Notre Dame Journal of Formal Logic
Volume 48, Number 1, 2007

# The Number of Countable Differentially Closed Fields

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**Abstract** We outline the Hrushovsk-Sokolović proof of Vaught's Conjecture for differentially closed fields, focusing on the use of dimensions to code graphs.

## 1 Introduction

Although there are no known natural examples of differentially closed fields, there are two strong reasons for the continued interest in their model theory.

1. Differentially closed fields provide useful universal domains for studying algebraic differential equations. The model theory of DCF has proved useful in studying differential Galois theory, differential algebraic groups, differential algebraic geometry and its applications to diophantine geometry.
2. Differentially closed fields exhibit many interesting model theoretic phenomena.

While much has been written on the former including [19], [17], [9], and [1], it is the latter that motivates this tutorial. In the introduction to *Saturated Model Theory* [21], Sacks describes differentially closed fields as the "least misleading" examples of $\omega$-stable theories. Many fundamental ideas from model theory have illustrative natural manifestations in the study of differential fields. These include

1. model theoretic algebra: quantifier elimination, model completeness;
2. classification theory: prime model extensions, forking, ranks, canonical bases, DOP, ENI-DOP;
3. geometric stability theory: geometry of strongly minimal sets, groups of definable automorphisms.

A perfect example of this is Vaught's Conjecture. In the early 1980s Shelah [22] proved that Vaught's Conjecture holds for $\omega$-stable theories. For nearly a decade we knew there were either $\aleph_0$ or $2^{\aleph_0}$ countable differentially closed fields, but the

Received June 19, 2006; accepted October 30, 2006; printed February 26, 2007
2000 Mathematics Subject Classification: Primary, 03C60, 03C45
Keywords: Vaught's conjecture, differentially closed fields, DOP

exact number was not known until Hrushovski and Sokolović [7] proved there were continuum many.[1] Indeed, there were plausible conjectures that would have implied there are only $\aleph_0$ countable models. Hrushovski and Sokolović proved that differentially closed fields have ENI-DOP and a key step in Shelah's proof is that $\omega$-stable theories with ENI-DOP have the maximal number of countable models.

My goal in this tutorial is to outline the proof of Vaught's Conjecture for differentially closed fields. In Section 1 I will describe in some simple cases the method of coding graphs into structures with ENI-DOP. My hope is that this method might be instructive to people working in computable model theory. Indeed, some of the examples below answer a question posed by Calvert during this workshop. In Sections 2 and 3 I will review some of the model theory of differentially closed fields and in Section 4 I will prove that there are continuum many countable models. I assume familiarity with model theory at the level of [13].

## 2 Coding Graphs with Dimensions

In [22] Shelah shows how to use the freedom to assign dimensions to code graphs into models. Since, for any infinite cardinal $\kappa$, there are $2^\kappa$ nonisomorphic graphs of size $\kappa$, this is a useful method for showing theories have the maximal number of models.

We will illustrate this in a very simple example. Let $\mathcal{L} = \{V, X, +, \pi, f\}$. We axiomatize an $\mathcal{L}$-theory $T_1$ as follows:

(i) models are the disjoint union of $V$ and $X$;
(ii) $(V, +)$ is a nontrivial torsion-free divisible Abelian group;
(iii) $\pi : X \to V$ is surjective;
(iv) $f : X \to X$ is a bijection with no cycles such that $\pi(x) = \pi(f(x))$.

Thus each fiber $\pi^{-1}(v)$ is isomorphic to a number of copies of the integers with a successor function. We let $\dim(\pi^{-1}(v))$ be the number of copies.

**Proposition 2.1** *Let $\kappa$ be an infinite cardinal. There are $2^\kappa$ nonisomorphic models of $T$ of cardinality $\kappa$.*

Let $\mathcal{M}_0$ be the prime model of $T_1$ over $A \subset V$, a linearly independent set of size $\kappa$. In $\mathcal{M}_0$, $\dim(\pi^{-1}(v)) = 1$ for all $v \in V$.

Let $G$ be an (irreflexive) graph with vertex set $A$ where every vertex has valence at least 2. Let

$$B = \{a + b : a, b \in A, (a, b) \in G\}.$$

We can directly construct $\mathcal{M}(G) \models T_1$ of cardinality $\kappa$ such that

$$\dim(\pi^{-1}(v)) = \aleph_0 \quad \text{for } v \in A \cup B, \text{ and}$$

$$\dim(\pi^{-1}(v)) = 1 \quad \text{for } v \in V \setminus (A \cup B).$$

We must show that $\mathcal{M}(G) \cong \mathcal{M}(H)$ if and only if $G \cong H$. We do this by showing that we can recover the graph $G$ from $\mathcal{M}(G)$.[2]

First note that

$$A \cup B = \{v \in V : \dim(\pi^{-1}(v)) = \aleph_0\}.$$

We say that $\{x, y, z\} \subseteq A \cup B$ is a *triangle* if $x, y, z$ are pairwise linearly independent but not linearly independent.

**Lemma 2.2** *Every triangle is of the form $\{a, b, a + b\}$ for some $a, b \in A$.*

**Proof** Suppose $\{x, y, z\}$ is a triangle; then $x, y, z \in A \cup B$ are pairwise linearly independent, but not independent. Clearly, any three elements of $A$ are linearly independent.

