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# Permutation Models and SVC

Eric J. Hall

**Abstract** Let $M$ be a model of ZFAC (ZFC modified to allow a set of atoms), and let $N$ be an inner model with the same set of atoms and the same pure sets (sets with no atoms in their transitive closure) as $M$. We show that $N$ is a permutation submodel of $M$ if and only if $N$ satisfies the principle SVC (Small Violations of Choice), a weak form of the axiom of choice which says that in some sense, all violations of choice are localized in a set. A special case is considered in which there exists an SVC witness which satisfies a certain homogeneity condition.

## 1 Introduction and Main Result

The principle SVC (Small Violations of Choice) is a weak form of AC (Axiom of Choice), introduced by Blass [1], which says that all failures of AC are localized in a set $S$:

> SVC: There is a set $S$ such that, for every set $a$, there is an ordinal $\alpha$ and a function from $S \times \alpha$ onto $a$.

When $S$ is such a set, we say that "SVC holds with $S$" and that $S$ is an *SVC witness*.

The main new result of this paper can be stated as follows: Let $M$ be a model of ZFA (ZF modified to allow a set of atoms) in which AC holds, and let $N$ be an inner model which has the same pure part and same set of atoms as $M$. If $N \models \mathrm{SVC}$, then $N$ is a permutation submodel of $M$.

**Definitions and Conventions** The theory *ZFA* is a modification of ZF allowing atoms, also known as urelements. See Jech [4] for a precise definition. A model of ZFA may have a proper class of atoms; however, for this paper we redefine ZFA to include an axiom which says that the class of atoms is a set (always denoted by $A$). Similarly, proper class forcing will not be considered in this paper; by *forcing* or *generic extension* it is to be understood that only a set of forcing conditions is permitted.

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In a model of ZFA, a *pure set* is a set with no atoms in its transitive closure, and the *pure part* or *kernel* is the class of all pure sets; the pure part is a model of ZF.

Our definition of *permutation submodel* will be almost the same as that given in [4] (or see Jech [5] for more detail), but generalized somewhat so as to make correct the statement of Theorem 1.1 below. A permutation model is determined by a model $M$ of ZFAC, a group $G$ of permutations of the set $A$ of atoms, and a normal filter $\mathcal{F}$ on $G$. Typically, it is assumed that $G$ is in $M$. In this paper, we only require that $G$ be in some generic extension of $M$ by a cardinal collapse (or by any almost homogeneous notion of forcing); the development of the basic theory is nearly unchanged. (See Hall [3] for an example of some $N \subset M$ where $N$ is a permutation submodel of $M$ which cannot be obtained by a group $G$ in $M$.)

From this point we will not work much directly with the definition of permutation model. Instead, we build on and generalize results in the paper [3], which gives a characterization of permutation submodels in terms of forcing. The new result stated above will be treated as part of the following main theorem.

**Theorem 1.1** *Let $M$ be a transitive model of* ZFAC*, and let $N \subseteq M$ be a transitive submodel of* ZFA *such that $N$ and $M$ have the same set of atoms and the same pure part. The following are equivalent:*

(a) *$N$ is a permutation submodel of $M$;*
(b) *$M$ is a generic extension of $N$;*
(c) *$N$ satisfies* SVC.

The equivalence between (a) and (b) is Theorem 4.1(a) of [3]. The implication from (a) to (c) is Theorem 4.2 of [1] (no serious changes are required to make the proof work for our slightly generalized permutation models). The implication from (b) to (c) follows immediately from Theorem 4.6 of [1], stated here (generalized slightly to allow a set of atoms).

**Theorem 1.2** *A model of* ZFA *satisfies* SVC *if and only if some generic extension satisfies* AC.

**Proof** **Hint:** If $\mathbb{P}$ is a notion of forcing such that $\Vdash_{\mathbb{P}}$ AC, then SVC holds with $\mathbb{P}$. □

The proof of Theorem 1.1 can therefore be completed by proving that (c) implies either (a) or (b). We will prove, in Section 3, that (c) implies (b); we have not found a nice proof that (c) implies (a) without in effect going through (b).

