Notre Dame Journal of Formal Logic
Volume 46, Number 3, 2005

Some Problems in Singular Cardinals Combinatorics

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Abstract This paper attempts to present and organize several problems in the theory of Singular Cardinals. The most famous problems in the area (bounds for the $\beth$-function at singular cardinals) are well known to all mathematicians with even a rudimentary interest in set theory. However, it is less well known that the combinatorics of singular cardinals is a thriving area with results and problems that do not depend on a solution of the Singular Cardinals Hypothesis. We present here an annotated collection of representative problems with some references. Where the problems are novel, attribution is attempted and it is noted where money is attached to particular problems.

Three closely related themes are represented in these problems: stationary sets and stationary set reflection, variations of square and approachability, and the singular cardinals hypothesis. Underlying many of them are ideas from Shelah's PCF theory. Important subthemes were mutual stationarity, Aronszajn trees, and superatomic Boolean Algebras.

The author notes considerable overlap between this paper and the unpublished report submitted to the Banff Center for the Workshop on Singular Cardinals Combinatorics, May 1–5, 2004.

1 The Singular Cardinals Hypothesis and Hilbert's First Problem

In 1873, Cantor showed that for every cardinal κ the cardinality of the collection of subsets of κ (which we call 2^κ) is at least the cardinal successor of κ (which we call κ^+). For infinite cardinals, it is independent of the usual assumptions of mathematics (the axioms "ZFC") whether $2^\kappa = \kappa^+$. Indeed the question of whether cardinality of all subsets of the natural numbers is equal to the first uncountable cardinal was the first problem on the famous list of problems presented by Hilbert at the 1900 International Congress of Mathematics. Partial information on this question is given by *König's Theorem* which says that the cofinality of 2^κ is at least κ^+.

Printed August 22, 2005
2000 Mathematics Subject Classification: Primary, 03E35, 03E55; Secondary, 03E05
Keywords: singular cardinals, Aronszajn trees, PCF theory, superatomic Boolean algebras, singular cardinals hypothesis, mutual stationarity, $I[\lambda]$

Gödel showed that in the Constructible Universe L, the *Generalized Continuum Hypothesis* holds; namely, for all infinite cardinals κ, $2^\kappa = \kappa^+$. For regular cardinals König's theorem is all one can say: it is a theorem of Easton that if $V \models$ GCH then for all monotone functions $f : OR \to OR$ such that $f(\alpha) \geq \alpha$ and $\mathrm{cf}(\aleph_{f(\alpha)}) > \aleph_\alpha$ there is a generic extension of V where $2^{\aleph_\alpha} = \aleph_{f(\alpha)}$ for all α where $\aleph_\alpha$ is regular (and moreover, all cardinals in V remain cardinals in the generic extension and cofinalities are not changed by the forcing).

At singular cardinals the situation turns out to be quite different. Silver [20] proved that if λ is a singular cardinal of uncountable cofinality and for a stationary collection of $\kappa < \lambda$, $2^\kappa = \kappa^+$ then $2^\lambda = \lambda^+$. This was improved by Galvin and Hajnal [9] to get general bounds on the power of a singular cardinal of uncountable cofinality in terms of the behavior of the power of smaller singular cardinals. At the conference, Gitik [10] announced recent results along this line, which are summarized in his paper for the proceedings.

This left the problem of cardinals with countable cofinality quite open. Magidor [12] showed that Silver's theorem is false for cardinals of countable cofinality: assuming large cardinals it is consistent for $2^{\aleph_\omega} > \aleph_{\omega+1}$ with the GCH holding below $\aleph_\omega$. After this result it was generally thought that the behavior of the power of singular cardinals of cofinality ω was as arbitrary as that of regular cardinals.

However, in the late 1980s Shelah proved a series of results getting cardinal bounds on the behavior of the power function at singular cardinals by studying reduced products of cardinals below the singular cardinal. This ultimately led to a powerful general tool known as PCF theory [19]. This theory has had many applications outside the study of cardinal arithmetic, constructing examples of Jonnson algebras on successors of singular cardinals, and providing interesting examples in set theoretic topology and algebra.

1.1 PCF theory problems We will say that a set A is an *interval of regular cardinals* if it is the intersection of an interval of cardinals with the regular cardinals. A will be called *progressive* if and only if $|A| < \min(A)$. If A is a set of regular cardinals then $\mathrm{PCF}(A)$ is defined to be

$$\{\mathrm{cf}(\prod A/D) : D \text{ is an ultrafilter on A}\}.$$

Shelah showed that if A is a progressive interval of regular cardinals with supremum λ then

$$\mathrm{cf}(\langle [\lambda]^{|A|^+}, \subset \rangle) = \max \mathrm{PCF}(A).$$

In particular $\max \mathrm{PCF}(A)$ always exists. As an immediate corollary one sees that if $|A| < \kappa < \lambda$ and κ is regular then

$$[\lambda]^\kappa = 2^\kappa \times \max \mathrm{PCF}(A).$$

In particular, if λ is a singular strong limit cardinal of cofinality κ that is not a cardinal fixed point and we take A to be a progressive tail of the regular cardinals below λ, then $2^\lambda = 2^\kappa \times \max \mathrm{PCF}(A)$.

