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The Operator Inequality $P^{2k} \le A^*P^{2k}A$

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1. Introduction

Let $B(H)$ denote the algebra of operators (i.e., bounded linear transformations) on an infinite-dimensional separable Hilbert space H into itself. Given a (nontrivial) operator $P \ge 0$, contraction operators A satisfying the positivity condition

$$A^*P^{2k}A - P^{2k} \ge 0, \quad 0 < k \le 1, \tag{1}$$

occur quite naturally. Thus, if T is a k-hyponormal operator ($0 < k \le 1$) with polar decomposition $T = UP$, then $UP^{2k}U^* \le P^{2k} \le U^*P^{2k}U$ [1; 6]. Again, if A is a contraction such that $s - \lim_n A^{*n}A^n = P^2$, then $P^2 = A^*P^2A$ and $P^{2k} = (A^*P^2A)^k \le A^*P^{2k}A$ for all $0 < k$. Operator inequality (1) has been considered by Douglas for the case $k = 1$ in [7], where it is shown that if P is a compact operator then $P = A^*PA$, $\overline{\operatorname{ran} P}$ reduces A, and $A|_{\overline{\operatorname{ran} P}}$ is unitary [7, Thm. 8, and Cor. 6.5].

Let $T \in B(H)$. Then T is said to be of the class $\mathcal{C}_\rho$ ($\rho > 0$) if there exists a unitary U on a Hilbert space $K \supset H$ such that $T^n = \rho P_H U^n|_H$ for $n = 1, 2, \dots$, where P_H denotes the orthogonal projection of K onto H (see [12, p. 45]). Operators $T \in \mathcal{C}_1$ are contractions and, if $\rho > 1$, then operators $T \in \mathcal{C}_\rho$ are similar to a contraction [5]. In this note we consider operators A which are similar to a contraction and which satisfy inequality (1) for some $P \ge 0$. We also prove the following theorem (and some of its consequences).

Theorem 1. *Let $A \in B(H)$ be such that either $A \in \mathcal{C}_\rho$ ($\rho > 1$) or A^m is a contraction for some integer $m > 1$, and let P be a positive operator such that inequality* (1) *is satisfied. Then there exists a positive invertible operator L and a positive operator Q such that*:

(i) $\|A^*P^{2k}A - P^{2k}\| \le M\frac{k}{\pi}\iint_{\sigma(LQL^{-1}A)} r^{2k-1}\,dr\,d\theta$, *where* $M = (2\rho - 1)^2$ *if* $A \in \mathcal{C}_\rho$ *and* $M = \|L\|^2\|L^{-1}\|^2$ *if A^m is a contraction*;
(ii) *the operators A and $QL^{-1}AL$ are not supercyclic.*

Now suppose further that P is compact and has dense range. Then

(iii) *A is unitary and $A^*PA - P = 0 = APA^* - P$ if $A \in \mathcal{C}_\rho$*;
(iv) *A is similar to a unitary and $A^*P^{2k}A - P^{2k} = 0$ if A^m is a contraction.*

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Before going on to prove the theorem, we explain our terminology and introduce some complementary results.

Let $T \in B(H)$, and let $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$ denote the orbit of $x \in H$ under T. We say that T is supercyclic with supercyclic vector x if the set of scalar multiples of elements in $\mathrm{Orb}(T, x)$ is dense in H. It has been known for some time that a normal operator cannot be supercyclic. Recently, Bourdon [4] has shown that a hyponormal operator cannot be supercyclic. Indeed, more is true.

LEMMA 2. *If $T \in B(H)$ satisfies the growth condition*

$$\|T^n x\|^2 \le \|T^{n+1}x\|\,\|T^{n-1}x\|$$

for $x \in H$ and $n = 1, 2, \dots,$ then T cannot be supercyclic.

Proof. See [4, p. 352]. □

The operator T is said to be *power bounded* if there exists an $M > 0$ such that $\sup_n \|T^n\| \le M$. Note that $\mathcal{C}_\rho$ operators are power bounded.