## 2 Questions

Let ZFAC$^K$ be the theory of ZFA (with a set of atoms) + "AC for pure sets." It was claimed in passing on the first page of [3] that SVC is a theorem of ZFAC$^K$. The claim was mistaken. No proof is known; also no disproof is known.

Consider a weaker version of the claim. If $N \models$ ZFAC$^K$ and $M$ is an extension with the same pure part and set of atoms as $N$ such that $M \models$ ZFAC, then we'll say that $M$ is a *choice extension* of $N$. By Theorem 1.1, the following questions are equivalent.

**Question 2.1** *For a given model of* ZFAC$^K$*, is every choice extension a generic extension?*

**Question 2.2** *Does* SVC *hold in every model of* $\mathrm{ZFAC}^K$ *that has a choice extension?*

If the answer is "yes," then the three equivalent conditions of Theorem 1.1 are simply true under the given hypotheses.

## 3 Proof of the Main Theorem

The following lemma is proved in Blass and Scedrov [2]; a sketch of the proof is included here because it contains an idea to be used later.

**Lemma 3.1** *Let $M$ be a model of* ZFAC *with pure part $K$, and let $f : A' \to A$ be a bijection from a pure set to the set of atoms. Then $M$ is the smallest model of* ZFAC *which contains $K$ and $f$.*

**Sketch of Proof** For convenience, assume that all the elements of $A'$ have the same rank, and let $X_0 \notin A'$ also be a pure set of that rank. As in the proof of Lemma 15.47 in [4], construct a model $M'$ of ZFAC inside $K$ whose set of atoms is $A'$. The elements of $M'$ are obtained by iterating the power set operation over $A'$, modified by letting $X_0$ stand in for the empty set each time. Now $\langle M', \in \rangle$ is a model of ZFAC.

There is a unique collapsing map from $M'$ onto $M$ whose restriction to $A'$ is $f$. This map is $\Delta_1$-definable using $f$ as a parameter, so $M$ is generated by $K$ and $f$. □

Note that the collapsing map $M' \to M$ in the proof above is an isomorphism.

To prove (c) implies (b) in Theorem 1.1, we will start with a model $M$ of ZFAC with a submodel $N$ as in the hypotheses of the theorem, and assume that $N$ satisfies SVC. As in the proof of Lemma 3.1, let $M'$ be a copy of $M$ contained in the pure part of $M$ (which is also the pure part of $N$), with $A'$ as its set of atoms. There is a copy $N'$ of $N$ contained in $M'$. In $N$, the set $A$ of atoms is not well-orderable (excepting the boring case $N = M$), and $N$ does not see that $N$ is isomorphic to $N'$ or to any other submodel of $M'$. We'll build a notion of forcing in $N$ out of certain partial embeddings from $N$ to $N'$.

Of all generic extensions of $N$ which add a well-ordering of $A$, $M$ is a minimal such model. (Other extensions which add the same well-orderings of $A$ that $M$ has must contain $M$ and also add new pure sets.) Intuitively, to get a "small" extension like $M$ generically, we want a notion of forcing whose conditions are as large as possible; perhaps a proper class containing arbitrarily large partial embeddings $N \rightharpoonup N'$. The assumption that SVC holds in $N$ turns out to ensure that a mere set of forcing conditions suffices, and also ensures, by way of the next lemma, that the dense subsets will be well-behaved.

In the following Lemma 3.2, think of $S$ as an SVC witness in a model of ZFA. A form of this lemma was first pointed out to me by Omar De la Cruz.