It remains to bound the cardinality of $\mathrm{PCF}(A)$. Shelah did this by proving the remarkable theorem that if A is a progressive interval of cardinals then

$$(\dagger) \qquad |PCF(A) \leq |A|^{+3}.$$

Putting these results together we get the following corollary.

Theorem 1.1 (Shelah) *Suppose that* $\lambda = \aleph_\alpha$ *is a singular cardinal of cofinality* κ *and is not a cardinal fixed point. Then*

$$\lambda^\kappa < \max((2^\kappa)^+, \aleph_{\alpha+4}).$$

In particular, if $\aleph_\omega$ is a strong limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Despite significant progress by Gitik, Shelah, Woodin, and others, it is not known if these bounds are optimal. Our first questions relate to this.

Question 1.2 *Is it consistent to have a progressive set A such that* $|\mathrm{PCF}(A)| > |A|$*?*

Question 1.3 *Is it consistent that*

$$\max \mathrm{PCF}\{\aleph_n : 1 \leq n < \omega\} > \aleph_{\omega_1}?$$

For the next two problems we need a new definition which expands our scope beyond the possible cofinalities of intervals. Let κ be a singular cardinal of cofinality λ. We put a cardinal $\mu \in PP(\kappa)$ if and only if there is a sequence $\langle \kappa_i : i < \lambda \rangle$ and ultafilter D on λ such that $\lim_D \langle \kappa_i \rangle = \kappa$ and $\mu = \mathrm{cf}(\prod_{i<\lambda} \kappa_i / D)$. As in the case of progressive intervals, $PP(\kappa)$ is an interval of regular cardinals and $pp(\kappa)$ is defined to be its supremum.

Question 1.4 *Is it possible that*

$$\{\kappa < \lambda : pp(\kappa) \geq \lambda\}$$

be uncountable?

Question 1.5 *Is it possible that*

$$\{\kappa : \mathrm{cf}(\kappa) > \omega \text{ and } pp(\kappa) \geq \lambda\}$$

be infinite?

The assumption that the answers to Questions 1.4 and 1.5 are "no" is known as the Shelah *weak hypothesis*.[1]

1.2 PCF structures There are several collections of axioms that have been proposed to capture the essence of PCF theory. Indeed Shelah's original bound (†) was proved by summarizing results about the behavior of real PCF structures and showing that any structure satisfying his summary had to have small cardinality.

Jech [11] found a very weak collection of axioms that suffice to prove Shelah's bound. Here our intention is different. We want to find as strong a collection of axioms as possible and see if they can prove a better bound.

This project then has two directions: the first is to establish whether a better bound on the size of PCF structures can be proved. The second is to find a "complete" axiomatization of PCF structures. We will use here an axiomatization due to Magidor (with aid from Foreman). It appeared in print in 1998 in the Ph.D. thesis of Ruyle [16].

1.2.1 The PCF topology Inherent in the axiomatization is the PCF topology. The operation $A \mapsto \mathrm{PCF}(A)$ is a closure operator and hence there is a natural topology associated with the PCF operation. For simplicity we will restrict ourselves to sets A of regular cardinals such that $\mathrm{PCF}(A)$ is a progressive set that has no limit points that are cardinal fixed points. (In particular, these properties hold for progressive intervals of cardinals.)

Explicity: $A \subset \mathrm{PCF}(A)$ and for all $B, C \subset \mathrm{PCF}(A)$,

1. if $B \subset C$ then $\mathrm{PCF}(B) \subset \mathrm{PCF}(C)$,
2. $\mathrm{PCF}(B \cup C) = \mathrm{PCF}(B) \cup \mathrm{PCF}(C)$,
3. $\mathrm{PCF}(\mathrm{PCF}(B)) = \mathrm{PCF}(B)$.

The PCF topology is compact Hausdorff, 0-dimensional, and scattered. Via Stone duality there is a direct connection between locally compact Hausdorff, 0-dimensional, scattered spaces and superatomic Boolean Algebras. Namely, given such a space X, the clopen sets form a superatomic Boolean algebra whose Stone space is the original space X.