LEMMA 3. *Let $T \in B(H)$ be power bounded. If T is supercyclic, then $T^n x \to 0$ as $n \to \infty$ for each $x \in H$.*

Proof. See [3, Thm. 2.2]. □

Let $A \in B(H)$. Then $L = \left\{\sum_{r=0}^{m-1} A^{*r}A^r\right\}^{1/2}$ is a positive invertible operator. Assume now that A^m is a contraction for some integer $m > 1$. Then $A^{*m}A^m \le 1$ and

$$L^{-1}A^*L^2AL^{-1} = L^{-1}\left(\sum_{r=0}^{m-1} A^{*r}A^r + A^{*m}A^m - 1\right)L^{-1} \le L^{-1}(L^2)L^{-1} = 1;$$

that is, LAL^{-1} is a contraction C. A similar result holds for the case in which $A \in \mathcal{C}_\rho$.

LEMMA 4 (see [5, Thm. 5]). *If $A \in B(H) \cap \mathcal{C}_\rho$ $(\rho > 1)$, then there exists a positive invertible operator L such that $\|L\|\,\|L^{-1}\| \le (2\rho - 1)$ and $A = L^{-1}CL$ for some contraction C.*

Recall from [1] that the operator T is said to be *k-hyponormal*, $0 < k \le 1$, if $(TT^*)^k \le (T^*T)^k$. (Thus a 1-hyponormal operator is hyponormal.) It is known that k-hyponormal operators satisfy a Putnam area inequality similar to that satisfied by hyponormal operators. Let $\sigma(T)$ denote the spectrum of T. One then has the following.

LEMMA 5 [6, Thm. 5]. *If $T \in B(H)$ is k-hyponormal for some $0 < k \le 1$, then*

$$\|(T^*T)^k - (TT^*)^k\| \le \frac{k}{\pi}\iint_{\sigma(T)} r^{2k-1}\,dr\,d\theta.$$

Let P be a compact injection, let $A \in \mathcal{C}_\rho$, and let U be a unitary such that $PA = UP$. Then A is power bounded, and it follows from an application of [2, Thm. 2] that U is singular unitary (i.e., the spectral measure of U is singular with respect to the linear Lebesgue measure on the unit circle). Applying [2, Thm. 1(ii)] to $PA = UP$ yields the following lemma.

LEMMA 6. *If $A \in \mathcal{C}_\rho$ is such that $PA = UP$ for some unitary U and compact injection P, then A is unitary.*

In the sequel we shall denote the closure of the range (resp., the orthogonal complement of the kernel) of an $A \in B(H)$ by $\overline{\operatorname{ran} A}$ (resp., $\ker^{\perp} A$). The restriction of A to an invariant subspace N will be denoted by $A|_N$. Recall that the operator X is said to be a *quasi-affinity* if X is injective and has dense range.

2. Proof of Theorem 1

Let $A \in B(H)$ be such that either $A \in \mathcal{C}_\rho$ or A^m is a contraction for some integer $m > 1$. Then, by Lemma 4 (and the argument preceding the statement of Lemma 4), there exists an invertible positive operator L and a contraction C such that $A = LCL^{-1}$ (with $\|L\|\|L^{-1}\| \le (2\rho - 1)$ if $A \in \mathcal{C}_\rho$). By hypothesis, $A^*P^{2k}A \ge P^{2k}$; hence $C^*LP^{2k}LC \ge LP^{2k}L$. Let $L_1 = L/\|L\|$; then L_1 is a contraction and $C^*L_1P^{2k}L_1C \ge L_1P^{2k}L_1$. The operator $L_1P^{2k}L_1$ is positive and thus has a unique positive $2k$th root. Denote this root by Q; then $C^*Q^{2k}C \ge Q^{2k} \ge (QCC^*Q)^k$. Recall now from Hansen's inequality [10] that, if Q is a positive (semi-definite) operator and C is a contraction, then $C^*Q^{2k}C \le (C^*Q^2C)^k$ for all $0 < k \le 1$. Hence $(QCC^*Q)^k \le (C^*Q^2C)^k$, that is, the operator QC is k-hyponormal. Using Lemma 5, we have that

