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TOPOLOGICAL GEOMETRIES AND A NEW CHARACTERIZATION OF R^m

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INTRODUCTION*. Mathematicians engaged in research in one area of study may find concepts which have already been defined and studied in another area, and they can therefore draw on those definitions and results which already exist. In this way, topology has already been used to study geometry; for example, in certain axiomatizations of Euclidean plane geometry, "natural" topologies, having as subbasis elements either half-planes or interiors of triangles, can be defined on the underlying set, and these topologies can then be used in the formulation of propositions, or further axioms.

Starting with chapter II in this paper, however, the underlying set on which a "geometry" is defined is assumed already to possess the structure of a topological space. The geometry is an additional structure with axioms to bind topological and geometric structures together. We may compare what has been done to the example of a topological group, where two structures, algebraic and topological, are related by the continuity of the group operation.

The usual manner by which a structure is given to a point set X is by the selection of certain distinguished subsets either of X itself, or of certain sets related to X, or of both. A topology on X is defined using a family of distinguished subsets of X, while an operation on X is defined by a subset of $(X \times X) \times X$. Classical geometries usually call for, either implicitly or explicitly, the existence of distinguished subsets called lines, planes, or k-dimensional subspaces. We define a geometry on a set X in terms of distinguished subsets of X called k-flats, which are generalizations of k-dimensional subspaces.

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Though the concept of geometry as defined in chapter I seems worthy of study in its own right, we are primarily concerned in this paper with properties generalized from the n-dimensional Euclidean geometry on $\mathbf{R}^m$, and our goal is a surprisingly simple characterization of $\mathbf{R}^m$ as the only space which admits a geometry having certain properties. We remark here that the n-sphere can also be characterized in terms of the geometries it admits, but this will be the subject of a future paper.

In $\mathbf{R}^m$, the usual Euclidean geometry, the standard metric, and the algebraic structure of $\mathbf{R}^m$ as a vector space are tightly bound together. In our initial abstraction we do away with coordinatization, metrics, numbers, and the algebraic structure of geometries derived from linear manifolds. We find, however, that if topology and geometry are related in a very natural way, then we begin to gain back many of the properties we discarded in order to get the essentials of a geometry. We never regain the algebraic structure, but we do regain many of its advantages in a more general setting.

Chapter I will essentially deal with those set theoretic definitions and propositions that will be of use in later chapters. Chapter II generalizes convexity, the link between geometry and topology, and deals in large part with assumptions which enable us to get the order properties of lines. Chapter III expands on a generalized notion of simplex and gives assumptions that insure simplices and flats will behave in a manner which resembles their behavior in $\mathbf{R}^m$. Chapter IV investigates what happens in an m-arrangement, the structure developed in chapter III, when the generalized lines have no end points. Chapter V is concerned with the effect of a parallel postulate on m-arrangements. This chapter shows that a good deal less than Euclidean geometry is needed to force the underlying space of a geometry to be a product space.

CHAPTER I

BASIC CONCEPTS

Definition 1: *Let X be a set. An element of X is called a point. We define $G = \{F^{-1}, F^0, \ldots, F^n\}$ to be a geometry on X if the following axioms are satisfied*:

1) *$F^i \subseteq \mathcal{P}(X)$, the power set of X, $-1 \leqslant i \leqslant n$. An element of F^i is called an i-flat, or merely just a flat.*

2) *$F^{-1} = \{\phi\}$.*

3) *If f is an i-flat, $-1 \leqslant i \leqslant n$, then $f \neq X$.*

4a) *Every set of $i+1$ points not all contained in some k-flat, $k < i$, is contained in at least one i-flat, $-1 \leqslant i \leqslant n$.*

4b) *Every set of $i+1$ points not all contained in some k-flat, $k < i$, is contained in at most one i-flat, $-1 \leqslant i \leqslant n$.*

Combining (4a) and 4b) we have

4) *Every set of $i + 1$ points not all contained in some k-flat, $k < i$, is contained in a unique i-flat, $-1 \leq i \leq n$.*

5) *The intersection of any two flats is again a flat.*

6) *If $k < i$, then no i-flat is contained in a k-flat.*

n is called the length of G, and we write $l(G) = n$. i is called the dimension of F^i, as well as the dimension of any flat f in F^i; we write $\dim f = i$. By G^* we denote $\{F^{-1}, \ldots, F^n, \widetilde{F}^{n+1}\}$, called the augment of G, where $\widetilde{F}^{n+1} = \{X\}$. X is then considered to be an $n + 1$-flat. G^* satisfies all the axioms for a geometry except 3); however, for simplicity, all propositions and definitions in this paper will refer to G^*, unless they specifically state that they apply only to G.

Lemma: 1)-6) *are independent.*

Proof: independence of 1): $X = \{0,1\}$. $F^{-1} = \{\phi\}$, $F^0 = \{\{0,2\}, \{1,3\}\}$.
independence of 2): $X = \{0,1\}$. $F^{-1} = \{\phi,\{0\}\}$, $F^0 = \{\{1\}\}$.
independence of 3): G^*, the augment of G, a geometry on X, if a geometry on X exists;
independence of 4a): $X = \{0,1\}$. $F^{-1} = \{\phi\}$, $F^0 = \{\{0\}\}$.
independence of 4b): $X = \{0,1,2\}$. $F^{-1} = \{\phi\}$, $F^0 = \{\{0\}, \{1\}, \{2\}, \{0,1\}\}$.
independence of 5): $X = \mathbf{R}^2$. $F^{-1} = \{\phi\}$, $F^0 = \{\{x\} | x \in \mathbf{R}^2\}$, $F^1 = \{l \,|\, l$ *is a straight line in* $\mathbf{R}^2\}$. Since every triple of non-collinear points of $\mathbf{R}^2$ determines a unique circle, we can set $F^2 = \{C | C$ *is a circle in* $\mathbf{R}^2\}$;
independence of 6): $X = \mathbf{R}^1$. $F^{-1} = \{\phi\}$, $F^i = \{H \subseteq \mathbf{R}^1 | 1 \leq \operatorname{card} H \leq i + 1\}$.

