}$, and $\operatorname{cof} A$ is the transpose of the adjugate $\operatorname{adj} A$ of A.

Let $\mathbf{V}$ be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ oriented by a unit vector $\mathbf{v}$ normal to $\mathbf{V}$. Then, for each linear mapping $L\colon \mathbf{V} \to \mathbb{R}^n$, there is a vector $\Lambda_{n-1}L \in \mathbb{R}^n$ such that

$$\Lambda_{n-1}L \cdot \mathbf{v} = (\operatorname{cof} \tilde{L})\mathbf{v}$$

whenever $\tilde{L}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear extension of L (cf. [16]).

The following result is due to Müller, Spector, and Tang [16].

PROPOSITION 2.1. *Let $G \subset \mathbb{R}^n$ be a domain with a smooth boundary and let $f \in \mathcal{C}(\bar{G}) \cap W^{1,p}(\partial G)$. Let $D_T f$ be the tangential derivative of f with respect to ∂G in the sense of distributional differentiation on manifolds. Let $h \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Assume that either*

(a) $p > n-1$, *or*
(b) $p \geq n-1$ *and* $\mathcal{L}^n(f(\partial G)) = 0$.

Then

$$\int_{\partial G} (h \circ f)(x) \Lambda^{n-1} D_T f(x) \cdot \mathbf{n}(x)\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^n} \operatorname{div} h(y) \deg(f, G, y)\, dy. \tag{2.1}$$

Proof. Part (a) is directly stated in [16]. For part (b), we can mimic the proof in [16], where the strict inequality $p > n-1$ is used only to prove assumption (b). □

The following proposition is stated in ultimate generality because it may be interesting in its own right. In the sequel we will use the assertion only under the stronger hypothesis that $|Df| \in L^p(\Omega)$, $p > n-1$. A reader interested in only this level of generality may skip the proof and realize that the conclusion easily follows from part (a) of Proposition 2.1.

PROPOSITION 2.2. *Suppose that $f\colon \Omega \to \mathbb{R}^n$ is a continuous mapping and that $|Df| \in \mathrm{L}^{n-1,1}(\Omega)$. Let $\eta \in C_c^\infty(\Omega)$, $\eta \geq 0$, and $h \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Then, for almost all $t > 0$, we have*

$$\mathcal{L}^n(f(\{\eta = t\})) = 0 \tag{2.2}$$

and

$$\int_{\{\eta=t\}} (h\circ f)(x)\cdot \operatorname{cof} Df(x)\,\mathbf{n}(x)\,d\mathcal{H}^{n-1}(x)$$
$$= \int_{\mathbb{R}^n} \operatorname{div} h(y)\deg(f,\{\eta>t\},y)\,dy, \tag{2.3}$$

where $\mathbf{n}(x)$ *denotes the outward unit normal to* $\{\eta=t\}$ *at* x.

Proof. According to Corollary 2.4 in [10], the property $|Df|\in \mathrm{L}^{n-1,1}(\Omega)$ implies that there is a nonnegative increasing function φ on $(0,\infty)$ such that

$$\int_0^\infty \varphi^{1/(n-1)}(s)\,dx < \infty$$

and

$$\int_{\{Df\neq 0\}} |Df|\varphi^{n/(1-n)}(|Df|)\,dx < \infty.$$

We call a level t *good* if $D\eta$ is bounded away from zero on $\{\eta=t\}$ (so that $\{\eta=t\}$ is a smooth manifold), if the trace of f belongs to $W^{1,n-1}(\{\eta=t\})$, and if, for $\mathcal{H}^{n-1}$-a.e. $x\in\{\eta=t\}$, the tangential derivative $D_T f(x)$ of the trace of f coincides with the restriction of $Df(x)$ to $T_x(\{\eta=t\})$ and

$$\int_{\{\eta=t\}\cap\{Df\neq 0\}} |Df|\varphi^{n/(1-n)}(|Df|)\,d\mathcal{H}^{n-1}(x) < \infty.$$

Using the Sard theorem, the co-area formula, and well-known behavior of traces, we observe that almost all levels t are good.