$$
\begin{aligned}
\|A^*P^{2k}A - P^{2k}\| &= \|L^{-1}(C^*LP^{2k}LC - LP^{2k}L)L^{-1}\| \\
&\le \|L^{-1}\|^2 \| \|L\|^2 (C^*Q^{2k}C - Q^{2k})\| \\
&= \|L\|^2 \|L^{-1}\|^2 \|C^*Q^{2k}C - Q^{2k}\| \\
&\le \|L\|^2 \|L^{-1}\|^2 \|C^*Q^{2k}C - (QCC^*Q)^{2k}\| \\
&\le \|L\|^2 \|L^{-1}\|^2 \|(C^*Q^2C)^k - (QCC^*Q)^{2k}\| \\
&\le \|L\|^2 \|L^{-1}\|^2 \frac{k}{\pi} \iint_{\sigma(QC)} r^{2k-1}\, dr\, d\theta \\
&\le \|L\|^2 \|L^{-1}\|^2 \frac{k}{\pi} \iint_{\sigma(LQL^{-1}A)} r^{2k-1}\, dr\, d\theta,
\end{aligned}
$$

since $\sigma(QC) = \sigma(QLAL^{-1}) = \sigma(LQL^{-1}A)$. This proves (i) of the theorem.

Postponing the proof of part (ii) momentarily, we next prove parts (iii) and (iv). Thus, let P be compact with dense range. Then P^{2k}, as also the unique positive $2k$th root Q of $L_1P^{2k}L_1$, is a compact quasi-affinity. Consequently, QC is a compact operator and $\sigma(QC)$ has planar Lebesgue measure zero. It follows that

$$A^*P^{2k}A - P^{2k} = 0 = C^*Q^{2k}C - Q^{2k} = C^*Q^{2k}C - (QCC^*Q)^k.$$

In particular, $Q^2 = QCC^*Q$; that is, C is a co-isometry that satisfies $QC = CQ$ and $C^*CQ^{2k} = Q^{2k}$. Thus C is a unitary that commutes with Q, and A is similar to a unitary. This proves (iv). Now if $A \in \mathcal{C}_\rho$, then an application of Lemma 6 implies that A is unitary. Since $A^*P^{2k}A = P^{2k}$, A commutes with P and $A^*PA - P = 0 = APA^* - P$. This completes the proof of (iii).

To prove part (ii), we start by proving that the operator $T = QL^{-1}AL = QC$ satisfies the growth condition of Lemma 2. Toward this end we recall the Hölder–McCarthy inequality [11], which states that if the operator $Z \geq 0$ then, for all $x \in H$,

$$(Z^r x, x) \leq \|x\|^{2(1-r)}(Zx, x)^r, \quad 0 < r \leq 1,$$

and

$$(Zx, x,)^r \leq \|x\|^{2(r-1)}(Z^r x, x), \quad 1 \leq r.$$

Let T $(= QC)$ have the polar decomposition $T = U|T|$. Then, since $0 < k \leq 1$ and $|T^*|^{2k} \leq |T|^{2k}$,

$$\begin{aligned}
\|Tx\|^{2(1+k)} &= (|T|^2 x, x)^{1+k} \\
&\leq \|x\|^{2k}(|T|^{2(1+k)}x, x) \\
&= \|x\|^{2k}(U|T|^{2k}U^* Tx, Tx) \\
&= \|x\|^{2k}((U|T|^2 U^*)^k Tx, Tx) \\
&= \|x\|^{2k}(|T^*|^{2k} Tx, Tx) \\
&\leq \|x\|^{2k}(|T|^{2k} Tx, Tx) \\
&\leq \|x\|^{2k}\|Tx\|^{2(1-k)}(|T|^2 Tx, Tx)^k \\
&= \|x\|^{2k}\|Tx\|^{2(1-k)}\|T^2 x\|^{2k}
\end{aligned}$$