To review: Let B be a Boolean Algebra. Define a transfinite sequence of ideals in B by setting

1. J_0 to be the ideal generated by the atoms of B,
2. $J_{\alpha+1}$ the ideal generated by the atoms of B/J_α and J_α,
3. for limit α, $J_\alpha = \bigcup_{\beta<\alpha} J_\beta$.

B is *superatomic* if and only if whenever J_α is a proper ideal, B/J_α is atomic. (We will use the jargon "SBA" for superatomic Boolean algebra.)

If one traces through the proof of Stone duality, it is immediate that the atoms of B/J_α correspond canonically with the isolated points in the αth Cantor-Bendixson derivative of the Stone space of B.

We now give some more definitions necessary to formulate the PCF axioms:

1. the *height* of B is the least α, $J_\alpha = B$;
2. the *rank* of $b \in B$ is the least α, $b \in J_\alpha$;
3. c_α is defined to be the cardinality of $\{b \in B : \text{rank of } b = \alpha\}$;
4. the *cardinal sequence* of B is $\langle c_\alpha : \alpha < \text{height of } B\rangle$.

There is a standard mechanism for building SBAs involving well-founded partial orderings. Let $<^*$ be a well-founded partial ordering on a set T. For $t \in T$, let $b_t = \{s : s <^* t\}$.

An SBA *ordering* will be a pair $(<^*, i)$ such that $<^*$ is a well-founded ordering on a set T and

$$i : [T]^2 \to [T]^{<\omega}$$

is such that

1. for all $s, t,\ i(s,t)$ is a minimal set such that
$$b_s \cap b_t = \bigcup_{u \in i(s,t)} b_u$$
(so if $i(s,t) = \{u_0, \dots u_n\}$ then
$$b_s \cap b_t = b_{u_0} \cup \cdots \cup b_{u_n}.)$$
2. for all $t \in T$, α less than the $<^*$-rank of t,
$$b_t \cap \{s : \mathrm{rank}(s) = \alpha\}$$
is infinite.

Other authors call SBA orderings "selectors" or "admissible partial orderings." Given an SBA ordering on a set T we can topologize T by taking basic open sets to be of the form

$$b_t \setminus (b_{u_0} \cup b_{u_1} \cup \cdots \cup b_{u_n}).$$

The following proposition is standard.

Proposition 1.6 *Let* $(<^*, i)$ *be an* SBA *ordering on a set* T *and endow* T *with the topology above. Then*

1. T *is locally compact, Hausdorf,* 0*-dimensional, and scattered,*
2. *if* $T \subset b_{u_0} \cup b_{u_1} \cdots \cup b_{u_n}$*, for some* u_i*s, then* T *is compact,*
3. *the* α*th Cantor-Bendixson derivative of* T *is* $\{t : \text{the } <^*\text{-rank of } t \text{ is at least } \alpha\}$*,*
4. *the algebra of clopen subsets of* T *is an* SBA *with cardinal sequence*

$$c_\alpha = |\{t : \text{ the rank of } t = \alpha\}|.$$

We are now in a position to give the PCF axioms.

Definition 1.7 A δ-PCF *structure* (or PCF *algebra*) is an SBA partial ordering $<^*$ on a successor ordinal θ satisfying

PCF1 $\nu <^* \mu$ implies $\nu \in \mu$;
PCF2 $\overline{\delta} = \theta$;
PCF3 if $I \subset \theta$ is an interval, then $\overline{I}$ is also an interval;
PCF4 for each $\nu < \theta$ of uncountable cofinality, there is a closed unbounded $C_\nu \subset \nu$ such that $\overline{C_\nu} \subset \nu + 1$;
PCF5 θ is compact with the $<^*$ topology.

The main point of the axioms is that the work of Shelah shows that the PCF axioms are true.

Theorem 1.8 (Shelah [19]) *Let* A *be a progressive interval of regular cardinals of order type* δ*. Then there is an ordering* $<^*$ *on* PCF(A) *which makes* PCF(A) *into a* δ-PCF *structure.*

(Hint: Suppose that PCF$(A) = \langle \kappa_\alpha : \alpha \leq \alpha^* \rangle$. To define $<^*$ on α^*, find a "transitive" collection of generators $\langle b_\kappa : \kappa \in \text{PCF}(A)\rangle$ for the PCF ideals on PCF(A) and define $\beta <^* \alpha$ if and only if $\kappa_\beta \in b_{\kappa_\alpha}$.)

We now are in a position to state the main open questions involving PCF structures.