**Claim 2.3** *Any three elements of $B$ are linearly independent. There are several cases to consider. Suppose $a, b, c, d, e, f$ are distinct elements of $A$.*

(i) *$a + b$, $c + d$, and $e + f$ are linearly independent.*

(ii) *$a + b$, $c + d$, and $a + d$ are linearly independent, since*

$$\text{span}(a + b, c + d, a + d, a) = \text{span}(a, b, c, d)$$

*has dimension 4.*

(iii) *$a + b$, $a + c$, $b + c$ are linearly independent, since*

$$2a = (a + b) + (a + c) - (b + c)$$

*and*

$$\text{span}(a + b, b + c, a + c) = \text{span}(a, b, c).$$[3]

**Claim 2.4** *If $x \in A$ and $y, z \in B$, then $x, y, z$ are linearly independent. There are two cases to consider. Let $x, a, b, c, d$ be distinct elements of $A$.*

(i) *$x, a + b, c + d$ are linearly independent.*

(ii) *$x, a + b, b + c$ are linearly independent.*

(iii) *$x, x + a, b + c$ are linearly independent.*

(iv) *$x, x + a, x + b$ are linearly independent.*

Thus we must have two elements $x, y \in A$ and one element $z \in B$. If $x, y, c, d$ are distinct elements of $A$, then $x, y, x+c$ and $x, y, c+d$ are linearly independent. Thus we must have $z = x + y$. □

Since every vertex has valence at least 2,

$$A = \{a \in S : a \text{ is in at least two triangles}\}$$

and $(a, b)$ is an edge if and only if there is a $c \in S$, $\{a, b, c\}$ is a triangle. Thus we can recover $G$ from $\mathcal{M}(G)$. If $G \ncong G'$, then $\mathcal{M}(G) \ncong \mathcal{M}(G')$.

During the Vaught's Conjecture Workshop, Calvert asked if there is an $\omega$-stable theory $T$ such that the isomorphism relation on computable models of $T$ is complete $\Sigma_1$. See [3] for a discussion of these concepts. The class of graphs is complete $\Sigma_1$, as is the class of graphs where each vertex has valence at least 2. We can construct $\mathcal{M}(G)$ from $G$ in a uniform computable manner. Thus isomorphism for models of $T_1$ is complete $\Sigma_1$.

We conclude with some variants on this example. Below, we let $I(T, \kappa)$ denote the number of nonisomorphic models of $T$ of cardinality $\kappa$.

**Exercise 2.5** Let $\mathcal{L} = \{V, X, +, \pi\}$. Let $T_2$ be the theory (i)–(iii) as above, but replacing (iv) with

(iv)′ $\pi^{-1}(v)$ is infinite for all $v \in V$.

Prove that $I(T_2, \kappa) = 2^\kappa$ for $\kappa$ uncountable, but $I(T, \aleph_0) = \aleph_0$.

The problem here is that in countable models we have no choices for $\dim(\pi^{-1}(v))$, while in uncountable models we can make this either countable or uncountable. This is an example that has DOP but not ENI-DOP. (See Definition 5.7.)

**Exercise 2.6** Modify the example to give an $\aleph_0$-categorical $\omega$-stable theory with $2^\kappa$ models of cardinality $\kappa$ for all uncountable cardinals $\kappa$.

**Exercise 2.7** Let $\mathcal{L} = \{V, X, +, \pi, f\}$. Let $T_3$ be the theory (i), (iii), (iv) as in $T_1$, but replacing (ii) with

(ii)$'$ $(V, +)$ is an infinite vector space over the two element field $\mathbb{F}_2$.

Prove that $I(T_3, \kappa) = 2^\kappa$ for all infinite $\kappa$. (Hint: Lemma 2.2 causes some problems that can be avoided by using triangle-free graphs.)

**Exercise 2.8** Let $\mathcal{L} = \{V, X, s, \pi\}$. Let $T_4$ be the theory (i), (iii), (iv) as above, but replacing (ii) with

(ii)$''$ $(V, s)$ is a model of the theory of $\mathbb{Z}$ with successor.

Prove that $I(T_4, \aleph_\alpha) \leq (\alpha + \aleph_0)^{(\alpha+\aleph_0)}$.

**Observations** For this method of coding graphs using dimensions to work, we seem to need

1. large family of types $(p_a : a \in A)$, $p_a \in S(a)$ to which we can assign dimensions (for building many countable models we would like to be able to assign different countable dimensions);
2. the ability to realize one type in the family while omitting others (orthogonality);
3. a good notion of independence in $A$ with lots of elements $a, b, c \in A$, pairwise independent but not independent (nontriviality).

## 3 Differentially Closed Fields

We will review without proofs most of the basic model theory of differentially closed fields. Proofs and further references can be found in [14].

A *differential ring* is a ring $R$ with a *derivation* $\delta : R \to R$, that is, a map such that

(i) $\delta(a + b) = \delta(a) + \delta(b)$;
(ii) $\delta(ab) = a\delta(b) + b\delta(a)$.

We will sometimes use the notation $a'$ for $\delta(a)$, and $a^{(n)}$ for the $n$th derivative of $a$.

If $(R, \delta)$ is a differential ring, we let $R\{X_1, \ldots, X_n\}$ be the ring

$$R[X_1, \ldots, X_n, X_1', \ldots, X_n', \ldots, X_1^{(m)}, \ldots, X_n^{(m)}, \ldots].$$

We extend the derivation to $R\{X_1, \ldots, X_n\}$ by defining $\delta(X_i^{(j)}) = X_i^{(j+1)}$. We call $R\{X_1, \ldots, X_n\}$ the ring of *differential polynomials* over $R$.

**Definition 3.1** We say that a differential field $K$ is differentially closed if for any differential field $L \supseteq K$ if $f_1, \ldots, f_n \in K\{X_1, \ldots, X_m\}$ and the system

$$f_1(X_1, \ldots, X_n) = \cdots = f_n(X_1, \ldots, X_n) = 0$$

has a solution in $L$, then it also has one in $K$.

We let DCF be the theory of differentially closed fields of characteristic zero.