Let t be a good level. Then, using again Corollary 2.4 in [10], we observe that $|D_T f|\in \mathrm{L}^{n-1,1}(\{\eta=t\})$ and thus by [10, Thm. C] we have

$$\mathcal{H}^{n-1}(f(\{\eta=t\}))=0;$$

in particular, (2.2) holds. Now formula (2.3) follows from Proposition 2.1. □

3. Sense-Preserving Mappings

Each sense-preserving mapping $f\colon\Omega\to\mathbb{R}^n$ satisfies the spherical monotonicity property

$$\operatorname{diam} f(B)\le \operatorname{diam} f(\partial B)\quad \text{for each}\ \ B\subset\subset\Omega. \tag{3.1}$$

Indeed, if $y\in f(B)\setminus f(\partial B)$ then y cannot belong to the unbounded component of $\mathbb{R}^n\setminus f(\partial B)$, since we would then have $\deg(f,B,y)=0$. Hence $f(B)$ is contained in the closed convex hull of $f(\partial B)$ and (3.1) holds.

If $f\in W^{1,p}(\Omega)$, $p>n-1$, satisfies (3.1), then the following well-known oscillation estimates hold: for each $x\in\Omega$ and $r\in\big(0,\frac12\operatorname{dist}(x,\partial\Omega)\big)$,

$$\left(\frac{\operatorname{diam} f(B(x,r))}{r}\right)^p \le Cr^{-n}\int_{B(x,2r)}|Df|^p\,dy.$$

The right-hand side is bounded as $r\to 0$ for all Lebesgue points of $|Df|^p$. By the Rademacher–Stepanov theorem, it follows that f is differentiable almost everywhere (cf. [6]) and thus, at almost every point x_0, $Df(x_0)$ is the classical (total) differential of f at x_0.

The following result is well known, but for the convenience of the reader we give a proof here.

LEMMA 3.1. *If $f \in W^{1,p}(\Omega, \mathbb{R}^n)$, $p > n-1$, is sense-preserving, then $J_f \geq 0$ a.e. in Ω.*

Proof. Fix x_0 such that $Df(x_0)$ is the classical differential of f at x_0 and $J_f(x_0) \neq 0$. It suffices to prove that $J_f(x_0) > 0$.

We may assume that $x_0 = 0 = f(x_0)$. Since $J_f(0) \neq 0$, there is a constant $c > 0$ such that

$$|Df(0)x| \geq c|x|$$

for all $x \in \mathbb{R}^n$. By the differentiability assumption, there exists an $r > 0$ for which $B(0,r) \subset\subset \Omega$ and

$$|f(x) - Df(0)x| < \tfrac{1}{2}cr$$

for all $x \in \partial B(0,r)$. It follows that

$$|f(x) - Df(0)x| < \operatorname{dist}(0, f(\partial B(0,r)))$$

for all $x \in \partial B(0,r)$. Then, by the properties of the topological degree (see e.g. [3, Thm. 2.3(2)]) we have

$$\deg(Df(0), B(0,r), 0) = \deg(f, B(0,r), 0) > 0,$$

whence $\det Df(0) > 0$. □

Let $q \geq 1$ and let q' be the conjugated exponent. If $f \in W^{1,q(n-1)}_{\text{loc}}(\Omega, \mathbb{R}^n) \cap L^{q'}_{\text{loc}}(\Omega, \mathbb{R}^n)$, then the weak Jacobian is the distribution $\operatorname{Det} Df$ defined by the rule

$$\langle \operatorname{Det} Df, \eta \rangle = -\int_\Omega f_n J(x, (f_1, \dots, f_{n-1}, \eta))\, dx$$

for each test function $\eta \in C^\infty_c(\Omega)$. Here $J(x, (f_1, \dots, f_{n-1}, \eta))$ is the determinant of the differential Dg of the mapping $g(x) = (f_1, \dots, f_{n-1}, \eta)$. Thus, in the language of differential forms,

$$J(x, (f_1, \dots, f_{n-1}, \eta))\, dx = df_1 \wedge \cdots \wedge df_{n-1} \wedge d\eta.$$

We need a result of Greco [5] according to which $J_f \in L^1_{\text{loc}}(\Omega)$ and

$$\operatorname{Det} Df(x) = J_f(x) := J(x, f)$$

whenever $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies (1.1) and either $J_f(x) \geq 0$ a.e. in Ω or $J_f(x) \leq 0$ a.e. in Ω. The regularity in this result is sharp in the sense that (1.1) cannot be replaced with (1.2).

