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A Quasi-Discursive System ND_2^+

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Abstract Discursive (or *discussive*) logic, D_2, introduced by Jaśkowski, is widely recognized as a first formal approach to paraconsistency. Jaśkowski applied a quite *extraordinary* technique at that time to describe his logic. He neither gave a set of the axiom schemata nor presented a direct semantics for D_2 but used a translation function to express his philosophical and logical intuitions. Discursive logic was defined by an interpretation in the language of S_5 of Lewis. The aim of this paper is to present a modified system of the discursive logic that allows some of the weaker versions of Duns Scotus's thesis to be valid. The initial idea is to consider a different characteristic of the connective of negation. We introduce both a direct semantics and an axiomatization of the new system, prove the key metatheorems, and describe labeled tableaux for the system.

1 Introduction

It is common in our daily life to encounter people who, discussing something, use vague or ambiguous terms. This in turn can result in fights over words (*logomachias*) and occasionally animate philosophical reflection on the source of apparent contradictions. Luckily enough, most of us can easily face up to the problem, and consistent interpretation of these terms or occurrences is not beyond the scope of modern logic. Discursive logic is a good example of that. Jaśkowski introduced it as a logical calculus in which contradictions do not entail triviality.

Definition 1.1 Let *var* denote a nonempty denumerable set of all propositional variables $\{p_1, p_2, \dots\}$. For_{D_2} is defined to be the smallest set for which the following holds:

(i) $\alpha \in var \Rightarrow \alpha \in For_{D_2}$;
(ii) $\alpha \in For_{D_2} \Rightarrow \sim \alpha \in For_{D_2}$;
(iii) $\alpha \in For_{D_2}$ and $\beta \in For_{D_2} \Rightarrow \alpha \bullet \beta \in For_{D_2}$, where $\bullet \in \{\vee, \wedge_d, \rightarrow_d\}$.[1]

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Jaśkowski applied a quite *exotic* logical apparatus to express his philosophical intuitions. He defined a translation function of the language of D_2 into the language of S_5 of Lewis, $f : For_{D_2} \Rightarrow For_{S_5}$, as follows:

(i) $f(p_i) = p_i$ if $p_i \in var$ and $i \in N$,
(ii) $f(\sim \alpha) = \sim f(\alpha)$,
(iii) $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$,
(iv) $f(\alpha \wedge_d \beta) = f(\alpha) \wedge \Diamond f(\beta)$,
(v) $f(\alpha \rightarrow_d \beta) = \Diamond f(\alpha) \rightarrow f(\beta)$,

and additionally,

(vi) $\forall \alpha \in For_{D_2} : \alpha \in D_2 \Leftrightarrow \Diamond f(\alpha) \in S_5$.[2]

Although it described how to transform any discursive formula into its modal counterpart, it would be an easy, but slightly time-consuming, task to check which of the formulas were valid in D_2. Here an intriguing question arises: what would happen if Jaśkowski had chosen the different definition of the discursive connectives and (vi)? Naturally, the deciding factor was to remove Duns Scotus's thesis from his calculus, but the calculus should also enable practical inference. It explains why we need the two-step translation process.

A small example to illustrate the point: assume that we check whether the formula $\sim ((p \wedge_d q) \vee r) \rightarrow_d (q \rightarrow_d \sim (p \vee r))$ is valid in D_2. We must first eliminate the discursive connectives, that is, $\Diamond \sim ((p \wedge \Diamond q) \vee r) \rightarrow (\Diamond q \rightarrow \sim (p \vee r))$ and then use (vi) to obtain $\Diamond\{\Diamond \sim ((p \wedge \Diamond q) \vee r) \rightarrow (\Diamond q \rightarrow \sim (p \vee r))\}$. The formula is valid in D_2 since its modal counterpart is valid in S_5. It is remarkable that we do need the item (vi) if the calculus is to come true in practice. But for the item, the calculus would be extremely poor and insufficient for inference. An attempt to get rid of the discursive implication also fails. It becomes obvious when we take a closer look at the detachment rule $\alpha, \alpha \rightarrow \beta / \beta$ being understood as $\Diamond \alpha, \Diamond(\alpha \rightarrow \beta) / \Diamond \beta$.[3]

In [10], Perzanowski mentioned (as a comment of the translator) that we could define a few additional discursive connectives, including discursive negation:

$$\sim_d \alpha = \Diamond \sim \alpha.$$

The idea to treat negation as 'possibly-not' is not quite new and was examined by many authors,[4] but hardly any of them studied it in relation to D_2 and even so, they neither axiomatized it nor gave a direct semantics for the resulting system.

Remark 1.2 $\sim_d \alpha = ((p_1 \vee \sim p_1) \wedge_d \sim \alpha)$.

Consequently, we define a new translation function of the language of the modified calculus, ND_2^+ for short, into the language of S_5 of Lewis, $g : For_{ND_2^+} \Rightarrow For_{S_5}$, in the following way:

(i)* $g(p_i) = p_i$ if $p_i \in var$ and $i \in N$,
(ii)* $g(\sim_d \alpha) = \Diamond \sim g(\alpha)$,
(iii)* $g(\alpha \vee \beta) = g(\alpha) \vee g(\beta)$,
(iv)* $g(\alpha \wedge_d \beta) = g(\alpha) \wedge \Diamond g(\beta)$,
(v)* $g(\alpha \rightarrow_d \beta) = \Diamond g(\alpha) \rightarrow g(\beta)$,

and introduce the key definition,

(vi)* $\forall \alpha \in For_{ND_2^+} : \alpha \in ND_2^+ \Leftrightarrow \Diamond g(\alpha) \in S_5$.

From now on we will treat the connective of $\sim_d$ as a primitive symbol. We start by eliminating the translation procedure. Instead, we introduce a direct semantics for ND_2^+.