Question 1.9 (PCF completeness) *Do the* PCF *axioms capture ALL of* PCF *theory?*

Question 1.10 *What* PCF *structures consistently exist?*

We need some more background to make these questions explicit. Let $(\theta, <^*)$ be a δ-PCF structure. Let $\langle c_\alpha : \alpha < \text{ht}(<^*)\rangle$ be the cardinal sequence of $(\theta, <^*)$. Then

1. ($|\delta|$-tightness/localization) if $A \subset \theta$ and $\alpha \in \overline{A}$ then there is a $B \in [A]^{|\delta|}$ such that $\alpha \in \overline{B}$ (in fact, using results of Todorčević, if $\delta = \omega$ the topology is "sequential");
2. if X is closed then $\sup X \in X$;
3. for $\xi < \text{ht}(<^*)$, $c_\xi \leq |\xi|$;
4. if $\theta = \kappa + 1$, then there is a closed unbounded set of $\xi < \kappa$ such that $c_\xi \leq |\delta|$.

These facts show a close connection between PCF structures and the literature about cardinal sequences for SBAs, especially those that have each $c_\alpha = \omega$. Using the work of Baumgartner and Shelah [1] and extending work of Velickovic, Ruyle proved that if $\langle c_\alpha : \alpha < \omega_2\rangle$ is a cardinal sequence with $c_\alpha = \omega$ on a closed unbounded set, then there is a cardinal preserving forcing for adding an SBA on $\omega_2 + 1$ with this cardinal sequence (and a little further). Moreover, if $\langle c_\alpha : \alpha < \gamma < \omega_2\rangle$ is a cardinal sequence where $c_\alpha = \omega$ for $\alpha < \omega_1$ and $|c_\alpha| \leq \omega_1$, then there is a PCF algebra of height $\gamma + 1$ with this cardinal sequence.

Question 1.11 *Is it consistent that there is an ω-PCF algebra of size ω_3? (If not, there is a better bound on $2^{\aleph_\omega}$.)*

This requires some new SBA techniques as there are no known examples of SBAs of height $\omega_3 + 1$ which have each countable level countable, and in which there is a closed unbounded collection of levels of cardinality ω_2 that are countable.

Question 1.12 *Is it consistent that there are ω-PCF algebras of height δ for all $\delta < \omega_3$? What about $\delta = \iota + 1$ where ι is the first indecomposible ordinal above ω_2?*

Question 1.12 may not require new SBA techniques, as Martínez, in work exposited at the workshop [14], has showed it consistent that there are thin SBA algebras of all heights less than ω_3.

The question of "PCF completeness" is a little vaguer and may involve all of the difficulties of the SCH itself. However, here is a concrete version of the question that may be somewhat easier.

Question 1.13 *Assuming large cardinals, is it true that if $\mathfrak{A}$ is a PCF structure then there is a forcing extension which produces a κ such that $\mathfrak{A}$ is isomorphic to a closed subset of $\mathrm{PCF}(\kappa \cap \{regular cardinals\}) \cap \{regular cardinals\}$? This subset should be of the form $\mathrm{PCF}(A)$ where A is a progressive subset of the regular cardinals of κ.*

We conclude with a problem of Todorčević about PCF structures. Topological results of Todorčević can be used to show that PCF structures are *sequential*. This leads to the following question.

Question 1.14 *What is the* sequential rank *of* $\mathrm{PCF}(\{\aleph_n : n > 1\})$?

In his talk, Martínez gave a collection of problems about the structure of SBAs that are not necessarily PCF algebras. These problems will appear in the proceedings of the conference.

2 Stationary Set Reflection, Variations of Square, Scales, and Aronszajn Trees

In 1989 Woodin and others asked whether the failure of the Singular Cardinals hypothesis at a cardinal κ of cofinality ω implied the existence of an Aronszajn tree on κ^+. The existence of special Aronszajn trees was proved by Jensen in the 1970s to be equivalent to the existence of a weak square sequence, so Woodin's question seems closely related to questions about square sequences of various types. Investigations of square properties in inner models for large cardinals led to the isolation of certain square properties weaker than conventional square [17]. These turned out to have direct relations to previously known combinatorial properties such as weak square and very weak square [7]. In this section we present some background and state some problems that remain open.

We begin first by motivating Woodin's question: As noted in the previous paragraph, Jensen showed that there is a special Aronszajn tree on κ^+ if and only if $\square^*_\kappa$ holds. Shelah showed that there are no Aronszajn trees on κ^+ if κ is a singular limit of strongly compact cardinals. Using this work, Magidor and Shelah [13] showed that if it is consistent that there is a 2-huge cardinal then it is consistent that there is no Aronszajn tree on $\aleph_{\omega+1}$.

Lacking any evidence to the contrary these results suggest that the failure of existence of Aronszajn trees on successors of cofinality ω cardinals is tied to being a limit of strongly compact cardinals. Since results of Solovay [21] show that the SCH holds above a strongly compact cardinal Woodin's question seems quite natural. We list it in the following form.