**Lemma 3.2** *If $f : S \times \alpha \to B$ is onto, then for every $D \subseteq B$, there is a pure set $y$ and a well-ordering $x$ of a subset of $\mathcal{P}(S)$ such that $D$ is $\Delta_0$-definable from the parameters $f$, $x$, and $y$.*

**Proof** Let $D \subseteq B$, and consider the set $f^{-1}[D] \subseteq S \times \alpha$. $D$ is $\Delta_0$-definable from $f$ and $f^{-1}[D]$; it remains to show that $f^{-1}[D]$ is $\Delta_0$-definable from some $x$ and $y$ as in the statement of the lemma. Define a one-to-one partial function $b : \mathcal{P}(S) \rightharpoonup \mathcal{P}(\alpha)$

by

$$b(T) = \begin{cases} \{\, \beta < \alpha \mid f^{-1}[D] \cap (S \times \{\beta\}) = T \times \{\beta\} \,\} & \text{if this is nonempty,} \\ \text{undefined} & \text{else.} \end{cases}$$

Observe that $\mathsf{Ran}(b)$ is a pairwise disjoint set of sets of ordinals, and hence is a well-orderable pure set. Let $y$ be a well-ordering of $\mathsf{Ran}(b)$, and let $x$ be the corresponding well-ordering of $\mathsf{Dom}(b)$. Then $b$ is $\Delta_0$-definable from $x$ and $y$, and $f^{-1}[D]$ is in turn $\Delta_0$-definable from $b$. $\square$

**Proof of Theorem 1.1, (c) implies (b)** Let $M$ be a transitive model of ZFAC and let $N$ be an inner model of ZFA, both with the same kernel $K$ and set of atoms $A$. As in the discussion above, let $M' \subset K$ be an isomorphic copy of $M$, with $A'$ as its set of atoms. Any bijection $A \to A'$ in $M$ can be extended uniquely to an isomorphism $M \to M'$. Let $j : M \to M'$ be such an isomorphism, and for $x \in M$ we'll write $x' = j(x)$, and $N' = j[N]$. Observe that if $j_1 : M \to M'$ is any other isomorphism, then $j$ and $j_1$ agree on $K$, since there is only one isomorphism $K \to K'$.

For a function $p$ whose range is contained in $M'$, define a new function $\tilde{p} : \mathsf{Ran}(p) \to K'$ by $\tilde{p}(r) = r'$; in other words, $\tilde{p} = j \restriction \mathsf{Ran}(p)$. It is immediate from the definition that if $p$ and $q$ are any two functions with the same range, then $\tilde{p} = \tilde{q}$. The remainder of this paragraph is optional, for readers interested in the motivation for defining $\tilde{p}$. Suppose that $p$ can be extended to an isomorphism $N \to N'$, and consider the function $p^+ = \bigcap\{\, i \supset p \mid i : N \to N' \text{ is an isomorphism} \,\}$, the intersection of all isomorphisms $N \to N'$ which extend $p$. Think of the domain of $p^+$ as the *extended domain* of $p$. For example, if $x \in \mathsf{Dom}\, p$, then $\{x\}$ is certainly in the extended domain of $p$. Each pure set $y$ is also in the extended domain (since all isomorphisms $N \to N'$ agree on pure sets); $p^+(y) = y'$. It turns out that $p$ is in its own extended domain; to show this, it suffices to show that each $\langle x, y\rangle \in p$ is in the extended domain of $p$. To this end, let $i : N \to N'$ be any isomorphism extending $p$. Clearly $i(x) = p(x) = y$. Since $y \in \mathsf{Ran}(p)$ is a pure set, $i(y) = y'$. Thus $i(\langle x, y\rangle) = \langle y, y'\rangle$ for any isomorphism $i$ extending $p$. It follows that $p \in \mathsf{Dom}(p^+)$, and $p^+(p) = \tilde{p}$.

Suppose $N$ satisfies SVC with $S$. Working in $N$, we will now define a notion of forcing $\mathbb{P}$. Let $T = \mathscr{P}(S)$. Let $F$ be the set of all functions from subsets of $T$ to $T'$; fix an ordinal $\alpha$ and a surjection $f : S \times \alpha \to F$. Note that although the priming function $j$ is not in $N$, the restriction $j \restriction K$ is in $N$ (it is the unique isomorphism $K \to K'$), so we may freely apply primes to pure sets. It follows that the tilde operation $p \mapsto \tilde{p}$ also makes sense in $N$ (when applied to functions whose ranges are pure sets). We will also refer to the two particular sets $T'$ and $f'$. Finally, the definition of $\mathbb{P}$ will also use the term $N'$. To avoid the implicit assumption that $N'$ is a definable class in $N$, one could replace $N'$ in the definition of $\mathbb{P}$ with some sufficiently large initial segment $N'_\xi$.