Robinson showed that DCF is axiomatizable, complete, model complete, and decidable. These results were extended by Blum who gave a surprisingly simple axiomatization using only differential polynomials in one variable.[4] For $f(X) \in K\{X\}$ the *order* of $f$ is the largest $m$ such that $X^{(m)}$ occurs in $X$. If $f$ is a constant, we say that the order of $f$ is $-1$. We let $\mathrm{ord}(f)$ denote the order of $f$.

**Theorem 3.2 (Blum)** *Let $K$ be a differential field of characteristic zero.*

(a) $K \models \mathrm{DCF}$ *if and only if for all nonzero $f, g \in K\{X\}$ where* $\mathrm{ord}(f) > \mathrm{ord}(g)$, *there is $a \in K$ with $f(a) = 0$ and $g(a) \neq 0$.*
(b) DCF *has quantifier elimination.*

For proofs see §2 of [14] or Theorem 4.3.32 of [13]. Quantifier elimination can be given a geometric interpretation. If $K$ is a differential field we can define a differential analog of the Zariski topology.

**Definition 3.3** We say $V \subseteq K^n$ is *Kolchin closed* if $V$ is a finite union of sets of the form

$$\{\overline{x} \in K^n : f_1(\overline{x}) = \cdots = f_m(\overline{x}) = 0\}$$

where $f_1, \ldots, f_m \in K\{\overline{X}\}$. We say that $X \subseteq \mathbb{K}^n$ is *$\delta$-constructible* if it is a finite Boolean combination of Kolchin closed sets.

Quantifier elimination says that the projection of a $\delta$-constructible set is $\delta$-constructible.

To show that the Kolchin closed sets are the closed sets of a topology we need to know that there are no infinite descending chains of Kolchin closed sets. This follows from the differential analog of Hilbert's Basis Theorem (see [14], 1.16).

**Theorem 3.4 (Ritt-Raudenbush Basis Theorem)** *If $R \supseteq \mathbb{Q}$ is a differential ring where every radical differential ideal is finitely generated, then every radical differential ideal in $R\{X\}$ is finitely generated.*

Quantifier elimination also leads to an algebraic description of types. Suppose $K \models \mathrm{DCF}$ and $k$ is a differential subfield of $K$. If $a_1, \ldots, a_n \in K^n$, then $\mathrm{tp}(\overline{a}/k)$ is determined by

$$\{f \in k\{X_1, \ldots, X_n\} : f(\overline{a}) = 0\}$$

a prime differential ideal.

Another consequence of the Ritt-Raudenbush Basis Theorem is that every prime differential ideal in $k\{X_1, \ldots, X_n\}$ is finitely generated. Thus there are only $|k|$ differential ideals, and hence $|k|$ $n$-types over $k$.

**Corollary 3.5** DCF *is $\omega$-stable.*

We can now use all of the tools for $\omega$-stable theories to study differentially closed fields.

**Definition 3.6** Let $k$ be a differential field of characteristic zero. If $K \models \mathrm{DCF}$ and $K \supseteq k$ we say that $K$ is a *differential closure* of $k$ if for any $L \models \mathrm{DCF}$ with $k \subseteq L$, there is a differential embedding of $K$ into $L$ fixing $k$.

A differential closure of $k$ is simply a prime model of DCF over $k$. We can apply Morley's existence and Shelah's uniqueness theorems in this context (see [13], 4.2.20 and 6.4.8).

**Corollary 3.7**

(i) *Every differential field has a differential closure.*
(ii) *If $K$ is a differential closure of $k$, and $\overline{a} \in K^n$, then $\mathrm{tp}(\overline{a}/k)$ is isolated.*
(iii) *If $K$ and $L$ are differential closures of $k$, then $K$ and $L$ are isomorphic over $k$.*

An interesting feature of differential closures is that they need not be minimal. For example, it was shown independently by Rosenlicht, Kolchin, and Shelah (see [14], §6) that if $K$ is the differential closure of $\mathbb{Q}$, then there is $L \subset K$ a proper differential subfield with $L \cong K$.

Our next goal is to survey some interesting definable sets and types. We work in a large saturated model $\mathbb{K}$ of DCF. The first interesting definable set is the *constant field*. Let $C = \{x \in \mathbb{K} : \delta(x) = 0\}$.

**Proposition 3.8** *$C$ is an algebraically closed field. Moreover, if $X \subseteq C^n$ is definable in $\mathbb{K}$, then $X$ is already definable in the field $(C, +, \cdot)$. We say that $C$ is a* pure *algebraically closed field.*

**Proof** First note that $\mathbb{K}$ is algebraically closed. Next, suppose $a \in \mathbb{K}$ and $\sum_{i=0}^{n} c_i X^i \in C[X]$ is the minimal polynomial of $a$ over $C$. Then

$$0 = \delta\left(\sum_{i=0}^{n} c_i a^i\right) = \left(\sum_{i=0}^{n} i c_i a^{i-1}\right)\delta(a).$$

Since we were considering the minimal polynomial, we must have $\delta(a) = 0$. Thus $C$ is algebraically closed.

Suppose $X \subseteq C^n$ is definable in $\mathbb{K}$. By quantifier elimination and the fact that $\delta$ is trivial on $C$, there is $Y \subseteq \mathbb{K}^n$ definable in $(\mathbb{K}, +, \cdot)$ such that $X = C^n \cap Y$. Using the stability of algebraically closed fields, $X$ is definable in $(C, +, \cdot)$. (See [13], 6.6.21) □

Recall that a definable $X \subseteq \mathbb{K}$ is *strongly minimal* if and only if it is infinite and every proper definable subset is either finite or cofinite. Since algebraically closed fields are strongly minimal, $C$ is an example of a strongly minimal set in $\mathbb{K}$.

Using $C$ we can analyze other important definable sets. We say that $f \in \mathbb{K}\{X\}$ is *linear* if and only if $f(X) = \sum_{n=1}^{n} a_m X^{(m)}$, for some $a_1, \ldots, a_m \in \mathbb{K}$. The usual theory of linear differential equations (see [14], §4) can be used to analyze zero sets of linear differential polynomials.