2 Semantics of ND_2^+

A frame (ND_2^+-frame) is a pair $\langle W, R\rangle$, where W is a nonempty set of points and R is the equivalence relation on W. By a model (ND_2^+-model) we mean a triple $\langle W, R, v\rangle$, where v is a function that each pair consisting of a formula and a point assigns an element of $\{1, 0\}$, $v : For_{ND_2^+} \times W \Rightarrow \{1, 0\}$, defined as follows:

$$\begin{array}{rrcl} (\sim_d) & v(\sim_d \alpha, x) = 1 & \Leftrightarrow & \exists y \in W(xRy \text{ and } v(\alpha, y) = 0); \\ (\vee) & v(\alpha \vee \beta, x) = 1 & \Leftrightarrow & v(\alpha, x) = 1 \text{ or } v(\beta, x) = 1; \\ (\wedge_d) & v(\alpha \wedge_d \beta, x) = 1 & \Leftrightarrow & v(\alpha, x) = 1 \text{ and } \exists y \in W(xRy \text{ and } v(\beta, y) = 1); \\ (\rightarrow_d) & v(\alpha \rightarrow_d \beta, x) = 1 & \Leftrightarrow & \forall y \in W(xRy \Rightarrow v(\alpha, y) = 0) \text{ or } v(\beta, x) = 1. \end{array}$$

A formula α is valid in ND_2^+, $\models \alpha$ for short, if and only if for any model $\langle W, R, v\rangle$, for every $x \in W$, there exists $y \in W$ such that xRy and $v(\alpha, y) = 1$.

The nonstandard definition of validity is a direct result of (vi)*, more explicitly, as follows.

Remark 2.1 $\forall \alpha \in For_{ND_2^+} : \models \alpha \Leftrightarrow \alpha \in ND_2^+ (\Leftrightarrow \Diamond g(\alpha) \in S_5)$.

Proof By induction. □

The translation procedure became redundant. The accessibility relation defined on ND_2^+-frames is reflexive, symmetric, and transitive. Any point is accessible from any other. It finally results in the simplified notion of the ND_2^+-model: A model (ND_2^+-model) is a pair $\langle W, v\rangle$, where W is a nonempty set (of points) and a function, $v : For_{ND_2^+} \times W \Rightarrow \{1, 0\}$, is inductively defined:

$$\begin{array}{rrcl} (\sim_d) & v(\sim_d \alpha, x) = 1 & \Leftrightarrow & \exists y \in W(v(\alpha, y) = 0); \\ (\vee) & v(\alpha \vee \beta, x) = 1 & \Leftrightarrow & v(\alpha, x) = 1 \text{ or } v(\beta, x) = 1; \\ (\wedge_d) & v(\alpha \wedge_d \beta, x) = 1 & \Leftrightarrow & v(\alpha, x) = 1 \text{ and } \exists y \in W(v(\beta, y) = 1); \\ (\rightarrow_d) & v(\alpha \rightarrow_d \beta, x) = 1 & \Leftrightarrow & \forall y \in W(v(\alpha, y) = 0) \text{ or } v(\beta, x) = 1. \end{array}$$

$\models \alpha$ if and only if for any model $\langle W, v\rangle$, there exists $y \in W$ such that $v(\alpha, y) = 1$.

The discursive equivalence is introduced as an abbreviation:

$$\alpha \leftrightarrow_d \beta = (\alpha \rightarrow_d \beta) \wedge_d (\beta \rightarrow_d \alpha).$$

It is worth mentioning that some of the *classically* valid formulas are not valid in ND_2^+, for example,

(1) $p \rightarrow_d (\sim_d p \rightarrow_d q)$,
(2) $p \rightarrow_d (\sim_d p \rightarrow_d \sim q)$,
(3) $(p \wedge_d \sim_d p) \rightarrow_d q$,
(4) $\sim_d (\sim_d p \wedge_d p)$,
(5) $(p \rightarrow_d q) \rightarrow_d ((p \rightarrow_d \sim_d q) \rightarrow_d \sim_d p)$,
(6) $(p \rightarrow_d q) \rightarrow_d \sim_d \sim_d (p \rightarrow_d q)$.

On the other hand, some ND_2^+-valid formulas are not *permissible* in many paraconsistent systems (after replacing $\rightarrow_d$ with $\rightarrow$, $\wedge_d$ with $\wedge$, and $\sim_d$ with $\sim$), for example,

(7) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d q)$,
(8) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d \sim_d q)$,
(9) $\sim_d (p \wedge_d \sim_d p)$,
(10) $\sim_d (p \rightarrow_d \sim_d\sim_d p) \rightarrow_d p$,
(11) $(\sim_d p \rightarrow_d \sim_d q) \rightarrow_d ((\sim_d p \rightarrow_d \sim_d\sim_d q) \rightarrow_d p)$,
(12) $\sim_d\sim_d p \rightarrow_d p$.[5]

One can perceive the similarities between ND_2^+ and P_1 of Sette. Indeed, the formulas (6), (10), and (11) (after replacing $\rightarrow_d$ with $\rightarrow$, $\wedge_d$ with $\wedge$, and $\sim_d$ with $\sim$) constitute the *negational* part of P_1.[6]

3 Axiomatization of ND_2^+

We present here an axiomatization of ND_2^+. It consists of the following axiom schemata and rule of inference:

(A$_1$) $\alpha \rightarrow_d (\beta \rightarrow_d \alpha)$
(A$_2$) $(\alpha \rightarrow_d (\beta \rightarrow_d \gamma)) \rightarrow_d ((\alpha \rightarrow_d \beta) \rightarrow_d (\alpha \rightarrow_d \gamma))$
(A$_3$) $(\alpha \wedge_d \beta) \rightarrow_d \alpha$
(A$_4$) $(\alpha \wedge_d \beta) \rightarrow_d \beta$
(A$_5$) $(\alpha \rightarrow_d (\beta \rightarrow_d (\alpha \wedge_d \beta))$
(A$_6$) $\alpha \rightarrow_d (\alpha \vee \beta)$
(A$_7$) $\beta \rightarrow_d (\alpha \vee \beta)$
(A$_8$) $(\alpha \rightarrow_d \gamma) \rightarrow_d ((\beta \rightarrow_d \gamma) \rightarrow_d ((\alpha \vee \beta) \rightarrow_d \gamma))$
(A$_9$) $\sim_d (\alpha \wedge_d \sim_d \beta) \rightarrow_d \sim_d\sim_d (\sim_d \alpha \vee \beta)$
(A$_{10}$) $\sim_d (\alpha \wedge_d \sim_d \alpha)$
(A$_{11}$) $(\alpha \vee \sim_d \beta) \rightarrow_d ((\alpha \vee \sim_d\sim_d \beta) \rightarrow_d \alpha)$
(A$_{12}$) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d (\alpha \vee \sim_d\sim_d \beta)$
(A$_{13}$) $\sim_d\sim_d \alpha \rightarrow_d \alpha$
(A$_{14}$) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\alpha \vee \beta \vee \gamma)$
(A$_{15}$) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\beta \vee \alpha)$
(MP)* $\alpha, \alpha \rightarrow_d \beta / \beta$.

The consequence relation $\vdash_{ND_2^+}$ is defined by the set of axioms and (MP)*.

Remark 3.1 Each of the axiom schemata of ND_2^+ becomes a schema of the thesis of the classical propositional calculus CPC, after replacing in (A$_i$), where $i \in \{1, \ldots, 15\}$, all the discursive connectives, $\sim_d, \wedge_d, \rightarrow_d, \leftrightarrow_d$, by their classical counterparts. The rule (MP)* becomes an admissible rule of CPC after replacing $\rightarrow_d$ with $\rightarrow$.

Let $\vdash_{f_c(ND_2^+)}$ denote the consequence relation of the system being received according to Remark 3.1 and let $\vdash_{\text{CPC}}$ stand for the classical consequence relation.

Remark 3.2 $\{\alpha : \vdash_{f_c(ND_2^+)} \alpha\} \subset \{\alpha : \vdash_{\text{CPC}} \alpha\}$.

Observe that (A$_1$), (A$_2$) constitute the implicational fragment of ND_2^+ and the system is closed under the detachment rule, which is the sole rule of inference of ND_2^+. It immediately follows that the proof of the deduction theorem is standard.

Theorem 3.3

$$\Phi \vdash_{ND_2^+} \alpha \rightarrow_d \beta \Leftrightarrow \Phi \cup \{\alpha\} \vdash_{ND_2^+} \beta,$$

where $\alpha, \beta \in For_{ND_2^+}$, $\Phi \subseteq For_{ND_2^+}$.

Remark 3.4 The formulas listed below are provable in ND_2^+:

(T$_1$) $(\alpha \vee \alpha) \leftrightarrow_d \alpha$
(T$_2$) $(\alpha \vee \beta) \leftrightarrow_d (\beta \vee \alpha)$
(T$_3$) $((\alpha \vee \beta) \vee \gamma) \leftrightarrow_d (\alpha \vee (\beta \vee \gamma))$
(T$_4$) $(\alpha \vee (\beta \wedge_d \gamma)) \leftrightarrow_d ((\alpha \vee \beta) \wedge_d (\alpha \vee \gamma))$
(T$_5$) $(\alpha \rightarrow_d \beta) \rightarrow_d ((\alpha \vee \gamma) \rightarrow_d (\beta \vee \gamma))$
(T$_6$) $(\beta \vee \alpha \vee \beta) \rightarrow_d (\alpha \vee \beta)$
(T$_7$) $(\alpha \wedge_d (\alpha \rightarrow_d \beta)) \rightarrow_d \beta$
(T$_8$) $\alpha \vee (\alpha \rightarrow_d \beta)$
(T$_9$) $\sim_d \sim_d (\alpha \vee \sim_d \alpha)$
(T$_{10}$) $\alpha \vee \sim_d \alpha$

and the set of $\{\alpha : \vdash_{ND_2^+} \alpha\}$ is closed under the rules:

(R$_1$) $\alpha, \beta / \alpha \wedge_d \beta$
(R$_2$) $\alpha \wedge_d \beta / \alpha(\beta)$
(R$_3$) $\alpha(\beta)/\alpha \vee \beta$.

Proof We prove (T$_1$) – (T$_8$) in much the same way as it is in the (positive) classical case. (T$_9$) is a direct result of (A$_9$), (A$_{10}$), (A$_{15}$), and (MP)*. In order to obtain (T$_{10}$), it is sufficient to apply (T$_9$), (A$_{13}$), and (MP)*. (R$_1$) – (R$_3$) are obvious due to (A$_5$), (A$_3$), (A$_4$), (A$_6$), (A$_7$), and (MP)*. □

4 Soundness and Completeness

Theorem 4.1 (Soundness) $\vdash_{ND_2^+} \alpha \Rightarrow \models \alpha$.

Proof By induction. □

Theorem 4.2 (Completeness) $\models \alpha \Rightarrow \vdash_{ND_2^+} \alpha$.

The initial idea of the proof we present below originates from [13]. The key point is to define a canonical valuation that falsifies a nonthesis. However, by contrast to the well-known method a la Henkin, we do not verify, but falsify, the sets of formulas we construct.

Proof Assume that $\nvdash_{ND_2^+} \alpha$ and $\models \alpha$. Define a sequence of all the formulas of ND_2^+ as follows:

$$\Gamma = \gamma_1, \gamma_2, \gamma_3, \ldots .$$

The sole restriction is that the first element of Γ is α (i.e., $\gamma_1 = \alpha$).

Next define a family of (finite) subsequences of Γ.