Question 2.1 *If there are no Aronszajn trees on $\aleph_{\omega+1}$ and $\aleph_\omega$ is a strong limit, is it true that $2^{\aleph_\omega} = \aleph_{\omega+1}$?*

Cummings, Foreman, and Magidor initiated a program of giving an affirmative answer to Woodin's question. The philosophy was to try to use PCF theory to construct Aronszajn trees. It has the following components:

1. isolate PCF properties that are consequences of square,
2. show that they imply the existence of Aronszajn trees,
3. show that they follow from the failure of SCH.

Figure 1 is a summary of the results of this program. This diagram includes results from [7], [4], [5], and [3]. Some of the arrows and nonarrows in the diagram were the main contents of the series of talks given by Cummings and Magidor at the workshop.

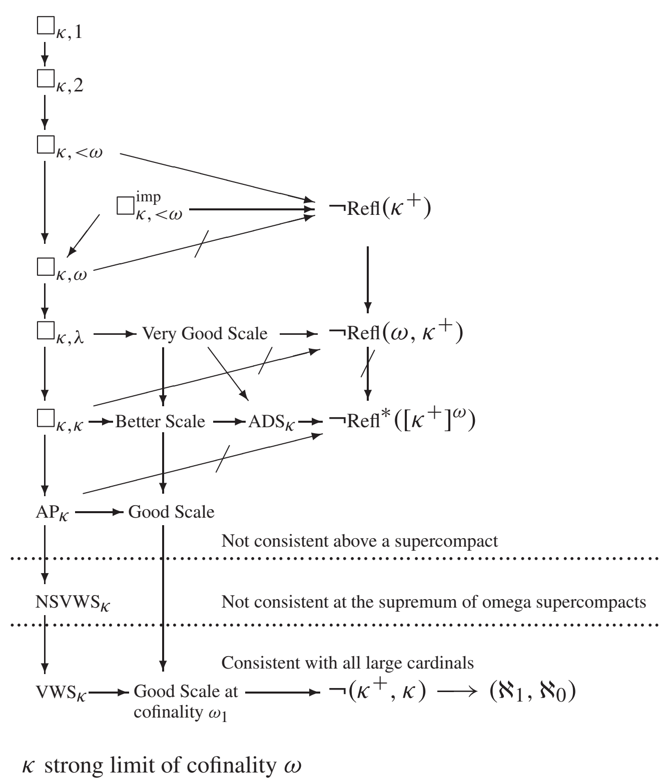

Figure 1 Known relationships between weak square properties, scales, and reflection properties.

As an aid to interpreting Figure 1, we note that

1. $\square_{\kappa,\lambda}$ is the version of square that allows λ many sets to cofinalize each ordinal, and these sets have to cohere,
2. $\square_{\kappa,\kappa}$ is the same as weak square, $\square^*$, and is equivalent to the existence of a special Aronszajn tree,
3. AP_κ is the *approachability property*, defined in Section 3,
4. VWS stands for a property called *very weak square* and NSVWS is *not so very weak square*,
5. Refl is the weak reflection property and Refl* is a slight strengthening of the reflection property.

Recent results of Gitik and Sharon deal a major blow to this program when they showed the following.

Theorem 2.2 (Gitik, Sharon) *From appropriate large cardinals it follows that*

1. *Con(λ is singular strong limit of cofinality ω, $2^\lambda > \lambda^+$ and the approachability property fails);*
2. *Con(There is a singular strong limit cardinal λ, and $\langle \lambda_i : i \in \omega\rangle$ cofinal in λ with* $\mathrm{PCF}(\lambda_i : i \in \omega) = \{\lambda_i : i \in \omega\} \cup \{\lambda^+\}$ *but no very good scale on $\langle \lambda_i \rangle$ of length λ^+);*
3. *Con(λ is a singular strong limit cardinal, $2^\lambda > \lambda^+$ and every stationary subset of λ^+ reflects).*

In particular, these results show that one cannot hope to prove (for example) that the failure of the SCH implies the approachability property or that there is a very good scale. Both of these latter propositions were viewed as candidates for a property intermediate between the failure of the SCH and the existence of Aronszajn trees.

There are some potential loopholes in the Gitik/Sharon results though. Their arguments can be improved to make λ into $\aleph_{\omega^2}$, but are not yet known to apply to $\aleph_\omega$. Thus, they may not be directly relevant to Question 2.1. There are examples of properties (such as the equivalent between the approachability property and very weak square) that hold at $\aleph_\omega$, but not at $\aleph_{\omega^2}$. A very strong conjecture might be that the following question has an affirmative answer.

Question 2.3 *If $\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} > \aleph_{\omega+1}$, then $\square^*_{\aleph_\omega}$ holds.*

We note that Gitik and Sharon have been able to show that there is a model where $pp(\aleph_\omega) > \aleph_{\omega+1}$, and the approachability property fails at $\aleph_{\omega+1}$, but in this model $\aleph_\omega$ is not a strong limit.