Let $\mathbb{P}$ be the set of all partial injections $p : T \rightharpoonup T'$ such that $\mathsf{Ran}(p)$ is well-orderable in $N'$, and for every $\Delta_0$ formula $\varphi$, every $y \in K$, and every (transfinite) sequence $\mathbf{x}$ of elements of $\mathsf{Dom}\, p$, we have $p(\mathbf{x}) \in N'$ and

$$N \models \varphi(\mathbf{x}, p, y, f) \quad \leftrightarrow \quad N' \models \varphi(p(\mathbf{x}), \tilde{p}, y', f'). \tag{$*$}$$

The domain of each $p \in \mathbb{P}$ is well-orderable (since $\mathsf{Ran}(p) \subset T'$ is always a pure set). Conversely, if $X \subset T$ is well-orderable, then $X$ is the domain of the function

$p = j \restriction X \in \mathbb{P}$. To see that $p \in N$, let $k_1 : X \to \kappa$ be a bijection in $N$ from $X$ to some ordinal $\kappa$. Since $j \in M$, there is clearly a $k_2 : \kappa \to \mathsf{Ran}(p)$ such that $p = k_2 \circ k_1$. But this $k_2$ would be a pure set, so $k_2$ and hence $p$ are in $N$.

Now, back out in $M$, define $G = \{ p \in \mathbb{P} \mid p \subset j \}$. It is not hard to see that $G$ is a filter in $\mathbb{P}$. It remains to show that $G$ is generic over $N$. This will suffice because $M \subseteq N[G]$ by Lemma 3.1, and since $N \subset M$ and $G \in M$ it must be that $M = N[G]$.

Toward showing that $G$ is $\mathbb{P}$-generic over $N$, let $D \in N$ be a dense subset of $\mathbb{P}$. Applying Lemma 3.2, we get a parameter $y \in K$, a parameter $\mathbf{x}$ which we can think of as a sequence of elements in $T$, and a $\Delta_0$ formula $\varphi$ such that for all $t$,

$$t \in D \quad \leftrightarrow \quad \varphi(\mathbf{x}, y, f, t).$$

Let $p \in G$ such that $\mathsf{Dom}\, p$ contains all elements of $\mathbf{x}$. Since $D$ is dense, let $d \leq p$ with $d \in D$. Next, we'll need a $c \in G$ which has the same range as $d$. Take $c = j \restriction z$, where $z = j^{-1}[\mathsf{Ran}(d)]$. By definition of $\mathbb{P}$, $\mathsf{Ran}(d)$ is well-orderable in $N'$. Since $j : N \to N'$ is an isomorphism, $z$ is well-orderable in $N$. It follows that $c = j \restriction z$ is in $\mathbb{P}$ (and hence in $G$). Observe that $\mathsf{Ran}(c) = \mathsf{Ran}(d)$, and consequently $\tilde{c} = \tilde{d}$. Also, $c \leq p$, since both are in $G$ and $\mathsf{Ran}(c) = \mathsf{Ran}(d) \supseteq \mathsf{Ran}(p)$.

Since $d \in D$, we have $N \models \varphi(\mathbf{x}, y, f, d)$, and hence $N' \models \varphi(d(\mathbf{x}), y', f', \tilde{d})$ by $(*)$. But $\tilde{d} = \tilde{c}$, and $d(\mathbf{x}) = c(\mathbf{x})$ since both $d$ and $c$ extend $p$. $N' \models \varphi(c(\mathbf{x}), y', f', \tilde{c})$ and $N \models \varphi(\mathbf{x}, y, f, c)$. Therefore, $c \in D \cap G$, which is what we needed to show that $G$ is generic. $\square$

## 4 Homogeneity

Suppose in Theorem 1.1 that we insist in condition (a) that $N$ be a permutation submodel of $M$ in the more traditional sense, with $G \in M$. How must (b) and (c) be restricted to preserve equivalence? The answer for (b) was determined in [3]; the result is as follows.