$\Delta_1 = \delta_1$ where $\delta_1 = \gamma_1 = \alpha$;
$\Delta_2 = \delta_1, \delta_2$ where $\delta_1 = \gamma_1 = \alpha$ and $\delta_2 = \gamma_2$ if $\nvdash_{ND_2^+} \delta_1 \vee \delta_2$, otherwise $\delta_2 \neq \gamma_2$ and $\Delta_2 = \Delta_1$;
$\vdots$
$\Delta_n = \delta_1, \delta_2, \delta_3, \ldots, \delta_n$ where $\delta_1 = \alpha, \delta_2 = \gamma_i, \delta_3 = \gamma_{i+k}, \ldots, \delta_n = \gamma_{i+s}$, if $\nvdash_{ND_2^+} \delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$; otherwise $\delta_n \neq \gamma_{i+s}$ and $\Delta_n = \Delta_{n-1}$;
$\vdots$

Now define

$$\begin{aligned}
\nabla_1 &= \Delta_1, \Delta_2, \Delta_3 \ldots\\
\nabla_2 &= \Delta_2, \Delta_3, \Delta_4 \ldots\\
\nabla_3 &= \Delta_3, \Delta_4, \Delta_5 \ldots\\
&\vdots\\
\nabla_n &= \Delta_n, \Delta_{n+1}, \Delta_{n+2} \ldots\\
&\vdots
\end{aligned}$$

In what follows, some properties of ∇_i will be of crucial importance.

Lemma 4.3

(i) $\nvdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_n$, *for any* $n \in N$.
(ii) *If* $\beta \notin \nabla_i$, *then* $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_k \vee \beta$, *for some* $k \in N$.

Proof (i) By the definition of ∇_i, where $i \in \{1, 2, 3, \ldots\}$.

(ii) From the fact that the set of $\{\alpha : \vdash_{ND_2^+} \alpha\}$ is closed under (R_3). □

Let ∇ stand for $\{\nabla_1, \nabla_2, \ldots, \nabla_i, \ldots, \nabla_n, \ldots\}$.

Lemma 4.4 *For every* $\beta \in For_{ND_2^+}$, *for any* $\nabla_i \in \nabla$,

$$\beta \in \nabla_i \Leftrightarrow \exists_{\Delta_k} \subset \nabla_i (\beta \in \Delta_k).$$

Proof Straightforward. □

Lemma 4.5 *For every* $\beta, \gamma \in For_{ND_2^+}$, *for any* $\nabla_i, \nabla_k \in \nabla$,

(i) $\beta \vee \gamma \in \nabla_i \Leftrightarrow \beta \in \nabla_i$ *and* $\gamma \in \nabla_i$,
(ii) $\beta \wedge_d \gamma \in \nabla_i \Leftrightarrow \beta \in \nabla_i$ *or* $\forall_{\nabla_k \in \nabla} (\gamma \in \nabla_k)$,
(iii) $\beta \rightarrow_d \gamma \in \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla} (\beta \notin \nabla_k)$ *and* $\gamma \in \nabla_i$.

Proof We show (ii) as an example.

(ii) $\Leftarrow$ Assume that (1) $\beta \in \nabla_i$ or $\forall_{\nabla_k \in \nabla} (\gamma \in \nabla_k)$, (2) $\beta \wedge_d \gamma \notin \nabla_i$.

Subcase (a) If (1a) $\beta \in \nabla_i$, (2a) $\beta \wedge_d \gamma \notin \nabla_i$, then for some $m \in N$: (3a) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m \vee (\beta \wedge_d \gamma)$ by Lemma 4.3(ii) and (2a). Apply (T_4) to deduce (4a) $\vdash_{ND_2^+} (\delta_1 \vee \cdots \vee \delta_m \vee \beta) \wedge_d (\delta_1 \vee \cdots \vee \delta_m \vee \gamma)$ and (R_2) to obtain (5a) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m \vee \beta$. Observe that $\delta_1, \ldots, \delta_m, \beta \in \nabla_i$. A contradiction due to Lemma 4.3(i).

Subcase (b) If (1b) $\forall_{\nabla_k \in \nabla} (\gamma \in \nabla_k)$, (2b) $\beta \wedge_d \gamma \notin \nabla_i$, then (3b) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m \vee (\beta \wedge_d \gamma)$, for some $m \in N$, and (4b) $\gamma \in \nabla_i$ by (1b). Next proceed analogously to Subcase (a).

(ii) $\Rightarrow$ Suppose that (1) $\beta \wedge_d \gamma \in \nabla_i$, (2) $\beta \notin \nabla_i$ and $\exists_{\nabla_k \in \nabla} (\gamma \notin \nabla_k)$. Obviously, $i \geq k$ or $k > i$.

Subcase (a) Let $i \geq k$. Since $\nabla_k = \Delta_k, \ldots, \overbrace{\Delta_i, \Delta_{i+1}, \ldots}^{\nabla_i}$ then $\nabla_i \subseteq \nabla_k$ and (3a) $\gamma \notin \nabla_i$. Now apply Lemma 4.3(ii), (R_3), (T_1), (T_2), (T_3), (R_1), (T_4), and (MP)* to finally obtain $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m \vee (\beta \wedge_d \gamma)$. But $\delta_1, \ldots, \delta_m, (\beta \wedge_d \gamma) \in \nabla_i$. This is in contradiction to Lemma 4.3(i).