In the second result, the sequence $\langle \lambda_i : i \in \omega\rangle$ is not the generator b_{λ^+}. In particular, the following remains open.

Question 2.4 *If λ has cofinality ω, is it true that there is some sequence $\langle \lambda_i : i \in \omega\rangle$ cofinal in λ which has a very good scale of length λ^+.*

The problem of the relation between scale properties and Aronszajn trees seems interesting on its own merits. A typical question here might be this.

Question 2.5 *If λ has cofinality ω and there is some sequence $\langle \lambda_i : i \in \omega\rangle$ cofinal in λ which has a very good scale of length λ^+ is it necessarily true that there is an Aronszajn tree on λ^+?*

Affirmative answers to both Questions 2.4 and 2.5 yield a solution to Woodin's question. Here is a variation of Questions 2.4 and 2.5.

Question 2.6 *If λ has cofinality ω and the approachability property holds at λ^+, is it necessarily true that there is an Aronszajn tree on λ^+?*

We note that the diagram leaves many problems open (and there are "obvious" arrows that we have not included in the diagram).

3 $I[\lambda]$ and Partial Squares.

Shelah's ideal $I[\lambda]$ was an important topic in the workshop. This ideal can be defined as follows.

Definition 3.1 Let λ be a regular cardinal. Let $\vec{X} = \langle a_\alpha : \alpha < \lambda \rangle$ be a sequence of bounded subsets of λ. Define $A(\vec{X})$ (the ordinals *approachable with respect to* X) as the collection of all singular $\beta < \lambda$ such that there is a set $C \subset \beta$ such that

1. C is unbounded in β and the order type of C is the cofinality of β,
2. for all $\gamma < \beta$ there is an $\alpha < \beta$ such that $C \cap \gamma = a_\alpha$.

The ideal $I[\lambda]$ is defined to be the ideal generated by all sets of the form $A(\vec{X})$ over the nonstationary ideal.

This ideal is normal and λ-complete and turns out to have close connections to forcing, especially for arguments that show (λ, ∞)-distributivity.

If $\lambda = \kappa^+$ and $[\kappa^+]^{<\kappa}$ has cardinality κ^+, then $I[\kappa^+]$ contains a stationary set S such that $I[\kappa^+]$ is generated by the nonstationary ideal restricted to $\kappa \backslash S$. Without the cardinal arithmetic assumption, it was a longstanding open problem whether $I[\kappa^+]$ contained a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$. This was recently settled by Mitchell [15] who showed that at ω_2 this need not be the case. His techniques also show that it is consistent that $I[\omega_2]$ is not generated by a single set over the nonstationary ideal. Mitchell's results will appear in the proceedings of this conference. While it appears promising it is not completely clear that Mitchell's techniques generalize to ω_3. Thus we ask the following question which might not remain open for long.

Question 3.2 *For regular $\kappa \geq \omega_2$ must $I[\kappa^+]$ contain a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$?*

Because of its close connection to forcing it would be very useful to know the answers to the following questions.

Question 3.3 *Can $I[\omega_2]$ be ω_3-saturated? Can $I[\omega_2] \subset J$ for some ω_3-saturated ideal J on ω_2?*

The *approachability property* mentioned above is the statement that $I[\lambda]$ is not a proper ideal. If square holds, then the square sequence itself is a witness to $\lambda \in I[\lambda]$. In general, $I[\lambda]$ can be viewed as those sets on which there is a defective square sequence with its timing out of order.

We now define a closely related notion. If $S \subset \lambda$ then a *partial square sequence* on S is a sequence of sets $\langle C_\alpha : \alpha \in S \rangle$ such that

1. C_α is an unbounded subset of α of order type the cofinality of α;
2. if β is a limit point of both C_α and C_γ ($\alpha, \gamma \in S$) then $C_\alpha \cap \beta = C_\gamma \cap \beta$.

Shelah showed that if $\mu < \kappa$ are regular then $\kappa^+ \cap \mathrm{cof}(\mu) = \bigcup_{\delta \in \kappa} S_\delta$ where each S_δ carries a partial square sequence. In particular, $\kappa^+ \cap \mathrm{cof}(\mu) \in I[\kappa^+]$.

At successors of singular cardinals, this type of question appears quite open. In particular we would like to know the following.