1.0: Examples of geometries

i) Let M_F be an n-dimensional vector space over a field F. Let $F^i = \{x + H | x \in M$; *H is an i-dimensional subspace of* $M_F\}$, $0 \leq i \leq n-1$. (It is assumed henceforth, in accordance with 2), that $F^{-1} = \{\phi\}$.) $G = \{F^{-1}, \ldots, F^{n-1}\}$ is a geometry on M, and is affine in sense of definition 6 of this chapter.

ii) $X = \mathbf{S}^2$. $F^0 = \{\{x,y\} | x$ *antipodal to* $y\}$, $F^1 = \{C | C$ *is a great circle on* $\mathbf{S}^2\}$. $G = \{F^{-1}, F^0, F^1\}$ is a semi-projective geometry (definition 7) on $\mathbf{S}^2$.
iii) Let X be an infinite set. Set $F^i = \{H \subseteq X | \operatorname{card} H = i + 1\}$, $0 \leq i < n + 1 < \infty$.

We assume throughout that the geometries with which we deal have finite length. Such an assumption will be necessary in later chapters, though not always in this one.

Prop. 1.1: *If f is a k-flat, $k \neq -1$, then $f \neq \phi$.*

Proof: By 6).

Prop. **1.2**: *If* $G = \{F^{-1}, \ldots, F^n\}$ *is a geometry on* X, $n > -1$, *then* $G' = \{F^{-1}, \ldots, F^{n-1}\}$ *is also a geometry on* X.

The proof is trivial.

Prop. **1.3**: *If* f *is an* i*-flat and* f' *is a* j*-flat, then* $f \cap f'$ *is a* k*-flat with* $k \leq min(i,j)$.

Proof: By 5), $f \cap f'$ is a flat. We may suppose $i \leq j$. Suppose $\dim(f \cap f') = k > i$. Then by **6**), $f \cap f' \not\subseteq f$, an obvious contradiction.

Definition **2**: *A set* $S = \{x_0, \ldots, x_k\}$ *is said to be linearly independent if there is no* i*-flat (in* G^**) containing* S *with* $i < k$.

Prop. **1.4**: *If* f *and* f' *are* k*-flats, and* $f \subseteq f'$, *then* $f = f'$.

Proof: *If* $k = -1$, *then* $f = f' = \phi$. Assume **1.4** has been proved for $-1 \leq i \leq k-1$, and suppose $k \geq 0$. By **1.1**, both f and f' are non-empty. Let $S = \{x_0, \ldots, x_q\}$ be a maximal linearly independent subset of f, and let $\mathbf{f}_q(S)$ be the q-flat that S determines in accordance with **4**). Suppose $q < k$. Then by **6**), $f - \mathbf{f}_q(S) \neq \phi$, hence choose y in $f - \mathbf{f}_q(S)$. Let $\mathbf{f}(S \cup \{y\})$ be a flat of smallest dimension which contains $S \cup \{y\}$. **1.3** implies then that $\mathbf{f}_q(S) \cap \mathbf{f}(S \cup \{y\})$ is a flat of dimension less than or equal to q which contains S, hence it must be that $\dim(\mathbf{f}_q(S) \cap \mathbf{f}(S \cup \{y\})) = q$, or else S would not be linearly independent. Therefore, by the induction assumption, $\mathbf{f}_q(S) \cap \mathbf{f}(S \cup \{y\}) = \mathbf{f}_q(S)$. If $\dim \mathbf{f}(S \cup \{y\}) = q$, then $\mathbf{f}_q(S) = \mathbf{f}(S \cup \{y\})$, also by the induction assumption, contradicting the choice of y in $f - \mathbf{f}_q(S)$. If $\dim \mathbf{f}(S \cup \{y\}) < q$, then by **1.3**, $\dim(\mathbf{f}_q(S) \cap \mathbf{f}(S \cup \{y\})) < q$, contradicting $\dim(\mathbf{f}_q(S) \cap \mathbf{f}(S \cup \{y\})) = q$. If $\dim \mathbf{f}(S \cup \{y\}) > q$, then S is not a maximal linearly independent subset of f. Therefore it must be that $q = k$. But if $f \neq f'$, then, since $S \subseteq f \subseteq f'$, we have a contradiction of **4**).

As an immediate consequence of the proof of **1.4** we have

Cor. **1.4.1**: *If* S *is a maximal linearly independent subset of a* k*-flat, then* S *contains* $k + 1$ *points*.

Prop. **1.5**: *If* $i \leq k$, *and* f *is a* k*-flat, and* f' *is an* i*-flat, then* $f \cap f'$ *is an* i*-flat iff* $f' \subseteq f$.

Proof: Case 1: $i = k$. By **1.4**, $f = f \cap f' = f'$. Case 2: $i < k$. Clearly if $f' \subseteq f$, then $f \cap f' = f'$ is an i-flat. Suppose $f \cap f'$ is an i-flat. Then since $f \cap f' \subseteq f'$, we have by **1.4** that $f \cap f' = f'$, i.e. $f' \subseteq f$.

Definition **3**: *A set* $S = \{x_0, \ldots, x_m\}$ *is a basis for a* k*-flat* f *if* i) S *is linearly independent,* ii) $S \subseteq f$, *and* iii) S *is not contained in any flat of lower dimension than* k.

At times we shall employ the notation $\mathbf{f}_q(S)$ to indicate the q-flat determined by the set S, i.e. $\mathbf{f}_q(S)$ is the minimal flat which contains S. To show $\mathbf{f}_q(S)$ is unique we suppose that $\mathbf{f}_q(S)$ and $\mathbf{f}'_p(S)$ are minimal flats which contain S. Then $\mathbf{f}_q(S) \cap \mathbf{f}'_p(S)$ is a flat contained both in $\mathbf{f}_q(S)$ and in $\mathbf{f}'_p(S)$, and which contains S, hence $\mathbf{f}'_p(S) = \mathbf{f}_q(S) \cap \mathbf{f}'_p(S) = \mathbf{f}_q(S)$ by the minimality of $\mathbf{f}'_p(S)$ and $\mathbf{f}_q(S)$. The letters g and h may also be used to designate flats for convenience of notation.

Prop. **1.6**: *Any k-flat f has a basis consisting of $k+1$ points.*

Proof: Let S be a maximal linearly independent subset of f. Then by **1.4.1**, card$S = k+1$. Clearly S is a basis for f.

Prop. **1.7**: *If f is any k-flat, then every basis of f contains exactly $k+1$ points.*

Proof: Let $S = \{x_0, \ldots, x_m\}$ be a basis for f. Since S is linearly independent and $S \subseteq \mathbf{f}_m(S)$, $m < k$ would imply that S is not a basis for f. If $m > k$, then S is not linearly independent since $S \subseteq f$, therefore $m = k$.

Cor. **1.7.1**: *We may replace definition* **3** *by*
Definition **3'**: *A set $S = \{x_0, \ldots, x_k\}$ is a basis for a k-flat f iff* i)*S is linearly independent, and ii) $S \subseteq f$.*

Cor. **1.7.2**: *Every linearly independent set of $k+1$ points is the basis of a unique k-flat.*

Proof: **4**) and **1.7.1**.