Subcase (b) Let $k > i$. Then $\nabla_k \subset \nabla_i$ and (3b) $\beta \notin \nabla_k$. To show that $\beta \wedge_d \gamma \in \nabla_k$, we use Lemma 4.4: (4b) $\exists_{\Delta_n \subset \nabla_i}(\beta \wedge_d \gamma \in \Delta_n)$. From above we have two possibilities, either $n \geq k$ or $k > n$. If $n \geq k$ it immediately follows that $\beta \wedge_d \gamma \in \nabla_k$ as $\nabla_k = \Delta_k, \ldots, \Delta_n, \Delta_{n+1}, \ldots$. The second case is more interesting. If $k > n$ then $\Delta_n \subset \Delta_k$ since $\Delta_1 \subset \cdots \subset \Delta_n \subset \cdots \subset \Delta_k \subset \cdots$. Finally, we receive $\beta \wedge_d \gamma \in \nabla_k, \beta \notin \nabla_k, \gamma \notin \nabla_k$ and may proceed analogously to Subcase (a). □

Let ∇_i be a sequence, $i \in \{1, 2, 3, \ldots\}$. Define

$$\nabla_i^\star = \delta_1^\star, \ldots, \delta_i^\star, \ldots$$

where

(a) $\delta_1^\star = \delta_1 = \gamma_1 = \alpha$;
(b) $(\delta_n = \delta_k^\star)$ if $\nvdash_{ND_2^+} \sim_d \sim_d (\delta_1^\star \vee \delta_2^\star \vee \cdots \vee \delta_n)$, for any $\delta_n \in \nabla_i$ and $n \geq k$, where $i, k, n \in N$; otherwise $\delta_n \neq \delta_k^\star$.

Definition 4.6 We call a formula β *discursive* if it contains at least one of the following symbols: $\rightarrow_d, \wedge_d, \leftrightarrow_d$. A formula β is a *discursive thesis* if it is a thesis and discursive.

Lemma 4.7

(i) $\nabla_i^\star \subseteq \nabla_i$, *for every* $i \in \{1, 2, 3, \ldots\}$,
(ii) $\nvdash_{ND_2^+} \sim_d \sim_d (\delta_1^\star \vee \cdots \vee \delta_n^\star)$, *for every* $n \in N$,
(iii) *if* β *is not a* discursive thesis, $\beta \notin \nabla_i$, *then*

$$\vdash_{ND_2^+} \sim_d \sim_d (\delta_1^\star \vee \cdots \vee \delta_k^\star \vee \beta), \text{ for some } k \in N.$$

Proof (i) – (ii) Straightforward.

(iii) By (T_9), (A_{14}), (A_{15}), (MP)*, and the fact that $\nabla_i^\star \subseteq \nabla_i$ holds for every $i \in \{1, 2, \ldots\}$. □

Lemma 4.8 *For every* $\beta \in For_{ND_2^+}$, *for any* $\nabla_i, \nabla_k \in \nabla$,

(i) $\sim_d \beta \in \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}(\beta \notin \nabla_k)$.

Proof (i) $\Rightarrow$ Assume that (1) $\sim_d \beta \in \nabla_i$ and (2) $\exists_{\nabla_k \in \nabla}(\beta \in \nabla_k)$. As before, either $i \geq k$ or $k > i$. Let $i \geq k$. It results in $\nabla_i \subseteq \nabla_k$ and $\sim_d \beta \in \nabla_k$ (if $k > i$, $\nabla_k \subset \nabla_i$, and $\beta \in \nabla_i$). Now apply Lemma 4.3(i) to obtain $\nvdash_{ND_2^+} \beta \vee \sim_d \beta$. A contradiction due to (T_{10}).

(i) $\Leftarrow$ Suppose (1) $\forall_{\nabla_k \in \nabla}(\beta \notin \nabla_k)$ and (2) $\sim_d \beta \notin \nabla_i$. In particular, (3) $\beta \notin \nabla_i$. Apply Lemma 4.3(ii) to (2) and (3) to deduce (4) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m \vee \sim_d \beta$, for some $m \in N$, (5) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_p \vee \beta$, for some $p \in N$. Notice that $m \geq p$ or $m < p$. Let $m \geq p$ (the case $m < p$ is similar to $m \geq p$). Now consider three subcases:

(a) neither β nor $\sim_d \beta$ is a *discursive thesis*;
(b) β is a *discursive thesis*, but $\sim_d \beta$ is not a *discursive thesis*;
(c) $\sim_d \beta$ is a *discursive thesis*, but β is not a *discursive thesis*.

Note that the fourth subcase (β is a *discursive thesis* and $\sim_d \beta$ is a *discursive thesis*) is impossible due to *Soundness*.

Subcase (a) Apply Lemma 4.7(iii) to (3) to obtain (6a) $\vdash_{ND_2^+} \sim_d \sim_d (\delta_1^\star \vee \cdots \vee \delta_j^\star \vee \beta)$, for some $j \in N$.[7] Use (A_{12}), (6a), and (MP)* to get (7a) $\vdash_{ND_2^+} \delta_1^\star \vee \cdots \vee \delta_j^\star \vee \sim_d \sim_d \beta$. Clearly, either $m > j$ or $m = j$ (if $m > j$ apply (R_3) to have $m = j$). Now use (R_3), (T_2), (T_3), and (MP)* to receive (8a) $\vdash_{ND_2^+} \delta_1^\star \vee \cdots \vee \delta_m^\star \vee \sim_d \sim_d \beta$ (where $\delta_1^\star = \delta_1, \delta_2^\star = \delta_2, \ldots, \delta_m^\star = \delta_m$) and finally (9a) $\vdash_{ND_2^+} \delta_1 \vee \cdots \vee \delta_m$ by (4), (8a), (A_{11}), and (MP)*. But $\delta_1, \ldots, \delta_m \in \nabla_i$. A contradiction due to Lemma 4.3(i).

Subcases (b) and (c) Proofs are similar to Subcase (a). Notice, however, that you are not allowed to apply Lemma 4.7(iii) to $\beta \notin \nabla_i$ (in Subcase (b)) and to $\sim_d \beta \notin \nabla_i$ (in Subcase (c)). □

Let $\langle \nabla, \upsilon_c \rangle$ be a canonical model for $\vdash_{ND_2^+}$. Define the canonical valuation $\upsilon_c : For_{ND_2^+} \times \nabla \Rightarrow \{1, 0\}$ as follows:

$$\upsilon_c(\beta, \nabla_i) = \begin{cases} 1, & \text{if } \beta \notin \nabla_i \\ 0, & \text{if } \beta \in \nabla_i. \end{cases}$$

We must show that the conditions ($\vee$), ($\wedge_d$), ($\rightarrow_d$), and ($\sim_d$) hold for υ_c.