LEMMA 3.2. *Let $f\colon \Omega \to \mathbb{R}^n$ be a sense-preserving mapping in $W^{1,p}(\Omega)$ with $p > n-1$, and assume that the weak Jacobian* $\operatorname{Det} Df$ *satisfies* $\operatorname{Det} Df = J_f$. *Then*

$$\mathcal{L}^n(f(G)) \leq \int_G J_f(x)\, dx$$

for all open $G \subset\subset \Omega$.

Proof. Let $s \in (0, 1)$. Choose $\eta \in C_c^\infty(G)$ such that $0 \le \eta \le 1$, $|\nabla \eta| \neq 0$ in $\{0 < \eta < 1\}$, and

$$s\mathcal{L}^n(f(G)) \le \mathcal{L}^n(f(\{\eta = 1\})).$$

Then, for almost every $t \in (0, 1)$,

$$f \in W^{1,p}(\{\eta = t\}, \mathbb{R}^n).$$

Thus, by choosing $h(y) = (0, \dots, 0, y_n)$ in Proposition 2.2 we have, since $\mathbf{n}(x) = -\nabla\eta(x)/|\nabla\eta(x)|$, that

$$\begin{aligned} s\mathcal{L}^n(f(G)) \le \mathcal{L}^n(f(\{\eta > t\})) &\le \int_{\mathbb{R}^n} \deg(f, \{\eta > t\}, y)\, dy \\ &= -\int_{\{\eta = t\}} \frac{f_n(x)}{|\nabla\eta(x)|} J(x, (f_1, \dots, f_{n-1}, \eta))\, d\mathcal{H}^{n-1}(x). \end{aligned} \tag{3.2}$$

Integrating (3.2) over $t \in (0, 1)$ via the co-area formula, we obtain (see e.g. [2, Thm. 3.2.12])

$$s\mathcal{L}^n(f(G)) \le -\int_G f_n(x) J(x, (f_1, \dots, f_{n-1}, \eta))\, dx = \int_G \eta J_f \le \int_G J_f(x)\, dx.$$

In the last inequality we have used the fact that $J_f \ge 0$ a.e. (Lemma 3.1). Now let $s \to 1$. □

4. Proofs of Theorem A and Corollaries B and C

The first part of the claim of Theorem A immediately follows from Lemma 3.2 since, by Lemma 3.1, $J_f \ge 0$ a.e. and thus by Greco's result $J_f \in L^1_{\mathrm{loc}}(\Omega)$ and $\operatorname{Det} Df = J_f$. The example of Section 5 gives the second part of Theorem A as well as the second parts of Corollaries B and C.

Corollary B follows immediately from Theorem A since, by [11, Thm. 1.5], a mapping f of finite distortion satisfying (1.1) is sense-preserving.

Under the assumptions of Corollary C,

$$\int_\Omega \frac{|Df|^n}{\log(e + |Df|)} < \infty$$

(see [7]), whence, by the results of Greco [5], (1.1) is satisfied. Thus Corollary C follows from Corollary B.

5. An Example

Let Ψ be as in Corollary C. We will construct a homeomorphism $f \colon Q_0 = [0, 1]^n \to Q_0$ ($n \ge 2$) that fixes the boundary ∂Q_0 and has the following properties.

(a) $f \in W^{1,1}(Q_0, \mathbb{R}^n)$, f is differentiable almost everywhere, and

$$\sup_{0 < \varepsilon \le n-1} \varepsilon \int_{Q_0} |Df(x)|^{n-\varepsilon}\, dx < \infty. \tag{5.1}$$

(b) The Jacobian determinant $J_f(x)$ is strictly positive for almost every $x \in Q_0$, and

$$\int_{Q_0} J_f(x)\,dx < \infty. \tag{5.2}$$

(c) The dilatation $K(x) = |Df(x)|^n/J_f(x)$ is finite almost everywhere, and

$$\int_{Q_0} \exp(\Psi(K(x)))\,dx < \infty. \tag{5.3}$$

(d) f does not satisfy Lusin's condition N.