**Lemma 3.9** *If $f(X) \in \mathbb{K}\{X\}$ is linear of order $n$, then $V = \{x \in \mathbb{K} : f(x) = 0\}$ is an $n$-dimensional vector space over $C^n$.*

Since $C$ is a pure algebraically closed field and $V$ is definably isomorphic to $C^n$, $V$ must have Morley rank exactly $n$. For nonlinear differential polynomials $g \in \mathbb{K}\{X\}$, the Morley rank of the zero set may not equal $\mathrm{ord}(g)$, but $\mathrm{ord}(g)$ is always an upper bound. For proofs and examples showing nonequality, see [14], §5.

If $k \subseteq \mathbb{K}$ and $\overline{a} \in \mathbb{K}^n$, we let $k\langle\overline{a}\rangle$ denote the differential subfield generated by $k(\overline{a})$. We let $\mathrm{td}(k\langle\overline{a}\rangle/k)$ denote the transcendence degree of $k\langle\overline{a}\rangle$ over $k$.

**Proposition 3.10** *If $\overline{a} \in \mathbb{K}^n$ and $\mathrm{td}(k\langle\overline{a}\rangle/k)$ is finite, then $\mathrm{RM}(a/k) \leq \mathrm{td}(k\langle\overline{a}\rangle/k)$.*

**Definition 3.11** We say that $a \in \mathbb{K}$ is *$\delta$-transcendental* over $k$ if $f(a) \neq 0$ for all nonzero $f \in k\{X\}$. Otherwise, we say $a$ is $\delta$-algebraic over $a$. More generally, we say that $A \subseteq \mathbb{K}$ is *$\delta$-independent* over $k$ if $f(a_1, \ldots, a_n) \neq 0$ for any $n$, any $a_1, \ldots, a_n \in A$, and any nonzero $f \in k\{X_1, \ldots, X_n\}$.

It is worth noting that $a$ may be $\delta$-algebraic over $k$ without being model theoretically algebraic. For example, if $a \in k$, then a generic solution to $X' = a$ will be $\delta$-algebraic but not algebraic.

Suppose $k \subseteq \mathbb{K}$ and $a \in \mathbb{K}$ is $\delta$-transcendental over $k$. Then $\mathrm{tp}(a/k)$ is the unique 1-type corresponding to the ideal $\{0\}$. By Proposition 3.10, all other 1-types over $k$ have finite Morley rank. By the analysis of linear differential polynomials, $\mathrm{RM}(a/k\langle a^{(n)}\rangle) = n$. Thus $\mathrm{RM}(a/k) = \omega$.

It follows from Theorem 3.7 that if $a$ is in the differential closure of $k$, then $a$ is $\delta$-algebraic over $k$. Suppose not. Then $\mathrm{tp}(a/k)$ is isolated, but $\mathrm{tp}(a/k)$ is determined by $\{f(v) \neq 0 : f \in k\{X\}\}$. Thus the isolating formula can be taken to be

$$f_1(v) \neq 0 \wedge \cdots \wedge f_n(v) \neq 0$$

for some $f_1, \ldots, f_n \in K\{X\}$. Let $m > \mathrm{ord}(f_n)$ for all $n$. There is $b$ in the differential closure of $k$ such that

$$b^{(m)} = 0 \wedge \prod_{i=1}^{m} f_i(b) \neq 0.$$

Since $f_1(b) \neq 0, \ldots, f_n(b) \neq 0$, $\mathrm{tp}(a/k) = \mathrm{tp}(b/k)$, a contradiction since $a^{(m)} \neq 0 = b^{(m)}$.

Proposition 3.10 is a special case of a more general relationship between model theoretic and algebraic independence. Recall that Morley rank gives us a natural notion of independence in any $\omega$-stable theory.

**Definition 3.12** We say that $\overline{a}$ is *independent* from $B$ over $A$ if

$$\mathrm{RM}(\overline{a}/A \cup B) = \mathrm{RM}(\overline{a}/A).$$

We write $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A B$.

**Example 3.13** If $a_0, \ldots, a_n$ are $\delta$-independent over $k$, then $a_0$ is $\delta$-transcendental over $k\langle a_1, \ldots, a_n\rangle$. Thus

$$\mathrm{RM}(a_0/k) = \omega = \mathrm{RM}(a_0/k, a_1, \ldots, a_n)$$

and $a_0 \mathop{\smile\!\!\!\!\!\!|}_k a_1, \ldots, a_n$.

**Example 3.14** Let $a$ be $\delta$-transcendental over $k$. Then $a \not\mathop{\smile\!\!\!\!\!\!|}_k a'$, since over $k\langle a'\rangle$, $a$ satisfies the rank 1 formula $X' = a'$.

The following is one of the basic properties of independence; for a proof see [13], 6.3.19.

**Lemma 3.15 (Symmetry)** *If $\overline{a} \mathop{\smile\!\!\!\!\!\!|}_A \overline{b}$, then $\overline{b} \mathop{\smile\!\!\!\!\!\!|}_A \overline{a}$.*

Independence has a concrete algebraic meaning in differentially closed fields.

**Definition 3.16** Let $k \subseteq l_1, l_2$ be fields. $l_1$ and $l_2$ are *free* over $k$ if any $a_1, \ldots, a_n \in l_1$ algebraically dependent over $l_2$ are already algebraically dependent over $k$.

**Theorem 3.17** *If $k$ is a differential field and $\overline{a}, B \subseteq \mathbb{K} \models \mathrm{DCF}$, then the following are equivalent*

(i) $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_k B$,
(ii) $k\langle\overline{a}\rangle$ *and* $k\langle B\rangle$ *are free over* $k$.