Case 1 $\beta = \varphi \vee \psi$.

(a) $\upsilon_c(\varphi \vee \psi, \nabla_i) = 1 \Leftrightarrow \varphi \vee \psi \notin \nabla_i \Leftrightarrow \varphi \notin \nabla_i$ or $\psi \notin \nabla_i \Leftrightarrow \upsilon_c(\varphi, \nabla_i) = 1$ or $\upsilon_c(\psi, \nabla_i) = 1$.

(b) $\upsilon_c(\varphi \vee \psi, \nabla_i) = 0 \Leftrightarrow \varphi \vee \psi \in \nabla_i \Leftrightarrow \varphi \in \nabla_i$ and $\psi \in \nabla_i \Leftrightarrow \upsilon_c(\varphi, \nabla_i) = 0$ and $\upsilon_c(\psi, \nabla_i) = 0$.

Case 2 $\beta = \varphi \wedge_d \psi$.

(a) $\upsilon_c(\varphi \wedge_d \psi, \nabla_i) = 1 \Leftrightarrow \varphi \wedge_d \psi \notin \nabla_i \Leftrightarrow \varphi \notin \nabla_i$ and $\exists_{\nabla_k \in \nabla}(\psi \notin \nabla_k) \Leftrightarrow \upsilon_c(\varphi, \nabla_i) = 1$ and $\exists_{\nabla_k \in \nabla} \upsilon_c(\psi, \nabla_k) = 1$.

(b) $\upsilon_c(\varphi \wedge_d \psi, \nabla_i) = 0 \Leftrightarrow \varphi \wedge_d \psi \in \nabla_i \Leftrightarrow \varphi \in \nabla_i$ or $\forall_{\nabla_k \in \nabla}(\psi \in \nabla_i) \Leftrightarrow \upsilon_c(\varphi, \nabla_i) = 0$ or $\forall_{\nabla_k \in \nabla} \upsilon_c(\psi, \nabla_i) = 0$.

Case 3 $\beta = \varphi \rightarrow_d \psi$.

(a) $\upsilon_c(\varphi \rightarrow_d \psi, \nabla_i) = 1 \Leftrightarrow \varphi \rightarrow_d \psi \notin \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}(\varphi \in \nabla_k)$ or $\psi \notin \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla} \upsilon_c(\varphi, \nabla_k) = 0$ or $\upsilon_c(\psi, \nabla_i) = 1$.

(b) $\upsilon_c(\varphi \rightarrow_d \psi, \nabla_i) = 0 \Leftrightarrow \varphi \rightarrow_d \psi \in \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla}(\varphi \notin \nabla_k)$ and $\psi \in \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla} \upsilon_c(\varphi, \nabla_k) = 1$ and $\upsilon_c(\psi, \nabla_i) = 0$.

Case 4 $\beta = \sim_d \varphi$.

(a) $\upsilon_c(\sim_d \varphi, \nabla_i) = 1 \Leftrightarrow \sim_d \varphi \notin \nabla_i \Leftrightarrow \exists_{\nabla_k \in \nabla}(\varphi \in \nabla_k) \Leftrightarrow \exists_{\nabla_k \in \nabla} \upsilon_c(\varphi, \nabla_k) = 0$.

(b) $\upsilon_c(\sim_d \varphi, \nabla_i) = 0 \Leftrightarrow \sim_d \varphi \in \nabla_i \Leftrightarrow \forall_{\nabla_k \in \nabla}(\varphi \notin \nabla_k) \Leftrightarrow \forall_{\nabla_k \in \nabla} \upsilon_c(\varphi, \nabla_k) = 1$.

Suppose that $\nvdash_{ND_2^+} \alpha$ and $\models \alpha$. Note that the formula α is the very first element of each i-sequence, where $i \in \{1, 2, 3, \ldots\}$. Since $\alpha \in \nabla_i$, then the formula α is not valid in $\langle \nabla, \upsilon_c \rangle$ and consequently $\not\models \alpha$. A contradiction.

5 Labeled Tableaux for ND_2^+

In Section 5, we depict a tableau-based proof technique that can be used for proving theorems in ND_2^+.[8] We will deal with signed labeled formulas such as $\sigma :: TP$ (or $\sigma :: FP$), where σ is a label and TP (or FP) is a signed formula (i.e., a formula prefixed with a T or an F). The phrase $\sigma :: TP$ is read as "P is true at the world σ" and $\sigma :: FP$ as "P is false at the world σ." By label, we understand a natural number. We call ρ a root label and always assume that $\rho = 1$. A tableau for a labeled formula P is a downward rooted tree, where each of the nodes contains a signed labeled formula, constructed using the branch extension rules defined below.

Disjunction

$$\frac{\sigma :: TP \vee Q}{\sigma :: TP | \sigma :: TQ}\ (\mathbf{T}\vee) \qquad\qquad \frac{\sigma :: FP \vee Q}{\begin{array}{c}\sigma :: FP\\ \sigma :: FQ\end{array}}\ (\mathbf{F}\vee)$$

The rule ($\mathbf{F}\vee$) is linear, but ($\mathbf{T}\vee$) is branching.

Discursive negation

$$\frac{\sigma :: T \sim_d P}{\tau :: FP}\ (\mathbf{T}\sim_{\mathbf{d}}) \qquad\qquad \frac{\sigma :: F \sim_d P}{\sigma' :: TP}\ (\mathbf{F}\sim_{\mathbf{d}})$$

where, for ($\mathbf{T}\sim_{\mathbf{d}}$), τ is a label that is *new* to the branch and, for ($\mathbf{F}\sim_{\mathbf{d}}$), σ' is a label that has been *already used* in the branch.