Besides the usual Euclidean norm $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, we will use the cubic norm $\|x\| = \max_i |x_i|$. Using the cubic norm, the x_0-centered closed cube with edge length $2r > 0$ and sides parallel to coordinate axes can be represented in the form

$$Q(x_0, r) = \{x \in \mathbb{R}^n : \|x - x_0\| \le r\}.$$

We then call r the *radius* of Q. We will use the notation $a \lesssim b$ if there is a constant $c = c(n) > 0$ depending only on n such that $a \le cb$, and we write $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

We will be dealing with radial stretchings that map cubes $Q(0, r)$ onto cubes. The following lemma can be verified by an elementary calculation.

LEMMA 5.1. *Let $\rho\colon (0,\infty) \to (0,\infty)$ be a strictly monotone and differentiable function. Then, for the mapping*

$$f(x) = \frac{x}{\|x\|}\rho(\|x\|), \quad x \ne 0,$$

we have for a.e. x

$$\max\left\{\frac{\rho(\|x\|)}{\|x\|}, |\rho'(\|x\|)|\right\} \approx |Df(x)|$$

and

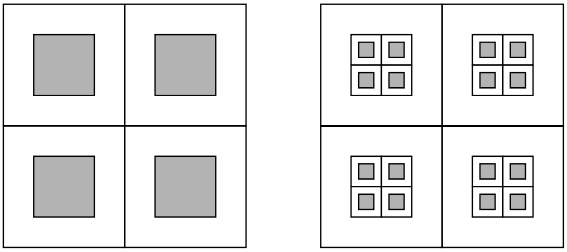
$$\frac{\rho'(\|x\|)\rho(\|x\|)^{n-1}}{\|x\|^{n-1}} \approx J_f(x).$$

We will first give two Cantor set constructions in Q_0. We define f as the limit of a sequence of piecewise continuously differentiable homeomorphisms $f_k\colon Q_0 \to Q_0$, where each f_k maps the kth step of the first Cantor set construction onto the second one. Then f maps the first Cantor set onto the second one. Choosing the Cantor sets so that the measure of the first one equals zero and the second has positive measure, we obtain property (d).

Let $V \subset \mathbb{R}^n$ be the set of all vertices of the cube $Q(0,1)$. Then sets $V^k = V \times \cdots \times V$ $(k = 1, 2, \dots)$ will serve as the sets of indices for our construction (with the exception of the subscript 0). If $w \in V^{k-1}$, we denote

$$V^k[w] = \{v \in V^k : v_j = w_j,\ j = 1, \dots, k-1\}.$$

Let $z_0 = \left[\frac{1}{2}, \dots, \frac{1}{2}\right]$ and $r_0 = \frac{1}{2}$. For $v \in V^1 = V$ let $z_v = z_0 + \frac{1}{4}v$, $P_v = Q\big(z_v, \frac{1}{4}\big)$, and $Q_v = Q\big(z_v, \frac{1}{8}\big)$. If $k \in 2, 3, \dots$ and $Q_w = Q(z_w, r_{k-1})$ is a cube



$k = 1$ $k = 2$

Figure 1 Cubes Q_v, $v \in V^k$

from the previous step of construction, $w \in V^{k-1}$, then Q_w is divided into 2^n subcubes P_v ($v \in V^k[w]$) with radius $r_{k-1}/2$, and inside them concentric cubes Q_v ($v \in V^k[w]$) are considered with radius $r_k = \frac{1}{4} r_{k-1}$. These cubes form the new families. Thus, if $v = (v_1, \ldots, v_k) \in V^k$ then

$$z_v := z_w + \tfrac{1}{2} r_{k-1} v_k = z_0 + \tfrac{1}{2} \sum_{j=1}^{k} r_{j-1} v_j,$$

$$P_v := Q(z_v, r_{k-1}/2), \qquad Q_v := Q(z_v, r_k).$$

See Figure 1. We thus obtain the families $\{Q_v : v \in V^k\}$, $k = 1, 2, 3, \ldots,$ for which the radius of Q_v is