Prop. **1.8**: *Let $S = \{x_0, \ldots, x_m\}$ be a linearly independent subset of a k-flat f. Then S can be extended to a basis of f.*

Proof: If $k = m$, then by **1.7.1**, we are finished. If $m > k$, then S is not linearly independent; therefore, suppose $m < k$. There is x_{m+1} in $f - \mathbf{f}_m(S)$ with $S_1 = \{x_{m+1}\} \cup S$ linearly independent, or else we would have $f \subseteq \mathbf{f}_m(S)$, a contradiction of **6**). If $m+1 = k$, we are finished; otherwise, we may continue in the obvious fashion until we arrive at $S_q = \{x_{m+q}\} \cup S_{q-1}$, a maximal linearly independent subset of f with $S \subseteq S_q$. S_q is clearly a basis for f, and $m+q = k$.

Prop. **1.9**: *If $T \subseteq f$, then $\mathbf{f}_q(T) \subseteq f$.*

Proof: $T \subseteq \mathbf{f}_q(T) \cap f \subseteq f$, but since $\mathbf{f}_q(T)$ is the minimal flat which contains T, and $\mathbf{f}_q(T) \cap f$ is a flat by 5), it must be that $\mathbf{f}_q(T) \cap f = \mathbf{f}_q(T)$, i.e. $\mathbf{f}_q(T) \subseteq f$.

Cor. **1.9.1**: *If $S = \{x_0, \ldots, x_m\}$ is a basis for f, and $S \subseteq f'$, then $f \subseteq f'$.*

Prop. **1.10**: *Every subset of a linearly independent set is linearly independent.*

Proof: Suppose $S = \{x_0, \ldots, x_k\}$ is a linearly independent set; let $S' = \{x_0, \ldots, x_q\} \subseteq S$. S can be assumed to be ordered so that $S'' = \{x_0, \ldots, x_m\}$, $m \leqslant q$, is a maximal linearly independent subset of S'; then $S' \subseteq \mathbf{f}_m(S'') \subseteq \mathbf{f}_k(S)$ by **1.9.1**. Let x_{q+j_1} be the first x_{q+j}, $1 \leqslant j \leqslant k-q$, such that x_{q+j_1} is not in $\mathbf{f}_m(S'')$. Then $S_1 = \{x_0, \ldots, x_m, x_{q+j_1}\}$ gives $\mathbf{f}_{m+1}(S_1)$ with $\mathbf{f}_m(S'') \subseteq \mathbf{f}_{m+1}(S_1) \subseteq \mathbf{f}_k(S)$. Proceeding similarly we eventually obtain

$S_r = \{x_0, \ldots, x_m, x_{q+j_1}, \ldots, x_{q+j_r}\}$, a linearly independent set with $S \subseteq \mathbf{f}_{m+r}(S_r)$ and $\mathbf{f}_k(S) \supseteq \mathbf{f}_{m+r}(S_r) \supseteq \ldots \supseteq \mathbf{f}_m(S'')$, but since S is a basis for $\mathbf{f}_k(S)$, by **1.7** it must be that $m + r = k$. Since $r \leqslant k - q$, it follows that $m = q$, and $r = k - q$.

Prop. **1.11**: *If $0 \leqslant i \leqslant l(G)$, and $x \varepsilon X$, then there is at least one i-flat containing x.*

Proof: $\{x\}$ is a basis for some 0-flat by **4**). Suppose $x \varepsilon f$, an i-1-flat. Choose y in $X - f$ ($\neq \phi$ by **3**)); then $x \varepsilon \mathbf{f}(\{y\} \cup f)$. If S is a basis for f, then $S \cup \{y\}$ is a basis for $\mathbf{f}(\{y\} \cup f)$, and card($S \cup \{y\}$) $= i + 1$, hence $\mathbf{f}(\{y\} \cup f)$ is an i-flat.

Given a geometry G on X, the flats of G^* can be partially ordered by inclusion to give a complete, finite dimensional lattice which we denote by $|G|$. We define $f \vee g$ to be $\mathbf{f}(f \cup g)$, i.e. l.u.b.(f, g) in $|G|$.

Prop. **1.12**: *$|G|$ is upper semi-modular.* (For the definition of upper semi-modular, see the appendix.)

Proof: Suppose f, a k-flat, and f', an i-flat, $i \leqslant k$, $f \neq f'$, cover $f \cap f'$, a q-flat. Since $q \leqslant i$, if $q < k - 1$, then f could not cover $f \cap f'$, therefore $q = k - 1$. By **6**), **1.4.1**, and **1.5**, we also have $i = k$. Let $\{x_0, \ldots, x_{k-1}\}$ be a basis for $f \cap f'$. By **1.8** we can find x and x' such that $\{x_0, \ldots, x_{k-1}, x\}$ and $\{x_0, \ldots, x_{k-1}, x'\}$ are bases for f and f' respectively. Then $\{x_0, \ldots, x_{k-1}, x, x'\}$ is a basis for $f \vee f'$, hence $\dim(f \vee f') = k + 1$, hence $f \vee f'$ covers both f and f'.

Cor. **1.12.1**: $\dim(f \vee g) + \dim(f \cap g) \leq \dim f + \dim g$.

Definition **4**: The following property **7**) will not be needed until chapter III, but its introduction comes naturally during a discussion of $|G|$:

7) *If* $\dim f = k$, *and* $\dim f' = i$, and $f \cap f' \neq \phi$, *then* $\dim(f \vee f') + \dim(f \cap f') = k + 1$.

Or, equivalently,

7') *If $f \neq \phi$, then the sublattice of $|G|$ consisting of the interval $[f, X]$ is modular.*

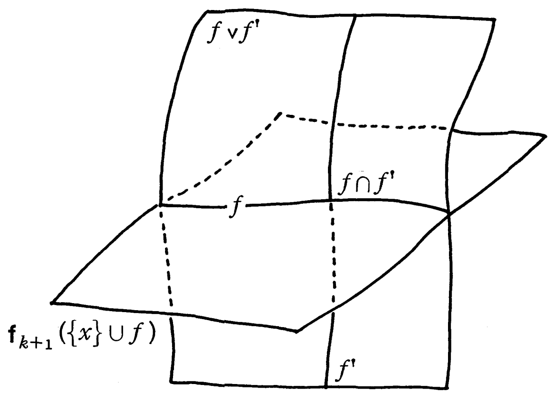
That **7**) is not an unnatural property is attested to by the fact that two of the three examples given in **1.0** satisfy **7**). Example i) satisfies **7**) because the subspaces of a vector space form a modular lattice; example ii) satisfies **7**) because it is semi-projective (cf. supra). In example iii) suppose that $n = 6$. Let S and S' be subsets of X each of which contain 6 points, and such that $S \cap S'$ contains a single point. Then $\dim(S \vee S') + \dim(S \cap S') = 7 + 0 \neq 6 + 6 = \dim S + \dim S'$. In general, example iii) does not satisfy **7**) if $n < \infty$, but if $n = \infty$, then **7**) is satisfied.