Discursive conjunction

$$\frac{\sigma :: TP \wedge_d Q}{\begin{array}{c}\sigma :: TP\\ \tau :: TQ\end{array}}\ (\mathbf{T}\wedge_{\mathbf{d}}) \qquad\qquad \frac{\sigma :: FP \wedge_d Q}{\sigma :: FP | \sigma' :: FQ}\ (\mathbf{F}\wedge_{\mathbf{d}})$$

where τ is a label that is *new* to the branch and σ' is a label that has been *already used* in the branch.

Discursive implication

$$\frac{\sigma :: P \rightarrow_d Q}{\sigma' :: FP \mid \sigma :: TQ}\ (\mathbf{T}\rightarrow_{\mathbf{d}}) \qquad\qquad \frac{\sigma :: FP \rightarrow_d Q}{\begin{array}{c}\tau :: TP\\ \sigma :: FQ\end{array}}\ (\mathbf{F}\rightarrow_{\mathbf{d}})$$

where σ' has been *already used* in the branch and τ is a label that is *new* to the branch.

Closure rule

$$\frac{\begin{array}{c}\sigma :: TP\\ \sigma :: FP\end{array}}{\text{closed}}\ (\mathbf{C})$$

A branch of a tableau is closed if we can apply ($\mathbf{C}$). Otherwise the branch is open. A tableau is closed if all of its branches are closed; otherwise the tableau is open.

As was mentioned in Section 1, Jaśkowski had proposed regarding a discussion as a set of opinions expressed by participants. Despite this fact, we do not know or care who is who in the discussion. Our knowledge of the participants is narrowed down to the following: *someone said something* or *someone stated that*. . . [9] The idea found expression in the characteristic of the translation function (especially, the item (vi)) and of the definition of '$\models$'. It is also reflected in our tableaux system.

Special rule

$$\frac{\rho :: FP}{\sigma' :: FP}\ \textbf{(S)}$$

where ρ is a root label and σ' is a label that has been *already used* in the branch. The application of the rule is *always* limited to root labels.

Let P be a formula. By a *tableau proof* of P (ND_2^+-tableau proof) we mean a closed tableau with $1 :: FP$.

Theorem 5.1 *A formula P has an ND_2^+-tableau proof if and only if P is valid in ND_2^+.*

Now we give a few examples to illustrate how the rules we depicted work in practice.

Example 5.2 Closed tableau for $\sim_d P \to_d (\sim_d\sim_d P \to_d Q)$.

(a)	$1 :: F \sim_d P \to_d (\sim_d\sim_d P \to_d Q)$	(start)
(b)	$2 :: T \sim_d P$	$(\mathbf{F\to_d})$, (a)
(c)	$1 :: F \sim_d\sim_d P \to_d Q$	$(\mathbf{F\to_d})$, (a)
(d)	$3 :: T \sim_d\sim_d P$	$(\mathbf{F\to_d})$, (c)
(e)	$1 :: FQ$	$(\mathbf{F\to_d})$, (c)
(f)	$4 :: FP$	$(\mathbf{T\sim_d})$, (b)
(g)	$5 :: F \sim_d P$	$(\mathbf{T\sim_d})$, (d)
(h)	$4 :: TP$	$(\mathbf{F\sim_d})$, (g)
	closed	(**C**), (f), (h)

Although this might seem a rather trivial example, it clearly demonstrates the usage of the *discursive* rules.

Example 5.3 Closed tableau for $\sim_d (P\wedge_d \sim_d P)$.

(a)	$1 :: F \sim_d (P\wedge_d \sim_d P)$	(start)
(b)	$1 :: TP\wedge_d \sim_d P$	$(\mathbf{F\sim_d})$, (a)
(c)	$1 :: TP$	$(\mathbf{T\wedge_d})$, (b)
(d)	$2 :: T \sim_d P$	$(\mathbf{T\wedge_d})$, (b)
(e)	$3 :: FP$	$(\mathbf{T\sim_d})$, (d)
(f)	$3 :: F \sim_d (P\wedge_d \sim_d P)$	(**S**), (a)
(g)	$3 :: TP\wedge_d \sim_d P$	$(\mathbf{F\sim_d})$, (f)
(h)	$3 :: TP$	$(\mathbf{T\wedge_d})$, (g)
(i)	$4 :: T \sim_d P$	$(\mathbf{T\wedge_d})$, (g)
	closed	(**C**), (e), (h)

Notice that we have applied the rule (**S**), the line (f), to close the tableau.

In the next example, we will use one of the branching rules to produce a closed tableau for (A_{11}).

Example 5.4 Closed tableau for $(P\vee \sim_d Q) \to_d ((P\vee \sim_d\sim_d Q) \to_d P)$.

(a)	$1 :: F\ (P\vee \sim_d Q) \to_d ((P\vee \sim_d\sim_d Q) \to_d P)$	(start)
(b)	$2 :: T\, P\vee \sim_d Q$	$(\mathbf{F\to_d})$, (a)
(c)	$1 :: F(P\vee \sim_d\sim_d Q) \to_d P$	$(\mathbf{F\to_d})$, (a)
(d)	$3 :: T\, P\vee \sim_d\sim_d Q$	$(\mathbf{F\to_d})$, (c)
(e)	$1 :: F P$	$(\mathbf{F\to_d})$, (c)

1st branch

(f.1)	$2 :: T P$	$(\mathbf{T}\vee)$, (b)
(f.2)	$2 :: F\ (P\vee \sim_d Q) \to_d ((P\vee \sim_d\sim_d Q) \to_d P)$	(**S**), (a)
(f.3)	$4 :: T\, P\vee \sim_d Q$	$(\mathbf{F\to_d})$, (f.2)
(f.4)	$2 :: F(P\vee \sim_d\sim_d Q) \to_d P$	$(\mathbf{F\to_d})$, (f.2)
(f.5)	$5 :: T\, P\vee \sim_d\sim_d Q$	$(\mathbf{F\to_d})$, (f.4)
(f.6)	$2 :: F P$	$(\mathbf{F\to_d})$, (f.4)
	closed	(**C**),(f.1), (f.6)