$$r_k = 2^{-2k-1}$$

and the number of cubes is $\# V^k = 2^{nk}$. Note that $r_k < r_{k-1}/2$ for all k. The measure of the resulting Cantor set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v$$

equals zero, since

$$\mathcal{L}^n \Big(\bigcup_{v \in V^k} Q_v \Big) = 2^{nk} 2^{-2kn} \to 0.$$

The second Cantor set construction is similar to the first except that now we denote the centers by z'_v and the cubes by P'_v and Q'_v ($v \in V^k$), with

$$z'_v := z'_w + \frac{1}{2} r'_{k-1} v_k = z_0 + \frac{1}{2} \sum_{j=1}^{k} r'_{j-1} v_j,$$

$$P'_v := Q(z'_v, r'_{k-1}/2), \qquad Q'_v := Q(z'_v, r'_k).$$

Here,

$$r'_k = \varphi(k) 2^{-k-1},$$

where $\varphi\colon \mathbb{N} \to (1/2, 1]$ is any fixed, strictly decreasing function such that $\varphi(0) = 1$. Note that $r'_k < r'_{k-1}/2$ for each k. We have

$$\mathcal{L}^n\left(\bigcap_{k=1}^{\infty} \bigcup_{v\in V^k} Q_v\right) = \lim_{k\to\infty} \mathcal{L}^n\left(\bigcup_{v\in V^k} Q_v\right) = \lim_{k\to\infty} 2^{nk}(2r'_k)^n \geq 2^{-n} > 0.$$

We are now ready to define the mappings f_k. Define $f_0 = \mathrm{id}$. We will give a mapping f_1 that stretches each cube Q_v $(v \in V^1)$ homogeneously so that $f_1(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f_1 is defined to be an appropriate radial map with respect to z_v in pre-image and z'_v in image to make f_1 a homeomorphism. The general step is as follows. If $k > 1$ then f_k is defined as f_{k-1} outside the union of all cubes Q_w, $w \in V^{k-1}$. Further, f_k remains equal to f_{k-1} at the centers of cubes Q_v $(v \in V^k)$. Then f_k stretches each cube Q_v $(v \in V^k)$ homogeneously so that $f(Q_v)$ equals Q'_v. On the annulus $P_v \setminus Q_v$, f is defined to be an appropriate radial map with respect to z_v in pre-image and z'_v in image to make f_k a homeomorphism (see Figure 2). Notice that the Jacobian determinant J_{f_k} will be strictly positive almost everywhere in Q_0.

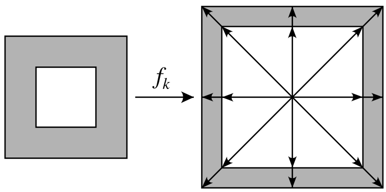


Figure 2 The mapping f_k acting on P_v, $v \in V^k$

To be precise, let $f_0 = \mathrm{id}|_{Q_0}$ and, for $k = 1, 2, 3, \ldots$, define

$$f_k(x) = \begin{cases} f_{k-1}(x) & \text{if } x \notin \bigcup_{v\in V^k} P_v, \\ f_{k-1}(z_v) + a_k(x - z_v) + b_k \frac{x - z_v}{\|x - z_v\|} & \text{if } x \in P_v \setminus Q_v \ (v \in V^k), \\ f_{k-1}(z_v) + c_k(x - z_v) & \text{if } x \in Q_v \ (v \in V^k). \end{cases}$$

Here a_k, b_k, c_k are chosen so that f_k maps each Q_v onto Q'_v, is continuous, and fixes the boundary ∂Q_0:

$$\begin{aligned} a_k r_k + b_k &= r'_k, \\ a_k r_{k-1}/2 + b_k &= r'_{k-1}/2, \\ c_k r_k &= r'_k. \end{aligned} \tag{5.4}$$

Clearly the limit $f = \lim_{k\to\infty} f_k$ is differentiable almost everywhere, its Jacobian determinant is strictly positive almost everywhere, and f is absolutely continuous on almost all lines parallel to coordinate axes. Continuity of f follows from the uniform convergence of the sequence (f_k): for any $x \in Q_0$ and $l \geq j \geq 1$, we have

$$|f_l(x) - f_j(x)| \lesssim r'_j \to 0$$

as $j \to \infty$.