**Theorem 4.1** *Let $M$ be a transitive model of* ZFAC*, and let $N$ be a transitive subclass of $M$ which is a model of* ZFA *such that $N$ and $M$ have the same set of atoms and the same pure part. The following are equivalent:*

(a′) *$N$ is a permutation submodel of $M$ obtained from a group $G \in M$,*
(b′) *$M$ is a generic extension of $N$ by an almost homogeneous notion of forcing.*

We now present a condition (c′) which is equivalent to (a′) and (b′), analogous to (c) in Theorem 1.1. This (c′) will say that $N$ has an SVC witness with a certain homogeneity property.

**Definition 4.2** Working in ZFA, let $T$ be a set, and let $\mathbf{x}_1, \mathbf{x}_2$ be (transfinite) sequences of elements of $T$. We say that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same *$T$-type* if they satisfy the same $\Delta_1$ formulas using $T$ and pure sets as parameters. We say $\mathbf{x}_1$ and $\mathbf{x}_2$ are *$T$-isomorphic* if there is an $\in$-automorphism $F$ of $T$ such that $F(\mathbf{x}_1) = \mathbf{x}_2$.

**Theorem 4.3** *Under the hypotheses of Theorem 4.1, conditions* (a′) *and* (b′) *are equivalent to*

(c′) *$N$ satisfies* SVC *with a set $S$ whose power set $T = (\mathcal{P}(S))^N$ has the following homogeneity property: Any two sequences in $N$ with the same $T$-type are $T$-isomorphic in $M$.*

**Example 4.4** We consider the basic Fraenkel model and the ordered Mostowski model; see [5] for precise descriptions. The basic Fraenkel model $N$ is the minimal model of ZFA for a given pure part and set of atoms. In $N$ the set $S$ of finite sequences of atoms is an SVC witness. (To see this, check that forcing with $S$ yields a generic extension satisfying AC, and use Theorem 1.2). Sequences in $N$ of elements of $T = \mathcal{P}(S)$ are finite, and so it is not hard to see that sequences in $N$ with the same $T$-type are $T$-isomorphic, not only in some $M$ where AC holds, but in $N$.

The above example is not typical. Suppose $N$ is the ordered Mostowski model, a minimal model of ZFA such that $A$ has a dense linear order $\prec$, obtained as a permutation submodel of some $M$ where $A$ is countable. The set $S$ of finite partial order embeddings $A \to \mathbb{Q}$ is an SVC witness. In $N$, there are no nontrivial automorphisms of $\langle A, \prec\rangle$. As a result, although sequences of elements of $S$ with the same $S$-type are $S$-isomorphic in $M$, they are not usually $S$-isomorphic in $N$ (and the same is true with $S$ replaced by $T = \mathcal{P}(S)$).

**Proof of Theorem 4.3** Let $M$ and $N$ be as in the hypotheses of Theorem 4.1. First, assume that (c′) holds: $N$ satisfies SVC with $S$, and $T = (\mathcal{P}(S))^N$ satisfies the given homogeneity condition. We'll prove (b′). In $N$, define a notion of forcing $\mathbb{P}_1$ consisting of partial embeddings $T \to T'$, just as in the proof of Theorem 1.1, but replace ($*$) with

$$N \models \varphi(\mathbf{x}, p, y, f, T) \quad \leftrightarrow \quad N' \models \varphi(p(\mathbf{x}), \tilde{p}, y', f', T'),$$

and further require that the above hold not only for $\Delta_0$ formulas, but rather all $\Delta_1$ formulas. The proof that $M$ is a $\mathbb{P}_1$-generic extension of $N$ works as before; it remains to show that $\mathbb{P}_1$ is an almost homogeneous notion of forcing. Observe that $T'$ is almost homogeneous in $M'$, and every $g \in \mathsf{Aut}(T', \in)$ induces a $\hat{g} \in \mathsf{Aut}(\mathbb{P}_1, \leq)$ by $(\hat{g}p)(x) = g(p(x))$.