Definition **5**: *If f and f' are both k-flats, then f and f' are said to be skew of order s, written $f/_s f'$ if* i) $f \cap f' = \phi$, *and* ii) $f \vee f'$ *is a $k + s$-flat.*

If $f = f'$, or if $f/_1 f'$, then we say f is parallel to f', and write $f \| f'$.

Definition **6**: *A geometry G on X is called affine if given any k-flat f, $0 \leq k \leq l(G)$, and any x in X, there is one and only one k-flat which contains x and is parallel to f.*

Prop. **1.13**: *Let G be a geometry on X such that for some integer k, $1 \leq k \leq l(G)$, the intersection of any two distinct k-flats is always a k-1-flat. Then $l(G) = k$.*



Proof: Suppose $l(G) \geq k + 1$, and let f and f' be distinct k-flats. Since it is assumed that $\dim(f \cap f') = k - 1$, we have by **1.12** that $\dim(f \cap f') = k + 1$. Choose $x \notin f \vee f'$; then $(f \vee f') \cap \mathbf{f}_{k+1}(\{x\} \cup f) = f$. Suppose g is a k-flat in $\mathbf{f}_{k+1}(\{x\} \cup f)$. Then $g \cap f' \subseteq \mathbf{f}_{k+1}(\{x\} \cup f) \cap f'$. Since $f' \nsubseteq \mathbf{f}_{k+1}(\{x\} \cup f)$, **1.3** together with **1.5** implies that $\dim(\mathbf{f}_{k+1}(\{x\} \cup f) \cap f') \leq k - 1$, but since $f \cup f' \subseteq \mathbf{f}_{k+1}(\{x\} \cup f) \cap f'$, it must be that $\dim(\mathbf{f}_{k+1}(\{x\} \cup f) \cap f') = k - 1$. Therefore, using **1.4**, we have $g \cap f' = \mathbf{f}_{k+1}(\{x\} \cup f) \cap f' = f \cap f'$. Since g was an arbitrary k-flat contained in $\mathbf{f}_{k+1}(\{x\} \cup f)$, it must be that any two distinct k-flats in $\mathbf{f}_{k+1}(\{x\} \cup f)$ intersect each other in $f \cap f'$. Choose y in $\mathbf{f}_{k+1}(\{x\} \cup f) - (f' \cap f)$, and let $S = \{y, x_1, \ldots, x_{k+1}\}$ be a basis of $\mathbf{f}_{k+1}(\{x\} \cup f)$. Then the distinct k-flats contained in $\mathbf{f}_{k+1}(\{x\} \cup f)$ determined by $S-\{x_1\}$ and $S-\{x_2\}$ contain y in their intersection, hence $y \varepsilon f' \cap f$, contradicting our choice of y.

Definition **7**: *A geometry G on X is called semi-projective if, given any two distinct k-flats, f and f, which are contained in the same $k+1$-flat in (G^*), then $f \cap f'$ is a k-1-flat.*

Prop. **1.14**: *A geometry G on X is semi-projective iff $|G|$ is modular.*

Proof: Assume G is semi-projective. Suppose $f \vee f'$ covers f and f', $f \neq f'$, in $|G|$. An argument similar to that used in **1.12** shows that $\dim f =$

$\dim f' = k$, and $\dim(f \vee f') = k+1$, hence $\dim(f \cap f') = k-1$, thus $|G|$ is lower semi-modular. Since $|G|$ is upper semi-modular by **1.12**, $|G|$ is modular. Assume G is modular, and f and f' are distinct k-flats contained in the same $k+1$-flat. Then $\dim(f \vee f') + \dim(f \cap f') = \dim f + \dim f' = k+1+\dim(f \cap f') = 2k$, hence $\dim(f \cap f') = k-1$, and G is semi-projective.

Cor. **1.14.1**: *Any semi-projective geometry satisfies* **7**) (*definition* **4**).

Definition **8**: *Let $Y \subseteq X$, and G be a geometry on X. The geometry induced on Y by G is defined as follows: $F_Y^{-1} = \{\phi\}$; for $k \geq 0$, $F_Y^k = \{\mathbf{f}_k(S) \cap Y | S = y_0, \ldots, y_k\}$, $S \subseteq Y$, and S linearly independent in $X\}$. Set $G_Y = F_Y^{-1}, \ldots, F_Y^M\}$ where there is some $M+1$-flat which contains Y, but no flat of dimension less than $M+1$ which contains Y. M is said to be the dimension of Y with respect to G, written $\delta(Y)M$.*

Prop. **1.15**: *If $Y \neq \phi$, then G_Y is a geometry on Y.*

Proof: **1**)-**3**) are clearly satisfied. **4**) follows since every set of points linearly independent in Y is linearly independent in X. Since $(f \cap Y) \cap (f' \cap Y) = (f \cap f') \cap Y$ where f and f' are arbitrary flats of G, **5**) holds. By definition of G_Y, every k-flat of G_Y contains at least one set of $k+1$ points which is linearly independent in X, hence no k-flat could be contained in some flat of lower dimension in G_Y, thus **6**) holds.

Cor. **1.15.1**: *If f is a k-flat, $k \neq -1$, then G_f is a geometry on f of length $k-1$; moreover, by* **1.9.1**, *f' is a flat in G_f^* iff $f' \subseteq f$, and f' is a flat in G^*.*

Definition **9**: *A property *) is hereditary if, given any geometry G having *), then, if f is any flat of G, $f \neq \phi$, G_f also has *).*

Prop. **1.16**: *Property* **7**) *is hereditary.*

Proof: Assume G has **7'**), and f is a non-empty flat of G. Then $|G_f| = [\phi, f] \subseteq |G|$. If $f' \varepsilon |G_f|$, and $f' \neq \phi$, then $[f', f]$ is an interval of the modular lattice $[f', X]$, hence is modular.