For a proof see [14], 5.6.

We will need the following lemma.

**Lemma 3.18** *If $a$ is $\delta$-transcendental over $k$ and $\mathrm{RM}(\overline{b}/k) < \omega$, then $a \mathop{\smile\!\!\!\!\!\!\mid}_k \overline{b}$.*

**Proof** We first note that $\mathrm{td}(k\langle\overline{b}\rangle/k)$ is finite. If not, then some $b_i$ is differentially transcendental over $k$ and $\mathrm{RM}(\overline{b}/k) \geq \omega$.

If $a \not\mathop{\smile\!\!\!\!\!\!\mid}_k \overline{b}$, then $k\langle a, \overline{b}\rangle$ has finite transcendence degree over $k\langle\overline{b}\rangle$. But then $k\langle a, \overline{b}\rangle$ has finite transcendence degree over $k$, a contradiction. $\square$

In our examples in Section 1 we needed the ability to realize some types while omitting others.

**Definition 3.19** Let $p \in S(A)$, $q \in S(B)$. We say $p$ is orthogonal to $q$ and write $p \perp q$ if $\overline{a} \mathop{\smile\!\!\!\!\!\!\mid}_M \overline{b}$ for any $M \supseteq A \cup B$, $a$ realizing $p$ and $b$ realizing $q$ with $a \mathop{\smile\!\!\!\!\!\!\mid}_A M$ and $b \mathop{\smile\!\!\!\!\!\!\mid}_B M$.

**Lemma 3.20** *Suppose $X$ is a strongly minimal set defined over $K \models \mathrm{DCF}$, $p$ is the generic type of $X$ over $K$, and $p \perp q$. Let $\overline{b}$ realize $q$. Then $p$ is omitted in the differential closure of $K\langle\overline{b}\rangle$.*

**Proof** Suppose $\overline{a}$ realizes $p$ in the differential closure of $K\langle\overline{b}\rangle$. There is $\varphi(\overline{v})$ isolating $\mathrm{tp}(\overline{a}/K\langle\overline{b}\rangle)$. Since $p \perp q$, $\overline{a}$, and $\overline{b}$ are independent over $K$. Thus $\overline{a}$ is not field-theoretically algebraic over $K\langle\overline{b}\rangle$ and $\mathrm{RM}(\varphi) = 1$. Since $X$ is strongly minimal, $\varphi$ holds of some elements of $X(K)$, a contradiction. $\square$

For strongly minimal sets $A$ and $B$ we say $A \perp B$ if their generic types are orthogonal.

## 4 Strongly Minimal Sets in DCF

To carry out a construction as in Section 1 we will need to find families of orthogonal types. The types we will consider are generic types of strongly minimal sets. We begin by reviewing some of the basics on the geometry of strongly minimal sets. Recall that if $X$ is a strongly minimal set we define a closure relation cl on $X$ such that for $A \subseteq X$, $\mathrm{cl}(A)$ is the model theoretic algebraic closure of $A$, that is, the set of all $b \in X$ such that there is a formula $\varphi(v)$ with parameters from $A$ such that $\varphi(b)$ and $\{x : \varphi(x)\}$ is finite. In differentially closed fields, $b$ is model theoretically algebraic over $A$ if and only if $b$ is in the field-theoretic algebraic closure of the differential field generated by $A$ (see [14], 5.1).

**Definition 4.1** A strongly minimal set $X$ is *trivial* if

$$\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(\{a\})$$

for all $A \subseteq X$.

Examples of trivial strongly minimal theories include

1. the theory of an infinite set with no additional structure, in this case
$$\mathrm{cl}(A) = A;$$
2. the theory of $(\mathbb{Z}, s)$ where $s(x) = x + 1$, in this case
$$\mathrm{cl}(A) = \{x : x \text{ is reachable from } A \text{ with finitely many iterations of } s \text{ or } s^{-1}\}.$$

**Definition 4.2** A strongly minimal set $X$ is *locally modular* if for all $b, c \in X$ and $A \subseteq X$ if $c \in \mathrm{cl}(A \cup \{b\})$, then there is $a \in \mathrm{cl}(A)$ such that $c \in \mathrm{cl}(a, b)$.

Any trivial strongly minimal set is locally modular. There are also nontrivial examples. For example, if $X$ is a divisible torsion-free Abelian group, then $\mathrm{cl}(A)$ is the span of $A$ as a $\mathbb{Q}$-vector space. If $c \in \mathrm{cl}(A \cup \{b\})$, then there are $a_1, \ldots, a_n \in A$ and rationals $m_1, \ldots, m_n, m$ such that
$$c = m_1 a_1 + \cdots + m_n a_n + mb.$$
Letting $a = m_1 a_1 + \cdots + m_n a_n \in \mathrm{cl}(A)$, we see that $c \in \mathrm{cl}(a, b)$. Thus $X$ is locally modular. A key result of Hrushovski shows that any nontrivial locally modular strongly minimal set is nonorthogonal to an interpretable strongly minimal group (see [16]).

Algebraically closed fields are the natural examples of nonlocally modular strongly minimal sets. In this case cl is field-theoretic algebraic closure. If $a_0, \ldots, a_{n-1}$ are algebraically independent and
$$b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = 0,$$
then $b \in \mathrm{cl}(a_0, \ldots, a_{n-1})$, but $b$ is not in the algebraic closure of any subfield of $\mathrm{cl}(a_0, \ldots, a_{n-1})$ lower transcendence degree.

For a time, algebraically closed fields were the only known examples of nonlocally modular strongly minimal sets. This led Zilber to conjecture that there were no others. Hrushovski [8] showed that there are many such examples, some of which even have no infinite interpretable groups. Nevertheless, Zilber's conjecture has proved to be true in several natural contexts including differentially closed fields.