It is easily seen that f is a one-to-one mapping of Q_0 onto Q_0. Since f is continuous and Q_0 is compact, it follows that f is a homeomorphism. One also verifies easily that

$$f\Big(\bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q_v\Big) = \bigcap_{k=1}^{\infty} \bigcup_{v \in V^k} Q'_v,$$

so that property (d) holds.

To finish the proof of properties (a)–(c), we next estimate $|Df(x)|$ and $J_f(x)$ at x in the interior of the annulus $P_v \setminus Q_v$ for $v \in V^k$ $(k = 1, 2, 3, \dots)$. Let $r = \|x - z_v\| \approx r_k$. In the annulus,

$$f(x) = f_{k-1}(z_v) + (a_k \|x - z_v\| + b_k) \frac{x - z_v}{\|x - z_v\|};$$

whence, denoting $\rho(r) = a_k r + b_k$, we have by Lemma 5.1 (it is easy to check that $b_k > 0$ for large k) that

$$|Df(x)| \approx a_k + b_k/r_k$$

and

$$J_f(x) \approx a_k(a_k + b_k/r_k)^{n-1}.$$

From the equations (5.4) it follows that

$$a_k = \frac{r'_{k-1}/2 - r'_k}{r_{k-1}/2 - r_k} = (\varphi(k-1) - \varphi(k))2^k$$

and

$$a_k + b_k/r_k = r'_k/r_k = \varphi(k)2^k \approx 2^k.$$

Therefore,

$$|Df(x)| \approx 2^k$$

and

$$J_f(x) \approx (\varphi(k-1) - \varphi(k))2^{nk},$$

whence for large k we have

$$K(x) = \frac{|Df(x)|^n}{J_f(x)} \le \frac{c_0}{\varphi(k-1) - \varphi(k)}, \tag{5.5}$$

where $c_0 = c_0(n) \ge 1$ depends only on n.

The measure of $\bigcup_{v \in V^k} P_v$ is $2^{nk} r_{k-1}^n \approx 2^{-nk}$ and so, for $0 < \varepsilon \le n - 1$,

$$\begin{aligned}\varepsilon \int_{Q_0} |Df(x)|^{n-\varepsilon}\, dx &\lesssim \varepsilon \sum_{k=1}^{\infty} 2^{-nk} 2^{k(n-\varepsilon)} \\ &\le \varepsilon \sum_{k=0}^{\infty} 2^{-\varepsilon k} = \frac{\varepsilon}{1 - 2^{-\varepsilon}} \le C,\end{aligned}$$

where $C < \infty$ does not depend on ε. This proves (5.1), and it follows that $f \in W^{1,1}(Q_0, \mathbb{R}^n)$. Similarly, we prove (5.2):

$$\begin{aligned}\int_{Q_0} J_f(x)\,dx &\lesssim \sum_{k=1}^{\infty} 2^{-nk}(\varphi(k-1)-\varphi(k))2^{nk} \\ &= \sum_{k=1}^{\infty}(\varphi(k-1)-\varphi(k)) = \varphi(0) - \lim_{k\to\infty}\varphi(k) < \infty.\end{aligned}$$

For what follows we need to define φ more explicitly. Let

$$\varphi(k) = \frac{1}{2}\left(1 + \frac{1}{\lambda}\int_k^{\infty}\frac{du}{\Psi^{-1}(u)}\right)$$

for large k, where $\lambda > 0$ is chosen so that $2\lambda c_0 = 1$. Then φ and $|\varphi'|$ are decreasing. By (5.5), for large k we have

$$K(x) \leq \frac{c_0}{|\varphi'(k)|} = 2\lambda c_0 \Psi^{-1}(k) = \Psi^{-1}(k)$$

and thus

$$\begin{aligned}\int_{Q_0}\exp(\Psi(K(x)))\,dx &\lesssim \sum_k 2^{-nk}\exp(\Psi(\Psi^{-1}(k))) \\ &= \sum_k 2^{-nk}e^k = \sum_k (2^{-n}e)^k < \infty.\end{aligned}$$

Thus (5.3) is proven.

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