Prop. **1.17**: *The property of being semi-projective is hereditary.*

Proof: If G is semi-projective, then by **1.14**, $|G|$ is modular, hence if $f \varepsilon |G|$ then $[\phi, f] = |G_f|$ is modular, hence by **1.14**, G_f is semi-modular.

Prop. **1.18**: *The property of being affine is hereditary.*

Proof: Let g be some k-flat, $k \neq -1$, of an affine geometry G. If $k = 0$, or 1, then G_g is trivially affine. Assume $k > 1$. If f is any k-flat of G, then f is a flat of G_g iff f is properly contained in g (**1.15.1**). Let f be an i-flat of G_g with $i \geq 1$, and let y be any point in $g - f$. There is a unique i-flat f' of G which contains y and is parallel to f. $f \vee f'$ is an $i+1$-flat of G^*, and, if S is a basis of f, then $S \cup \{y\} \subseteq g$ is a basis of $f \vee f'$, hence by **1.9.1**, $f \vee f' \subseteq g$. Since f' is the only i-flat of G containing y and parallel to f, it is the only i-flat of G_g^* containing y and parallel to f, hence G_g is affine.

CHAPTER II

BASIC PROPERTIES OF TOPOLOGICAL GEOMETRIES. A STUDY OF *1*-FLATS

For the remainder of this paper X is assumed to be a topological space.

Definition **1**: *A subset W of X is convex with respect to G, a geometry on X, if the intersection of W with any flat of G is connected. Where no confusion can result, convex will stand for convex with respect to the particular geometry (or any member of a class of geometries) under consideration.*

Prop. **2.1**: *A subset W of X is convex iff* i) *given any 0-flat g, $g \cap W$ is connected, and* ii) *given any 1-flat f, $f \cap W$ is connected.*

If W is convex, then the intersection of W with any flat is connected, hence i) and ii) hold. Assume i) and ii) hold. By assumption then the intersection of W with any flat of dimension less than or equal to 1 is connected. Suppose f' is a k-flat $k \geqslant 2$, such that $W \cap f' = A \cup B$, $A \cap B = \phi$, A and B non-empty and relatively open in $W \cap f'$. Choose $x \varepsilon A$, $y \varepsilon B$. Because of i), $\{x,y\}$ is linearly independent. Then $f_1(x,y) \cap W = [A \cap \mathbf{f}_1(x,y)] \cup [B \cap \mathbf{f}_1(x,y)]$, hence $\mathbf{f}_1(x,y) \cap W$ is not connected, contradicting ii). It must be then that $W \cap f'$ is connected, hence W is convex.

Cor. **2.1.1**: *If $F^0 = \{\{x\} | x \varepsilon X\}$, then a subset W of X is convex iff $f \cap W$ is connected, where f is any 1-flat.*

Cor. **2.1.2**: *If each 0-flat consists of a discrete set of points, then a subset W of X is convex iff $f \cap W$ is connected, where f is any 1-flat, and W contains no more than one point from any 0-flat.*

Prop. **2.2**: *A subset W of a 1-flat f is convex iff i) W is connected, and ii) if g is any 0-flat, then $W \cap g$ is connected.*

Proof: Suppose i) and ii) hold. Then by assumption, the intersection of W with any 0-flat is connected. Let f' be any 1-flat. If $f' = f$, then $f' \cap w = W$, which is connected. If $f' \neq f$, then $f' \cap f$ is either ϕ, or some 0-flat, and, in either case, $f' \cap W$ is connected, hence W is convex by **2.1**. If W is convex, then by **2.1**, i) and ii) hold.

Cor. **2.2.1**: *If $F^0 = \{\{x\} | x \varepsilon X\}$, then a subset of a 1-flat is convex iff it is connected.*

Cor. **2.2.2**: *A subset of a 1-flat (great circle) in $\mathbf{S}^2$ is convex iff it is connected and contains no two points which are antipodal (cf.* **1.0**, example ii).

Cor. **2.2.3**: *If each 0-flat consists of a discrete set of points, then a subset of a 1-flat is convex iff it is connected, and contains no more than one point from any 0-flat.*

Prop. **2.3**: *If W is a convex set in X, and if f is any 1-flat, then $f \cap W$ is convex.*

Proof: Let f' be any 1-flat, or 0-flat of X. Then $f' \cap (f \cap W) = (f' \cap f) \cap W$. If $f' = f$, then $f' \cap f = f$. If $f' \neq f$, then $f' \cap f$ is either ϕ, or some 0-flat. In any case, by **2.1**, $(f' \cap f) \cap W$ is connected, hence by **2.1**, $f \cap W$ is convex.

Prop. **2.4**: *If W is convex, then W is connected.*

Proof: Since X is a flat of G^*, $X \cap W = W$ is connected if W is convex.

Definition **2**: *A geometry G on a space X is said to be topological if* i) *each flat is a closed subset of X, and* ii) *if $\{W_\lambda\}_{\lambda \varepsilon \Lambda}$ is any family of convex subsets of X, then $\bigcap_\Lambda W_\lambda$ is convex.*

Prop. **2.5**: *If $\phi \neq Y \subseteq X$, then if G is a topological geometry on X, and Y is given the subspace topology, then G_Y is a topological geometry on Y.*

Proof: $W \subseteq Y$ is convex with respect to G_Y iff it is convex with respect to G, hence ii) of definition **2** is satisfied. Since every flat of G_Y is the intersection of Y with a closed subset of X, each flat of G_Y is closed in Y.

Cor. **2.5.1**: *The property of being topological is hereditary* (*cf.* chapter I, definition **9**).

Prop. **2.6**: *A geometry G on X is topological iff* i) *every flat is closed;* ii) *given any 0-flat g and any family $\{W_\lambda\}_{\lambda \varepsilon \Lambda}$ of convex subsets of X, $g \cap (\bigcap_\Lambda W_\lambda)$ is connected; and* iii) *given any 1-flat, f, G_f is a topological geometry on f (with the subspace topology).*

Proof: If G is topological, then i), ii), and iii) hold by definition **2** and **2.5**. Suppose i), ii), and iii) hold. Because of **2.1** it only remains to be shown that, given any 1-flat f and any family $\{W_\lambda\}_{\lambda \varepsilon \Lambda}$ of convex subsets of X, $f \cap (\bigcap_\Lambda W_\lambda)$ is connected. $f \cap (\bigcap_\Lambda W_\lambda) = \bigcap_\Lambda (f \cap W_\lambda)$. By **2.3**, each $f \cap W_\lambda$ is convex with respect to G, hence each $f \cap W_\lambda$ is convex with respect to G_f. By iii), therefore, $\bigcap_\Lambda (f \cap W_\lambda)$ is convex in f, hence is connected by **2.4**.