One of the fascinating properties of differentially closed fields is that all species of strongly minimal sets arise. The first natural example is the constant field $C$. Hrushovski and Sokolović [7] proved that, up to nonorthogonality, $C$ is the only nonlocally modular strongly minimal set.

**Theorem 4.3** *If $X \subseteq \mathbb{K}^n$ is a nonlocally modular strongly minimal set, then there is a definable finite-to-one function $f : X \to C$. In particular, $X \not\perp C$.*

Hrushovski and Sokolović's original proof used the work of Hrushovski and Zilber [10] on Zariski geometries. Pillay and Ziegler [18] later gave a more straightforward geometric proof avoiding Zariski geometries.

We also know of many examples of trivial strongly minimal sets. For example,
$$X' = X^3 - X \text{ and } X' = \frac{X}{X+1}$$
define infinite sets on indiscernibles with no structure (see [14], §6). These examples arose independently in proofs by Rosenlich, Kolchin, and Shelah that the differential closure of $\mathbb{Q}$ is not minimal. Hrushovski and Itai [6] and Rosen [20] have also constructed interesting families of trivial strongly minimal sets living on curves of

genus $g \geq 2$. While there are many orthogonal trivial strongly minimal sets, they won't be helpful in building countable models as all of the known examples are infinite dimensional.[5] Indeed, the following conjecture remains open.

**Conjecture 4.4** *In any differentially closed field, every trivial strongly minimal set is infinite dimensional.*

Thus we need to find families of nontrivial locally modular strongly minimal sets. These sets already arose in differential algebraic geometry in the work of Manin [11] and Buium [2]. Before describing these sets we review some algebraic geometry. We refer to [5] for facts about Abelian varieties.

Let $K$ be an algebraically closed field of characteristic 0.

**Definition 4.5** An *Abelian variety* is a subvariety $A \subseteq \mathbb{P}^n(K)$ such that there is a rational map $m : A \times A \to A$ making $A$ into a group.

The simplest example is an elliptic curve

$$Y^2 = X^3 + aX + b$$

together with a point $O = (0, 1, 0)$ at infinity. For an elliptic curve $E$ the group law is given so that

1. $O$ is the identity of the group;
2. $(x, -y)$ is the inverse of $(x, y)$;
3. for distinct points $A, B, C \in E$

$$A + B + C = O$$

   if and only if $A, B, C$ are colinear.

**Proposition 4.6** *Every Abelian variety is a divisible commutative group. If A has dimension d, then there are $n^{2d}$ points of order n.*

**Definition 4.7** We say $A$ is *simple* if $A$ has no proper infinite Abelian subvarieties.

**Definition 4.8** Abelian varieties $A$ and $B$ are *isogenous* if there is a rational group homomorphism $f : A \to B$ with finite kernel.

We will need more detailed information about elliptic curves. Consider the elliptic curve $E$

$$Y^2 = X^3 + aX + b.$$

The *j-invariant* of the curve $j(E)$ is $\frac{6912a^3}{4a^3+27b^2}$.

**Theorem 4.9**

(i) *Let L be an algebraically closed field of characteristic 0. For $j \in L$ there is E defined over L with $j(E) = j$.*
(ii) *$E \cong E_1$ if and only if $j(E) = j(E_1)$.*
(iii) *If E and $E_1$ are isogenous, then $j(E)$ and $j(E_1)$ are interalgebraic over $\mathbb{Q}$.*

In classical algebraic geometry, Abelian varieties are very different from linear algebraic groups. Indeed any rational homomorphism from an Abelian variety to a linear algebraic group is constant. However, using the derivation we are able to get differential-algebraic group homomorphisms and these frequently give rise to strongly minimal sets. For proofs see [12] and [15].

**Theorem 4.10 (Manin-Buium)** *Let $K$ be a differentially closed field. If $A$ is an Abelian variety defined over $K$, there is a definable homomorphism $\mu : A \to K^n$ such that the kernel of $\mu$ is the Kolchin closure of the torsion points of $A$.*

For example, if $E$ is the elliptic curve

$$Y^2 = X^3 + aX + b$$

where $a, b \in C$ then $\mu(x, y) = \dfrac{x'}{y}$.

Let $A^\sharp$ be the Kolchin closure of the torsion points. If $A$ is defined over $C$, then $A^\sharp = A(C)$, the points of $A$ in $\mathcal{P}_n(C)$. We can now give a complete characterization of locally modular strongly minimal sets in DCF.

**Theorem 4.11 (Hrushovski–Sokolović)** *If $A$ is a simple Abelian variety that is not isomorphic to an Abelian variety defined over the constants, then $A^\sharp$ is a locally modular strongly minimal set. If $B$ is another Abelian variety that is not isomorphic to an Abelian variety defined over the constants, then $A^\sharp$ and $B^\sharp$ are nonorthogonal if and only if $A$ and $B$ are isogenous. Moreover, if $X \subseteq \mathbb{K}^n$ is any nontrivial modular strongly minimal set, then $X$ is nonorthogonal to $A^\sharp$ for some simple Abelian variety $A$.*

## 5 Constructing Many Models

Let $\kappa$ be an infinite cardinal. We are now ready for the proof that there are $2^\kappa$ non-isomorphic differentially closed fields of cardinality $\kappa$.

For $a \in \mathbb{K}$, let $E(a)$ be the elliptic curve with $j$-invariant $a$, let $E(a)^\sharp$ be the $\delta$-closure of the torsion points and let $p_a \in S(a)$ be the generic type of $E(a)^\sharp$. Let $r$ be the type of a $\delta$-transcendental element.

For $k \subseteq \mathbb{K}$ we let $k^{\mathrm{alg}}$ be the algebraic closure of $k$ and $k^{\mathrm{dif}}$ be a differential closure of $k$ in $\mathbb{K}$. We make several observations.