for all $x \in H$. Hence $\|Tx\|^2 \leq \|T^2 x\|\|x\|$ and

$$\|T^n x\|^2 = \|T(T^{n-1}x)\|^2 \leq \|T^2(T^{n-1}x)\|\|T^{n-1}x\| = \|T^{n+1}x\|\|T^{n-1}x\|$$

for all $x \in H$ and natural numbers n. Applying Lemma 2, we conclude that T is not supercyclic. To prove that A is not supercyclic, we argue by contradiction. Thus, suppose that A is supercyclic. Then A has a supercyclic vector $(0 \neq)\ x \in H$. Since A is power bounded, Lemma 3 implies that $\lim_n A^n x = 0$ (i.e., A is of the class C_0. [13]). But then $P^{2k} \leq A^*P^{2k}A$ implies that

$$\begin{aligned}
\|P^k x\|^2 &\leq (A^{*n}P^{2k}A^n x, x) \\
&\leq \|P^{2k}\|\|A^n x\|^2 \to 0 \text{ as } n \to \infty
\end{aligned}$$

and hence that $x \in \ker P$. Since the collection of supercyclic vectors of a supercyclic operator in $B(H)$ is dense in H (recall that $cA^n x$ is a supercyclic vector for A for every supercyclic vector x of A [3; 4]), we must have that $P = 0$—a contradiction. Hence A is not supercyclic, and the proof is complete. □

REMARK. It is clear from the foregoing proof that a power bounded operator A satisfying inequality (1) can not be supercyclic. The proof also implies that PA cannot be supercyclic when A is a contraction.

3. Applications

Theorem 1 has a number of consequences, amongst them the following. The operator T is said to be *completely polynomially bounded* if there exists a constant c such that, for all natural numbers n and $n \times n$ matrices (p_{ij}) with polynomial entries, $\|p_{ij}(T)\|_{B(l_2^n(H))} \leq c \sup_{z \in \mathcal{T}} \|p_{ij}(z)\|_{M_n}$. Here $\mathcal{T}$ denotes the unit circle in the complex plane, $(p_{ij}(T))$ is identified in a natural way with an operator in $l_2^n(H)$, and M_n is identified with $l_2^n(H)$. Paulsen [12] has shown that a completely polynomially bounded operator is similar to a contraction. Again, if T is power bounded and spectraloid (i.e., the spectral radius $r(T)$ of T equals the numerical radius $w(T)$ of T [9, p. 117]), then $w(T) = r(T) \leq 1$. This implies that $T \in \mathcal{C}_2$ [12, Prop. 11.2, p. 48] and hence that T is similar to a contraction. Thus Theorem 1 applies to operators A that are either completely polynomially bounded or power bounded and spectraloid. The following corollary generalizes [7, Cor. 6.5]. (We note here that if A is similar to a contraction D then there exists a positive invertible operator L and a contraction C, unitarily equivalent to D, such that $A = L^{-1}CL$.)

COROLLARY 7. *Let $A = L_1^{-1}C_1L_1$ and $B = L_2C_2L_2^{-1}$ for some positive invertible operators L_1, L_2 and contractions C_1, C_2. If $AXB = X$ for some compact quasi-affinity X, then C_1 and C_2^* are unitarily equivalent unitaries. If also A and B are spectraloid, then A and B^* are unitarily equivalent unitaries.*

Proof. Let $L_1XL_2 = Y$. Then Y is a compact quasi-affinity such that $C_1YC_2 = Y$, $|Y^*|^2 \leq C_1|Y^*|^2C_1^*$, and $|Y|^2 \leq C_2^*|Y|^2C_2$. Applying Theorem 1 it follows that C_1, C_2 are unitaries, $C_1^*YC_2^* = Y$, and C_1, C_2^* are unitarily equivalent. Now, if A is spectraloid then $r(A) = w(A) \leq 1$ and $A \in \mathcal{C}_2$ (see [12, p. 48]). Applying Lemma 6 to $L_1A = C_1L$, it follows that A is unitary. Since a similar argument applies when B is spectraloid, B^* is unitary and unitarily equivalent to A. □