Cor. **2.6.1**: *If each 0-flat consists of a discrete set of points, then a geometry G on X is topological iff conditions* i) *and* iii) *of* **2.6** *apply.*

Proof: If each 0-flat consists of a discrete set of points, then no convex set can contain more than a single point of any 0-flat, hence in ii) of **2.6**, $g \cap (\bigcap_\Lambda W_\lambda)$ could consist of at most one point.

Cor. **2.6.2**: *If* $F^0 = \{\{x\} \mid x \in X\}$, *then G is topological iff* i) *every flat is closed, and* ii) *given any family* $\{W_\lambda\}_{\lambda \in \Lambda}$ *of connected subsets of any given 1-flat, then* $\bigcap_{\Lambda} W_\lambda$ *is connected.*

Proof: **2.6** and **2.2.1**.

Prop. **2.7**: *If G is a topological geometry on X such that each 0-flat consists of a discrete set of points, and if f is any 1-flat of* G^*, *and if W is any convex set which is contained in f, then any minimal disconnecting subset of W consists of exactly one point.*

Proof: Suppose M is a minimal disconnecting subset of W. Since W is connected, card $M \geq 1$. Suppose card$M \geq 2$; choose $x \in M$. Setting $T = M - \{x\}$, we have $W - T$ is a connected subset of f. Since W contains no more than one point from any 0-flat, $W - T$ also has this property, hence $W - T$ is convex by **2.2.3**. Similarly, $W - \{x\}$ is convex, but $(W-T) \cap (W-\{x\}) = W - M$ is not connected, hence is not convex, contradicting definition **2**, ii).

Cor. **2.7.1**: *The usual geometry on* **RP**(2), *the real projective plane, is not topological.*

Proof: Since $F^0 = \{\{x\} \mid x \in \mathbf{RP}(2)\}$, and each line is connected, by **2.2.1**, each line is convex, hence it would have to be that a minimal disconnecting subset of any given line consists of only one point, but this is clearly not true.

We remark, however, that the great circle geometry on $\mathbf{S}^2$ is topological. We thus see that even though a geometry G on X may be topological, the "identification" geometry on the identification space obtained by identifying all points in the same 0-flat need not be topological.

We assume for the remainder of this chapter:

A1: $F^0 = \{\{x\} \mid x \in X\}$.

Cf. **2.1.1**, **2.2.1**, and **2.6.2**.

A2: G(any geometry under consideration) *is topological.*

Definition **3**: *Let* $S \subseteq X$. *Set* $\Delta(S) = \{W \subseteq X$ i) *W is convex, and* ii) $W \supseteq S\}$. *The convex hull of S, written C(S), is defined by* $C(S) = \bigcap_{W \in \Delta(S)} W$. *If x and y are distinct points of X, then* $C(\{x,y\})$ *is denoted by* $\overline{xy}$, *and is called the segment joining x and y.*

Prop. **2.8**: *The following statements are equivalent*:

a) *every 1-flat in* G^* *is connected*;

b) *every flat of* G^* *is convex*;

c) $\Delta(S) \neq \phi$ *for any* $S \subseteq X$.

Proof: a) implies b): Let f be an arbitrary 1-flat of G^*, and suppose f' is any flat of G^*. Then $\dim(f \cap f') \leq 1$ by **1.3**, hence $f \cap f'$ is connected. By **2.1.1** we have, therefore, that f' is convex. b) implies c): Since X is a flat of G^*, X is convex, hence $X \in \Delta(S)$ for any $S \subseteq X$. c) implies a): Setting

$S = X$, since $\Delta(X) \neq \phi$, X is convex, hence every 1-flat must be connected by **2.1.1**.

To **A1** and **A2** we add

A3: *Every 1-flat in G^* is connected.*

If $X \neq \phi$ is T_1, then X admits at least one topological geometry with $F^0 = \{\{x\} | x \varepsilon X\}$, namely the trivial geometry where any k-flat is merely a subset of X which contains $k + 1$ distinct points. The only sets convex with respect to this trivial geometry are ϕ and the one point subsets of X. No general condition has yet been found for deciding whether an arbitrary (T_1 and connected) space admits a topological geometry which satisfies **A3** and **A1**. The following properties of $C(\ )$ are trivial and are presented without proof.

Prop. **2.9:** a) $S \subseteq C(S)$.
b) *If* $S \subseteq T$, *then* $C(S) \subseteq C(T)$.
c) $C(\bigcup_{i \varepsilon I} S_i) \supseteq \bigcup_{i \varepsilon I} C(S_i)$.
d) *If* S *is convex, then* $C(S) = S$.
e) $C(C(S)) = C(S)$.
f) *If* W *is convex, and* $S \subseteq W$, *then* $C(S) \subseteq W$.

Prop. **2.10:** *A subset W of X is convex iff $\{x,y\} \subseteq W$, $x \neq y$, implies $\overline{xy} \subseteq W$.*

Proof: If W is convex and $\{x,y\} \subseteq W$, then by **2.9f)**, $\overline{xy} \subseteq W$. Suppose $\{x,y\} \subseteq W$ implies $\overline{xy} \subseteq W$. Let f be any 1-flat. If card$(f \cap W) = 0$, or 1, then $f \cap W$ is connected. Suppose card$(f \cap W) \geqslant 2$. Choose any two distinct points x and y from $f \cap W$. Since f is convex, by **2.9f)** $\overline{xy} \subseteq f$. $\overline{xy} \subseteq W$ by assumption, hence $\overline{xy} \subseteq f \cap W$. We have then by **2.4** that x and y are both in the same component of $f \cap W$, hence, since x and y were arbitrary points of $f \cap W$, $f \cap W$ is connected. W is therefore convex by **2.1.1**.