1. $E(a)^\sharp$ is strongly minimal and the type $p_a$ is determined by $\overline{x} \in E(a)^\sharp$, $\overline{x} \notin \mathbb{Q}\langle a\rangle^{\mathrm{alg}}$.
2. The torsion points of $E(a)$ are contained in $\mathbb{Q}(a)^{\mathrm{alg}}$. Hence $\mathbb{Q}\langle a\rangle$ contains infinitely many points of $E(a)^\sharp$.
3. If $p_a \not\perp p_b$, then $E(a)$ and $E(b)$ are isogenous and $a$ and $b$ are interalgebraic over $\mathbb{Q}$.
4. By Lemma 3.18, $p_a \perp r$ for all $a \in \mathbb{K}$.

**Lemma 5.1** *$p_a$ is not realized in $\mathbb{Q}\langle a\rangle^{\mathrm{dif}}$.*

**Proof** Suppose $\overline{b} \in \mathbb{Q}\langle a\rangle^{\mathrm{dif}}$ realizes $p_a$. Let $\varphi(v)$ isolate $\mathrm{tp}(b/\mathbb{Q}\langle a\rangle)$. Since $\overline{b} \notin \mathbb{Q}\langle a\rangle^{\mathrm{alg}}$, $\varphi(v)$ defines an infinite subset of $E(a)^\sharp$, but then it must contain a torsion point of $E(a)$. But the torsion points are in $\mathbb{Q}(a)^{\mathrm{alg}}$, a contradiction. □

Let $G$ be a graph with vertex set $A$ where $|A| = \kappa$ and every vertex has valence at least 2. Let $K_0 = \mathbb{Q}\langle A\rangle^{\mathrm{dif}}$ where the elements of $A$ are $\delta$-independent. Let $B = \{a + b : a, b \in A, (a, b) \in G\}$. Note that the elements of $B$ are also $\delta$-transcendental. The types $\{p_c : c \in A \cup B\}$ are pairwise orthogonal.

**Lemma 5.2** *If $a \in A \cup B$, then $p_a$ is omitted in $K_0$.*

**Proof** Suppose $a \in A$ (the other case is similar). By Lemma 5.1, $p_a$ is omitted in $\mathbb{Q}\langle a\rangle^{\mathrm{dif}}$. Then $p_a$ is omitted in $K_0 \cong (\mathbb{Q}\langle a\rangle^{\mathrm{dif}})\langle A \setminus \{a\}\rangle)^{\mathrm{dif}}$, since $r \perp p_a$. □

**Lemma 5.3** *There is* $K(G) \models \mathrm{DCF}$ *with* $K(G) \supset K_0$, $|K(G)| = |G|$ *where, if* $c \in A \cup B$, $\dim(p_c, K(G)) = 0$ *while if* $c$ *is* $\delta$*-transcendental and* $p_c \perp p_a$ *for all* $a \in A \cup B$, *then* $\dim(p_a, K(G)) = \aleph_0$.

**Proof** We build $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_\alpha \ldots$. Suppose $c \in K_\alpha$ and $p_c \perp p_a$ for all $a \in A \cup B$. We can build $K_{\alpha+1} \supseteq K_\alpha$ realizing a new realization of $p_c$ and adding no new realizations of $p_a$ for $a \in A \cup B$. With careful bookkeeping we construct $K(G) = \bigcup K_\alpha$. $\square$

We now must show that we can reconstruct the graph $G$ from $K(G)$.

**Lemma 5.4** $\not\perp$ *is an equivalence relation on realizations of* $r$.

**Proof** For $a, b$ realizing $r$, $a \not\perp b$ if $a$ is differentially algebraic over $k\langle b\rangle$. If $a \not\perp b$ and $b \not\perp c$, then $\mathbb{Q}\langle a, b, c\rangle$ is differentially algebraic over $\mathbb{Q}\langle a, b\rangle$ which is differentially algebraic over $\mathbb{Q}\langle a\rangle$. Thus $a \not\perp c$. $\square$

Let $[a]$ be the $\not\perp$-class of $a$. Let $S = \{[a] : a \text{ realize } r, \dim(p_a, K(G)) = 0\}$. For each $[a] \in S$ there is a unique $c \in A \cup B$ such that $[c] = [a]$, since if $p_c \not\perp p_a$ for some $a \in A \cup B$, then $E(c)$ and $E(a)$ are isogenous and $a$ and $c$ are field-theoretically interalgebraic.

We say that $\{[a], [b], [c]\} \in S^3$ is a *triangle* if $a, b, c$ are pairwise independent but not independent. This does not depend on choice of representative. If, say, $a_1 \not\perp b_1$, then $a \not\perp a_1 \not\perp b_1 \not\perp b$, and, since $\not\perp$ is an equivalence relation, $a \not\perp b$. Since

$$\mathbb{Q}\langle a_1, b_1\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b, c\rangle \subseteq \mathbb{Q}\langle a_1, b_1, a, b, c, c_1\rangle$$

and each of these extensions is of finite transcendence degree, the transcendence degree of $\mathbb{Q}\langle a_1, b_1, c_1\rangle$ over $\mathbb{Q}\langle a_1, b_1\rangle$ is finite and $c_1 \not\perp a_1, b_1$. Hence $a_1, b_1, c_1$ are pairwise independent but not independent.

**Proposition 5.5** *Every triangle is of the form* $\{[a], [b], [a+b]\}$ *where* $a, b \in A$.

**Proof** There are $x, y, z \in A \cup B$ where the triangle is $([x], [y], [z])$. We consider four cases.

**Case 1** $x, y, z \in A$. Any three distinct elements of $A$ are independent, so this is not a triangle.

**Case 2** $x, y, z \in B$. Up to permutation, we may assume that there are independent $a, b, c, d, e, f \in A$ such that $x = a + b$ and one of the following cases occurs.