Let $p$ and $q$ be conditions in $\mathbb{P}_1$, and let $\mathbf{d}$ and $\mathbf{e}$ be well-orderings of their respective domains. Following the proof of Theorem 1.1, we have a fixed isomorphism (the priming function) $j : M \to M'$ in $M$. Now $j(\mathbf{d})$ and $p(\mathbf{d})$ must have the same type in $T'$, so let $g$ be an automorphism of $T'$ such that $g(p(\mathbf{d})) = j(\mathbf{d})$. Likewise, find $h$ such that $h(q(\mathbf{e})) = j(\mathbf{e})$. Then $\hat{g}(p)$ maps $\mathbf{d}$ to $j(\mathbf{d})$, and $\hat{h}(q)$ maps $\mathbf{e}$ to $j(\mathbf{e})$. Thus $\hat{g}(p)$ and $\hat{h}(q)$ are compatible, which shows that $\mathbb{P}_1$ is almost homogeneous.

Conversely, assume that (a′) holds: $N$ is a permutation submodel of $M$, by a group $G \in M$. It is shown in [3] (Lemma 4.8) that $M$ is a generic extension of $N$ by a notion of forcing called the *generator poset*, which we'll denote by $\mathbb{P}_G$. The rest of the proof will make use of several facts about $\mathbb{P}_G$ proved in [3]; hereafter *italicized* lemma and theorem numbers refer to that paper. There is a $\mathbb{P}_G$-name $\dot{f}$ which is $\Delta_0$-definable (from $\mathbb{P}_G$) such that $\mathbf{1}_{\mathbb{P}_G} \Vdash$ "$\dot{f} : \check{A} \to \check{A}'$ is a bijection" (*Lemma 4.8*), where $A'$ is a pure set as in Lemma 3.1; this $\dot{f}$ can be thought of as a name for an isomorphism $N \to N'$. SVC holds in $N$ with $\mathbb{P}_G$ (follows easily from *Theorem 4.13*), and thus SVC holds with $S = \mathbb{P}_G \cup \{\langle \mathbb{P}_G, \leq\rangle\}$. Let $T = (\mathcal{P}(S))^N$, and let $\mathbf{x}_1, \mathbf{x}_2 \in N$ be sequences of elements of $T$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same $T$-type. We want to show that $\mathbf{x}_1$ and $\mathbf{x}_2$ are $T$-isomorphic.

Let $y$ be a pure set such that $p_1 \Vdash \dot{f}(\check{\mathbf{x}}_1) = \check{y}$ for some $p_1 \in \mathbb{P}_G$. By definition of "same $T$-type," it must also be true that there is a $p_2 \in \mathbb{P}_G$ such that $p_2 \Vdash \dot{f}(\check{\mathbf{x}}_2) = \check{y}$. By *Lemma 4.10(c)*, there are filters $\Gamma_1$ and $\Gamma_2$ in $\mathbb{P}_G$, generic over $N$, such that $p_i \in \Gamma_i \in M$. Let $f_i = \mathsf{Val}_{\Gamma_i}(\dot{f})$; then $g = f_2^{-1} \circ f_1$ is in $G$ (by *Lemma 4.10(d)*).

Since $G$ is a group of $\in$-automorphisms of $N$ and acts on $\langle \mathbb{P}_G, \leq \rangle$ (*Lemma 4.6*), we have that $g$ is an $\in$-automorphism of $T$, and observe that $g(\mathbf{x}_1) = \mathbf{x}_2$. $\square$

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Department of Mathematics and Statistics
University of Missouri—Kansas City
Kansas City MO 64110-2499
halle@umkc.edu