The following corollaries are now clear:

Cor. **2.10.1:** *A subset W of* $\mathbf{R}^m$ *is convex with respect to the usual Euclidean geometry of* $\mathbf{R}^m$ *iff it is convex in the usual sense.*

Cor. **2.10.2:** *The usual Euclidean geometry on* $\mathbf{R}^m$ *is topological.*

Prop. **2.11:** *If f is a flat in G, then f is nowhere dense in X.*

Proof: Since f is closed we must show that f does not contain any non-empty open set. Suppose $U \subseteq f$, $U \neq \phi$, and U is open in X. Choose $x \varepsilon U$, and $y \varepsilon X - f$. $U \cap \mathbf{f}_1(x,y)$ is an open neighborhood of x in $\mathbf{f}_1(x,y)$. Since $\mathbf{f}_1(x,y)$ is connected, $\mathrm{Fr}(U \cap \mathbf{f}_1(x,y))$ in $\mathbf{f}_1(x,y)$ is non-empty. Since f is closed, $\mathrm{Cl}\, U \subseteq f$. If $z \varepsilon \mathrm{Fr}(U \cap \mathbf{f}_1(x,y))$ in $\mathbf{f}_1(x,y)$, then $z \varepsilon \mathbf{f}_1(x,y)$, and $z \varepsilon \mathrm{Cl}\, U \subseteq f$, but $\{x,z\}$ is a basis for $\mathbf{f}_1(x,y)$, hence by **1.9.1** $\mathbf{f}_1(x,y) \subseteq f$. Since y was an arbitrary point of X we have that every point of X must be in f, i.e. $f = X$, a contradiction of **3**), definition **1**, chapter I.

Cor. **2.11.1:** *If $S = \{x_0, \ldots, x_k\}$ is linearly independent, and $k \leqslant l(G)$, then $C(S)$ is nowhere dense in X.*

Proof: $C(S) \subseteq \mathbf{f}_k(S)$ by **2.9f)** and **2.8**. By **2.11**, $\mathbf{f}_k(S)$ is nowhere dense in X. Since $\mathbf{f}_k(S)$ is closed, $\mathrm{Cl}\,C(S) \subseteq \mathbf{f}_k(S)$, hence $C(S)$ is nowhere dense in X.

Cor. **2.11.2**: *If $i < k$, and f is an i-flat contained in a k-flat f', then f is nowhere dense in f'.*

Proof: **1.15.1**, and **2.5.1** and **2.8**, i.e. **A1-A3** are hereditary properties.

Prop. **2.12**: *Suppose $f \subseteq f'$ where f and f' are flats in G^*. If f disconnects f', then f is a closed, minimal disconnecting subset of f'.*

Proof: Set $T = (f' - f) \cup \{x\}$, where x is any point in f. Let $w, z \in T$, $w \neq z$. Then since any 1-flat not contained in f intersects f in at most one point, $\overline{wx} \cup \overline{xz}$ lies entirely in T and is connected. Since w and z were arbitrary points in T, T must be connected, which shows f is a minimal disconnecting subset of f'. f is also closed in f' since both f and f' are closed.

Since any subset of a 1-flat is convex iff it is connected (**2.2.1**), topologically a 1-flat is a connected T_1 space in which the intersection of any family of connected subsets is again connected. If x and y are distinct points of a 1-flat f, then xy is the intersection of all connected subsets of f which contain both x and y, thus we see

Prop. **2.13**: *$\overline{xy}$ is the unique subset of f which is irreducibly connected about x and y.*

We set $\mathrm{Int}\,\overline{xy} = \overline{xy} - \{x,y\}$.

Prop. **2.14**: *If $z \in \mathrm{Int}\,\overline{xy}$, then* a) $\overline{xy} = \overline{xz} \cup \overline{zy}$, *and* b) $\overline{xz} \cap \overline{zy} = \{z\}$.

Proof: a) By **2.9f)**, $\overline{xz} \cup \overline{zy} \subseteq \overline{xy}$. $\overline{xz} \cup \overline{zy}$ is a connected (hence convex) subset of $f_1(x,y)$ which contains both x and y, hence again by **2.9f)**, $\overline{xy} \subseteq \overline{xz} \cup \overline{zy}$, b) follows directly from Wilder [9], chapter 1, **10.15**.

Prop. **2.15**: *If C is any connected subset of $\overline{xy}$ which contains either x or y, then $\overline{xy} - C$ is connected.*

Proof: *Cf.* Wilder [9], chapter 1, **11.2**.

Cor. **2.15.1**: *x and y are the only non-cut points of $\overline{xy}$.*

Proof: Since $\overline{xy}$ is irreducibly connected about x and y, if $z \in \mathrm{Int}\,\overline{xy}$, then $\overline{xy} - \{z\}$ is disconnected, hence z is a cut point of $\overline{xy}$. Since $\{x\}$ and $\{y\}$ are connected subsets of $\overline{xy}$ which contain x and y respectively, $\overline{xy} - \{x\}$ and $\overline{xy} - \{y\}$ are connected by **2.15**, hence both x and y are non-cut points of $\overline{xy}$.

Prop. **2.16**: *z is a cut point of a 1-flat f iff $z \in \mathrm{Int}\,\overline{xy}$ for some pair of distinct points x and y of f.*

Proof: Assume z is a cut point of f. Then $f - \{z\} = A \cup B$, A, B non-empty, and relatively open in f with $A \cap B = \phi$. Choose $x \in A$, $y \in B$, If $z \notin \overline{xy}$, then $\overline{xy} = (\overline{xy} \cap A) \cup (\overline{xy} \cap B)$, hence $\overline{xy}$ would not be connected, a contradiction to **2.13**. Suppose z is not a cut point of f. Then $f - \{z\}$ is convex since it is connected, hence if x and y are distinct points of $f - \{z\}$, then $\overline{xy} \subseteq f - \{z\}$ by **2.9f)**.

Cor. **2.16.1**: *If* $z \varepsilon \mathrm{Int}\overline{xy}$, *then* z *is a cut point of both* $\overline{xy}$(**2.15.1**) *and* $f_1(x,y)$ (**2.16**).

Cor. **2.16.2**: *The set of cut points of any 1-flat is non-empty.*

Proof: Let f be a *1*-flat; then card$f \geqslant 2$. Choose $x,y \varepsilon f$, $x \neq y$. $\overline{xy}$ is connected and T_1, hence card(Intxy) $\geqslant \aleph_0$, but every point of Int$\overline{xy}$ is a cut point of f by **2.16**.

Definition **4**: *A space X with topological geometry G is said to be locally convex if every point of X has an open neighborhood basis consisting of sets convex with respect to G.*

2.2.1 shows

Prop. **2.17**: *A 1-flat is locally convex iff it is locally connected.*

Prop. **2.18**: *A locally convex space is locally connected.*

Proof: **2.4**.

Prop. **2.19**: *If Y is a convex subset of a locally convex space X, then Y with the subspace topology is locally convex with respect to* G_Y.

The proof is clear.