The hypothesis that A and B are spectraloid in Corollary 7 is not required if A and B are $\mathcal{C}_\rho$. Furthermore, if A and B are in $\mathcal{C}_\rho$ and X is (simply) compact, then $AXB = X$ implies that $A_1X_1B_1 = X_1$, where $A_1 = A|_{\overline{\operatorname{ran} X}}$, $B_1^* = B^*|_{\ker^\perp X}$ (are $\mathcal{C}_\rho$), and $X_1 \colon \ker^\perp X \to \overline{\operatorname{ran} X}$ is the compact quasi-affinity defined by setting $X_1x = Xx$ for each $x \in H$. This in turn implies that A_1, B_1^* are unitarily equivalent unitaries and that $A^*XB^* - X = 0$. A better result is possible for the case in which A and B are contractions.

COROLLARY 8. *Let A, B be contractions. If $AXB = X$ for some operator X such that either X is compact or $\sigma(A|X^*|)$ is countable, then $A^*XB^* = X$, $\overline{\operatorname{ran} X}$ reduces A, $\ker^\perp X$ reduces B^*, and $A|_{\overline{\operatorname{ran} X}}$ and $B^*|_{\ker^\perp X}$ are unitarily equivalent unitaries.*

Proof. Define A_1, B_1, and X_1 as before. Then $\sigma(A|X^*|) = \sigma(A_1|X_1^*|) \cup \{0\}$ and $|X_1^*|^2 \leq A_1|X_1^*|^2 A_1^*$. Applying Theorem 1, it follows that A_1 is unitary and $A_1|X_1^*|A_1^* = |X_1^*|$. Let X_1 have the polar decomposition $X_1 = U_1|X_1|$; then $|X_1^*| = U_1|X_1|U_1^*$ and it follows from $A_1|X_1^*|A_1^* = |X_1^*|$ that $A_1 X_1 = X_1 U_1^* A_1 U_1$. Because the operator $U_1^* A_1 U_1$ is unitary, it follows from $X_1 = A_1 X_1 B_1 = X_1 U_1^* A_1 U_1 B_1$ that B_1 is unitary. The rest of the proof is now obvious. □

We conclude this note with the following range-kernel orthogonality result. (Recall that if $\mathcal{V}$ is a normed linear space with norm $\|\cdot\|$, then $x \in \mathcal{V}$ is said to be *orthogonal* to $y \in \mathcal{V}$ if $\|x - ty\| \geq \|ty\|$ for all complex numbers t.)

COROLLARY 9. *Let X be a compact operator such that $AXB = X$ for some operators $A, B \in \mathcal{C}_\rho$ $(\rho > 1)$. Then*

$$\|AYB - Y + X\| \geq \|X\|$$

for all $Y \in B(H)$.

Proof. Defining A_1, B_1, and X_1 as before, it follows that $\overline{\text{ran}\, X}$ reduces A, $\ker^\perp X$ reduces B^*, and A_1, B_1 are unitaries such that $A_1 X_1 B_1 = X_1$. Let $Y \in B(H)$, and let $Y : \ker^\perp X \oplus \ker X \to \overline{\text{ran}\, X} \oplus \overline{\text{ran}\, X}^\perp$ have the matrix representation $Y = [Y_{ij}]_{i,j=1}^2$. Then it follows from an application of [8, Cor. 3] that

$$\|A_1 Y_{11} B_1 - Y_{11} + X_1\| \geq \|X_1\| = \|X\|.$$

Recall now that the norm of the leading entry of an operator matrix is less than or equal to the norm of the operator matrix. Since

$$AYB - Y + X = \begin{bmatrix} A_1 Y_{11} B_1 - Y_{11} + X_1 & * \\ * & * \end{bmatrix},$$

the proof follows. □

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