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# A THEOREM ON S4.2 AND S4.4

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Theorem. If *ML* is abbreviated as *R*, the pure *C*-*N*-*R*-fragment of S4.2 can be axiomatized and contains a model of S4.4.

Proof. (i) The following theses and rule are in S4.2:

*R1.* *CRCpqCRpRq*
*R2.* *CRpRRp*
*R3.* *CNRpRNRp*
*R4.* *CRNpNRp*
*R5.* *From α to infer Rα*

Indeed all but *R1*, *R4* are in S4. Let **PC**, *C*-detachment, substitution, *R1*-*R5* be denoted as {*R*}. Taking {*R*} as primitive and the definition

Df.*L* $L\alpha = K\alpha R\alpha$

we can obtain the theses and rule

*L1.* *CLpp*
*L2.* *CLpLLp*
*L3.* *CpCNLpLNLp*
*L4.* *From α to infer Lα*
*L5.* *CLCpqCLpLq*
*L6.* *CNLNLpRp*
*L7.* *CRpNLNLp* .

*L1*-*L5* constitute a model of S4.4.

(ii) {*R*} is complete for pure *C*-*N*-*R*-theses in S4.2. For let *α* be such a thesis; then there is a corresponding *ML*-thesis provable from **PC**, *L1*-*L5*, since S4.4 contains S4.2. But then by *L6*, *L7* the *R*-thesis is provable in the *L*-system, and so from {*R*}. (i) and (ii) prove the theorem. It follows that the matrix of S4.2 can be used to decide S4.4—just eliminate *L* in the expression under test, by Df.*L*, and see whether the result is provable in S4.2.

It is worth noting that *L1*-*L6* follow from *R1*-*R3* and *R5*. But *R4* is independent (take *R* as Verum) and is needed for *L7*.

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*R1-3*, *R5* also contains a model of S5, in the sense that $R\alpha$ is provable here if and only if $\alpha$ is provable in S5. The key to this is that if *R4* is replaced by *CRpp* we have S5, but *RCRpp* is provable without *R4*.

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