2nd branch

(g.1)	$2 :: T \sim_d Q$	$(\mathbf{T}\vee)$, (b)
(g.2)	$6 :: F Q$	$(\mathbf{T\sim_d})$, (g.1)

1st subbranch

(g.3)	$3 :: T P$	$(\mathbf{T}\vee)$, (d)
(g.4)	$3 :: F\ (P\vee \sim_d Q) \to_d ((P\vee \sim_d\sim_d Q) \to_d P)$	(**S**), (a)
(g.5)	$7 :: T\, P\vee \sim_d Q$	$(\mathbf{F\to_d})$, (g.4)
(g.6)	$3 :: F(P\vee \sim_d\sim_d Q) \to_d P$	$(\mathbf{F\to_d})$, (g.4)
(g.7)	$8 :: T\, P\vee \sim_d\sim_d Q$	$(\mathbf{F\to_d})$, (g.6)
(g.8)	$3 :: F P$	$(\mathbf{F\to_d})$, (g.6)
	closed	(**C**), (g.3), (g.8)

2nd subbranch

(g.9)	$4 :: T \sim_d\sim_d Q$	$(\mathbf{T}\vee)$, (d)
(g.10)	$9 :: F \sim_d Q$	$(\mathbf{T\sim_d})$, (g.9)
(g.11)	$6 :: T Q$	$(\mathbf{F\sim_d})$, (g.10)
	closed	(**C**), (g.2), (g.11)

Now we will generate an infinite tableau for $\sim_d (\sim_d P \wedge_d P)$.

Example 5.5 Infinite tableau for $\sim_d (\sim_d P \wedge_d P)$.

(a)	$1 :: F \sim_d (\sim_d P \wedge_d P)$	(start)
(b)	$1 :: T \sim_d P \wedge_d P$	$(\mathbf{F}\sim_\mathbf{d})$, (a)
(c)	$1 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(d)	$2 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(e)	$3 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (c)
(f)	$2 :: F \sim_d (\sim_d P \wedge_d P)$	$(\mathbf{S})$, (a)
(g)	$2 :: T \sim_d P \wedge_d P$	$(\mathbf{F}\sim_\mathbf{d})$, (f)
(h)	$3 :: T \sim_d P \wedge_d P$	$(\mathbf{F}\sim_\mathbf{d})$, (f)
(i)	$2 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (g)
(j)	$4 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (g)
(k)	$5 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (i)
(l)	$3 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (h)
(m)	$6 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (h)
(n)	$7 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (l)
	$\vdots$	

Observe that our attempts to close the tableau fail. The only *gain* we receive comes in the form of new labels and just the same formulas that continuously occur in the branch. The whole procedure goes ad infinitum. Here is the last example.

Example 5.6 Infinite tableau for $(P \wedge_d \sim_d P) \rightarrow_d Q$.

(a)	$1 :: F(P \wedge_d \sim_d P) \rightarrow_d Q$	(start)
(b)	$2 :: T P \wedge_d \sim_d P$	$(\mathbf{F}\rightarrow_\mathbf{d})$, (a)
(c)	$1 :: T Q$	$(\mathbf{T}\rightarrow_\mathbf{d})$, (a)
(d)	$2 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(e)	$3 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(f)	$4 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (e)
(g)	$2 :: F(P \wedge_d \sim_d P) \rightarrow_d Q$	$(\mathbf{S})$, (a)
(b)	$2 :: T P \wedge_d \sim_d P$	$(\mathbf{F}\rightarrow_\mathbf{d})$, (a)
(c)	$1 :: T Q$	$(\mathbf{T}\rightarrow_\mathbf{d})$, (a)
(d)	$2 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(e)	$3 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (b)
(f)	$4 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (e)
(h)	$3 :: F(P \wedge_d \sim_d P) \rightarrow_d Q$	$(\mathbf{S})$, (a)
(i)	$5 :: T P \wedge_d \sim_d P$	$(\mathbf{F}\rightarrow_\mathbf{d})$, (h)
(j)	$3 :: T Q$	$(\mathbf{T}\rightarrow_\mathbf{d})$, (h)
(k)	$5 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (i)
(l)	$6 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (i)
(m)	$7 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (l)
(n)	$4 :: F(P \wedge_d \sim_d P) \rightarrow_d Q$	$(\mathbf{S})$, (a)
(o)	$8 :: T P \wedge_d \sim_d P$	$(\mathbf{F}\rightarrow_\mathbf{d})$, (n)
(p)	$4 :: T Q$	$(\mathbf{T}\rightarrow_\mathbf{d})$, (n)
(q)	$8 :: T P$	$(\mathbf{T}\wedge_\mathbf{d})$, (o)
(r)	$9 :: T \sim_d P$	$(\mathbf{T}\wedge_\mathbf{d})$, (o)
(s)	$10 :: F P$	$(\mathbf{T}\sim_\mathbf{d})$, (r)
	$\vdots$	

Just as in the previous example, we will never manage to close the tree.

Notes

1. The symbols $\sim$, $\vee$, $\wedge_d$, $\rightarrow_d$ denote negation, disjunction, discursive conjunction, and discursive implication, respectively.

2. Cf. [8], p. 44 and [7], p. 57. In what follows, we use the English translations of the Jaśkowski papers that appeared in *Logic and Logical Philosophy*. See References for details.

3. In both cases, '$\rightarrow$' stands for the classical implication. It is a bare outline. For more detailed description, see related publications, e.g., [2], [3], [8], [7], and [12].

4. See, e.g., [1], [6], and [15].

5. Cf. [5], [8], [7].

6. Cf. [14].

7. Since neither β nor $\sim_d \beta$ is a *discursive thesis* it is our choice to use Lemma 4.7(iii) either to (2) or to (3).

8. Cf. [4].

9. Cf. [8], pp. 41–43.

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