Question 3.4 *Is it provable in* ZFC *that there is a partial square sequence on a stationary subset of* $\aleph_{\omega+1} \cap cof(\omega_1)$*? On other cofinalities?*

In contrast to the successors of regular cardinals, it is always the case that $I[\kappa^+]$ contains a stationary set: if κ is singular and $\mu < \kappa$ is regular, then $I[\kappa^+]$ contains a stationary subset of cof(μ). Indeed in most cofinalities it is not known if $I[\kappa^+]$ can be a proper ideal. At $\aleph_{\omega+1}$ it is consistent that there is a stationary subset of $\aleph_{\omega+1} \cap \text{cof}(\omega_1)$ that does not belong to $I[\aleph_{\omega+1}]$, but this is not known at other cofinalities. This is our next question.

Question 3.5 *Does* $I[\aleph_{\omega+1}]$ *contain a closed unbounded set relative to cofinality* ω_2*?*

Here is a related question.

Question 3.6 *At successors of singular cardinals, is* $I[\lambda]$ *generated by a single set over the nonstationary ideal?*

In the same vein, it would be interesting to understand the relationship between the collection of approachable points in successors of singular cardinals and other natural stationary sets. A typical question here might be described as follows. If $b_{\aleph_{\omega+1}}$ is the generator for PCF$(\{\aleph_n : n \in \omega\})$ at $\aleph_{\omega+1}$, then relative to a closed unbounded set any two continuous scales agree on the collection of good points. Hence the collection of "good points" forms a well-defined stationary set (modulo the closed unbounded filter). An extreme form of a question relating canonical structure would be this.

Question 3.7 *Is* $I[\aleph_{\omega+1}] = \text{NS} \restriction \{\textit{Good Points}\}$*?*

We note that it is known that $I[\aleph_{\omega+1}]$ includes $\text{NS} \restriction \{\text{Good Points}\}$ and that if square holds below $\aleph_\omega$, then the two ideals coincide ([19], [5], and [3]).

At the workshop Eisworth [6] gave a collection of problems involving a "recipe" for generating ideals from squarelike principles and his contribution to the proceedings will list these questions.

4 Stationary Sets

In [8] Foreman and Magidor began to develop a theory of stationary sets for singular cardinals of countable cofinality. We work on the $\aleph_n$s for simplicity. Since a subset $A \subset \aleph_\omega$ naturally gives to a sequence of subsets $S_n = A \cap \omega_n$ we deal with sequences of subsets of the ω_ns directly.

Let θ be a large regular cardinal and $S \subset \mathcal{P}\mathcal{P}(\theta)$. Let $\langle S_n : m \leq n < \omega \rangle$ be a sequence of sets with $S_n \subset \omega_n$. Then the sequence S_n is S-*stationary* if and only if

$$\{N : \text{for all } n \geq m, \sup N \cap \omega_n \in S_n\} \in S.$$

Define $\chi_N(n) = \sup N \cap \omega_n$. Then we can restate this as saying that $\chi_N \in \prod_{m \leq n} S_n$. To illustrate the definition we give two important examples.

Example 4.1 $S = \{A \subset \theta : A \text{ is stationary}\}$. For this example we call the sequence *mutually stationary*.

Example 4.2 $S = \{A \subset \theta : A \text{ is stationary and consists of tight structures}\}$, where N is *tight* if and only if $N \cap \prod \omega_n$ is cofinal in $\prod (N \cap \omega_n)$ (i.e., $N \cap \prod \omega_n$ is cofinal below χ_N). This is called *tight stationarity*.

We note that there are many other interesting examples taken by varying S. One is obtained by taking S to be the internally approachable structures.

The theory of mutual stationarity and its variants is still in its infancy despite some success. In particular, there are a large number of embarassing problems still completely open. (Welch [22] gives another collection.)

Question 4.3 *Is there a* ZFC *example of a sequence of stationary sets* $\langle S_n \subset \omega_n : n \in \omega\rangle$ *such that* $\langle S_n\rangle$ *is not mutually stationary? For concreteness we may demand that* $S_n \subset \mathrm{cof}(\omega_1)$. *Find a* combinatorial *property that implies the existence of such a set.*

Foreman and Magidor showed that such a sequence exists in L and Welch, Schindler, and others have extended their results to certain inner models for large cardinals. The question of the existence of such sequences is open even in many well-studied inner models.

Solovay showed that every stationary subset of a regular cardinal κ can be split into κ many disjoint stationary subsets. Foreman and Magidor showed that a tightly stationary sequence of sets consisting of ordinals of a fixed cofinality μ can be split into μ many disjoint tightly stationary sequences. For mutual stationarity we do not know if we can split a sequence into even two disjoint mutually stationary sequences.

Question 4.4 *Suppose that* $\langle S_n : n \in \omega\rangle$ *is mutually stationary. Are there* $\langle S_n^0, S_n^1 : n \in \omega\rangle$ *such that*

1. *S is the disjoint union of* S_n^0, S_n^1,
2. $\langle S_n^i\rangle$ *is mutually stationary for* $i = 0, 1$.