(i) $y = a + c$ and $z = a + d$. Since $d$, and hence $z$, is $\delta$-transcendental over $\mathbb{Q}\langle a, b, c\rangle \supset \mathbb{Q}\langle x, y\rangle$, $x, y, z$ is independent, a contradiction.
(ii) $y = a + c$ and $z = b + d$. Similar to (i).
(iii) $y = a + c$ and $z = b + c$. In this case $\mathbb{Q}\langle a, b, c\rangle = \mathbb{Q}\langle x, y, z\rangle$, a contradiction.
(iv) $y = a + c$ and $z = d + e$. Since $\mathbb{Q}\langle z, e\rangle = \mathbb{Q}\langle d, e\rangle$ has $\delta$-transcendence degree 2 over $\mathbb{Q}\langle a, b, c\rangle$, $z$ is differentially transcendental over $\mathbb{Q}\langle x, y\rangle$, a contradiction.
(v) $y = c + d$ and $z = e + f$. Similar to (iv).

**Case 3** $x \in A$, $y, z \in B$. There are four possibilities to consider.

(i) $y = x + a$, $z = x + b$ where $a, b \in A$ are distinct. In this case $x, y, z$ are interdefinable with $x, a, b$ and hence independent.

(ii) $y = x + a$, $z = b + c$ where $a, b, c \in A$ are distinct. In this case $b$ and $c$ and hence $z$ are independent from $a, x$ which is interalgebraic with $x, y$. Thus $x, y, z$ are independent.

Similar arguments work to rule out the latter two cases.

(iii) $y = a + b$, $z = a + c$ where $a, b, c \in A$ are distinct.

(iv) $y = a + b$, $z = c + d$ where $a, b, c, d \in A$ are distinct.

**Case 4** $x, y \in A$ and $z \in B$. There are three possibilities: $z = x + a$ where $a \in A \setminus \{x, y\}$, $z = a + b$ where $a, b \notin \{x, y\}$, or $z = x + y$. In the first two cases $z \mathop{\smile\!\!\!\!\!\!|}\, x, y$, contradicting the assumption that $x, y, z$ is not independent. Thus we have $z = x + y$. □

We can now recover the graph. Let $V = \{[a] \in S :$ there are at least two triangles containing $[a]\}$. Since every vertex has valence at least two, $V = \{[a] : a \in A\}$.

Let $E = \{([a], [b]) :$ there is a triangle $\{[a], [b], [c]\}\}$. Then $(V, E) \cong G$.

**Theorem 5.6** *$\kappa \geq \aleph_0$. There are $2^\kappa$ nonisomorphic* DCF *of cardinality $\kappa$.*

For $\kappa > \aleph_0$, this was proved by Poizat using trivial strongly minimal sets instead of $E(a)^\sharp$.

Differentially closed fields also provide an example for Calvert's question about $\omega$-stable theories where the isomorphism problem is complete $\Sigma_1$. This is proved by carefully doing the preceding construction using Harrington's result [4] that the differential closure of a recursive differential field is recursive.

**DOP and ENI-DOP** We conclude by describing the general results that we have illustrated in Sections 1 and 4.

**Definition 5.7** A theory $T$ has the *Dimension Order Property* (DOP) if there are models $\mathcal{M}_0 \subseteq \mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{M}$ with $\mathcal{M}$ prime over $M_1 \cup M_2$, $p \in S(M)$ such that $p \perp \mathcal{M}_1$ and $p \perp \mathcal{M}_2$.

In our case we could take $K_0$ differentially closed, $a, b$ $\delta$-independent over $K_0$, $K_1 = K_0\langle a\rangle^{\mathrm{dif}}$, $K_2 = K_0\langle b\rangle^{\mathrm{dif}}$, $K = K_0\langle a, b\rangle^{\mathrm{dif}}$, and $p = p_{a+b}$.

We say that $T$ has ENI-DOP if we can choose the type $p$ to be strongly regular, nonisolated (as in our case), or more generally, nonisolated after adding finitely many parameters. Here ENI stands for "eventually nonisolated."

1. In DCF, the type $p_a$ is nonisolated over $a$ (since there are infinitely many torsion points algebraic over $a$), so we have ENI-DOP.
2. In $T_1$ of Section 1 the generic type is isolated over $a$, but once we have a realization $b$ it is nonisolated over $a, b$, so we have ENI-DOP.
3. In $T_2$ the generic type remains nonisolated, even if we add finitely many realizations. In this case we have DOP but not ENI-DOP.

**Theorem 5.8 (Shelah)** *Let $T$ be an $\omega$-stable theory with* DOP. *If $\kappa \geq \aleph_1$, there are $2^\kappa$ nonisomorphic models of cardinality $\kappa$. Further, if $T$ has* ENI-DOP*, then there are also $2^{\aleph_0}$ countable models.*

## Notes

1. Their influential paper [7] has never appeared—though the proof of Vaught's Conjecture appears in Pillay's survey paper [15].

2. The referee has pointed out that, for the examples given in this paper, we can use a much simpler coding, by letting $\dim(\pi^{-1}(v)) = 1$ for $v \in A$ and $\dim(\pi^{-1}(v)) = 2$ for $v \in B$.

3. Note, for Exercise 2.7, that this is not true if $V$ is a vector space over a field of characteristic 2.

4. Instead of looking at a single derivation we could look at several commuting derivations. This theory still has a well-behaved model theory, but there is no analog of the Blum axiomatization.

5. Poizat showed trivial strongly minimal sets can be used to build $2^\kappa$ models of size $\kappa \geq \aleph_1$ (see [14], §7).

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**Acknowledgments**

Partially supported by NSF grant DMS-0200393. I would like to thank the Notre Dame logic group for organizing a thought provoking workshop around Vaught's Conjecture.

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