A subproblem for Question 4.4 would be to isolate the appropriate Fodor's Theorem. We note that the natural conjecture would be that if $\langle S_n : m \leq n \in \omega\rangle$ is mutually stationary, then each S_n can be partitioned into ω_n disjoint subsets $\langle S_n^\alpha : \alpha < \omega_n\rangle$ such that for every function $f \in \prod_{m\leq n\in\omega} \omega_n$ the sequence $\langle S_n^{f(n)} : m \leq n\rangle$ is mutually stationary.

There are a whole host of related problems. We note the following definitions, which we give for sets of cardinality ω_1, again for concreteness. Let $N \prec H(\lambda)$ have cardinality ω_1. Then N is

1. N is *internally unbounded* iff $N \cap [N]^{\aleph_0}$ is unbounded in $[N]^{\aleph_0}$;
2. N is *internally stationary* iff $N \cap [N]^{\aleph_0}$ is stationary in $[N]^{\aleph_0}$;
3. N is *internally club* iff $N \cap [N]^{\aleph_0}$ contains a closed unbounded set in $[N]^{\aleph_0}$;
4. N is *internally approachable* iff $N = \bigcup_{\alpha<\omega_1} N_\alpha$ where each N_α is countable and for $\beta \in \omega_1$, $\langle N_\alpha : \alpha < \beta\rangle \in N$.

Under certain circumstances, such as the CH, these properties are all equivalent. It is not clear in general what the relation is.

Question 4.5 *Give examples separating the properties (1)–(4).*

Many properties in set theory propagate through successor cardinals but require special hypothesis to pass through limit cardinals. (This is one of the main reasons for the workshop.) There are, however, some properties where the propagation is not clear. We give one example that would seem to require useful new ideas.

Question 4.6 *Suppose that* κ *is regular,* $N \prec H(\theta)$ *and* $N \cap [N \cap \kappa]^{\aleph_0}$ *is stationary. Is* $N \cap [N \cap \kappa^+]^{\aleph_0}$ *stationary?*

5 General Combinatorial Problems

We list here several problems that were asked at the conference. The first is due to Hajnal who announced a \$250 (US) prize for *any significant progress* on the problem.

Question 5.1 *Does* $\omega_2 \to (\alpha)^2_\omega$ *for* $\omega_1 + 1 < \alpha < \omega_2$*?*

We note that it is also an interesting problem to determine what happens at successors of singular cardinals.

Cummings reminded the audience of the following 2 closely related questions, the first appeared in [2].

Question 5.2 *Is it consistent that there is a forcing that makes* $\aleph_{\omega+1}$ *into* ω_2*?*

Question 5.3 *Is it consistent that* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\omega_2, \omega_1)$*?*

In the presence of Woodin cardinals a positive answer to Question 5.3 yields a positive answer to Question 5.2.

Schimmerling (as explicated in his contribution to the Proceedings [18]) noted the following question.

Question 5.4 *Is it consistent to have the* GCH*, weak square, and no Suslin trees on* $\aleph_{\omega+1}$*? What about* $\square_{\aleph_\omega,\omega}$*?*

Question 5.5 (Steel) *Let M be the canonical minimal iterable extender model with a Woodin limit of Woodin cardinals* λ*. Let N be a derived determinacy model obtained by forcing over M with the Levy collapse making* $\lambda = \omega_1^N$*. (Thus N satisfies* AD_R*.) Prove or refute:* Θ *is regular in N.*

Reward: *\$200*

The next two questions were asked with significant cash prizes.

Question 5.6 (Steel) *Working in ZFC, either*

(a) *show that if* ZFC *plus "there is a singular strong limit* κ *such that* $\neg\square_\kappa$*" is consistent, then so is* ZFC *plus "there is a superstrong cardinal", or*
(b) *show that if there is a superstrong cardinal, then* ZFC *plus "there is a singular strong limit* κ *such that* $\neg\square_\kappa$*" is consistent.*

Reward*: For (b), \$300. For (a), \$4000−500x, where x is the time in years from May 1, 2004 to the submission of a manuscript with a correct, complete proof. UC Berkeley faculty are not eligible for the reward.*

Question 5.7 (Woodin) *Suppose that there is an extendible cardinal. Must* HOD *compute the successor correctly for some (uncountable) cardinal?*

Prize:

$$\$1000[\max(\min(n, 10 - n), 1)]$$

where

$$n = (\textit{calender year of submission}) - 2004.$$

Terms: *Collect if a correct proof is given for either "yes", or if a correct proof is given that the failure implies the consistency with* ZFC *of the large cardinal I0 of Kanamori's book. (Details: Clay rules)*

Note

1. These questions are well known, but relayed to the author by Gitik.

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Acknowledgments

Partially Supported by NSF Grant DMS 445860

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