Definition **5**: *Let* $Y \subseteq X$, *a locally convex space. Then x and y are said to be polygonally connected in Y if there are points* $x_0 = x, x_1, \ldots, x_n = y$, *with* $n < \infty$, *such that* $\overline{x_0 x_1} \cup \overline{x_1 x_2} \cup \ldots \cup \overline{x_{n-1} x_n} \subseteq Y$. *Y is said to be polygonally connected if every two points of Y are polygonally connected in Y.*

Prop. **2.20**: *If Y is polygonally connected, then Y is connected.*

Proof: if $x,y \varepsilon Y$, since any set of the form $\overline{x_0 x_1} \cup \overline{x_1 x_2} \cup \ldots \cup \overline{x_{n-1} x_n}$ is connected, x and y belong to the same component of Y. Since x and y were arbitrary, Y is connected.

Perhaps the most important property of locally convex spaces is given by

Prop. **2.21**: *If U is an open, connected subset of X, a locally convex space, then U is polygonally connected.*

Proof: Let $x \varepsilon U$. Let $A = \{y \varepsilon U \mid x$ *is polygonally connected to* y *in* $U\}$, and $B = U - A$. A is open since if $y \varepsilon A$, then some convex neighborhood V of y is contained in U. If $w \varepsilon V$, then $\overline{wy} \subseteq V$ by **2.10**, therefore $w \varepsilon A$, hence $V \subseteq A$. By a similar argument B is open. Since $U = A \cup B$, and $A \cap B = \phi$, either A or B must be empty, but $x \varepsilon A$, thus $B = \phi$.

To **A1-A3** we add assumptions

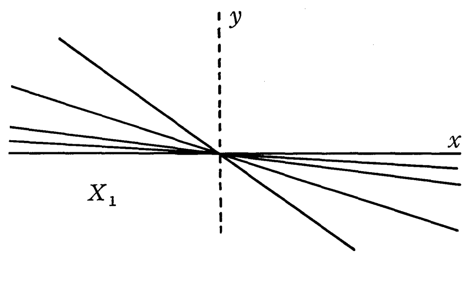
A4: *X is locally convex.*

A5: *If x,y, and z are distinct points of some 1-flat, then* $\overline{xy} \cup \overline{yz} = \overline{xy}$, $\overline{yz}$, *and/or* $\overline{xz}$.

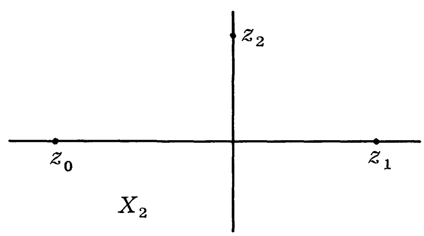
RP(*2*) is an example of a space with geometry which satisfies **A1**, **A3**, and **A4**, but not **A2** (**2.7.1**). The great circle geometry on $\mathbf{S}^2$ satisfies **A2**, A3, and A4, but neither A1, nor A5. If $\mathbf{R}^2$ with its usual geometry is given the discrete topology, we have an example of a space with geometry which satisfies A1, A2, and A4, but neither A3, nor A5; on the other hand, if $\mathbf{R}^2$ is given the indiscrete, or trivial topology, then we fall short of satisfying A1-A5 only in that each flat is not closed, and A5 does not hold. It should be noted that each of these examples also satisfy property 7) (chapter 1, definition 4). A1, A2, and A3 are necessary prerequisites for A5 for if any one of these is not satisfied, then the existence of $\overline{xy}$ for any given points x is uncertain.

The following examples, all subspaces of $\mathbf{R}^2$, with geometry $G = \{F^{-1}, F^0\}$, $F^0 = \{\{x\} | x \varepsilon X\}$, illustrate both the independence of A4 and A5, and what "pathological" condition each assumption is intended to eliminate. Each example satisfies property 7) and A1-A3. Example 1 satisfies neither A4, nor A5; example 2 satisfies A4, but not A5; and example 3 satisfies A5, but not A4.

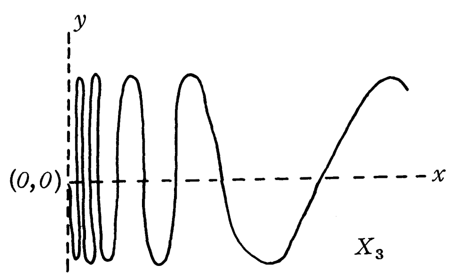
Ex. 1: $X_1 = \{(x,y) | y = 0, \text{ or } y = \frac{-1}{n} x, \text{ for } n \text{ a natural number}\}$.



Ex. 2: $X_2 = \{(x,y) | y = 0, \text{ or } x = 0\}$. Clearly $\overline{z_0 z_2} \cup \overline{z_2 z_1} \neq \overline{z_0 z_2}$, $\overline{z_2 z_1}$, or $\overline{z_0 z_1}$.



Ex. 3: $X_3 = \{(x,y) | (x,y) = (0,0), \text{ or } y = \sin\frac{1}{x}, x > 0\}$. X_3 can be totally ordered by $\prec$, where $(x_0, y_0) \prec (x_1, y_1)$ iff $x_0 < x_1$, but the order topology is not induced topology on X_3, for if it were, then the map $p: X_3 \to \{(x,y) | y = 0, x \geqslant 0\} = \mathbf{R}^+$ defined by $p((x,y)) = x$ would be 1-1, onto, and order-preserving between two spaces with the order topology, hence p would be a homeomorphism, but $\mathbf{R}^+$ is locally connected, and X_3 is not at $(0,0)$. Nevertheless, with this ordering, $z_0 \prec z_1$ implies $\overline{z_0 z_1} = \{z \varepsilon X_3 | z_0 \leqslant z \leqslant z_1\}$.



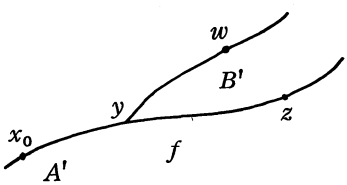
We now proceed to show that assuming A1-A5, we can order any 1-flat f in such a way that the order topology is the subspace topology of f.

Prop. 2.22: *If y is any cut point of a 1-flat f, then there are A and B, closed, connected subsets of f such that $A \cap B = \{y\}$, and $A \cup B = f$.*

Proof: Set $T = f - \{y\}$. Fix $x_0 \varepsilon T$. T is open in f. Set $A' = \{x \varepsilon T | x$ *is polygonally connected to* x_0 *in* $T\}$, and $B' = T - A'$. A' is connected by **2.20**. Suppose $w \varepsilon A'$, and U is an open, convex neighborhood of w in f which excludes y. Then if $z \varepsilon U$, $\overline{wz} \subseteq U$, hence $U \subseteq A'$, which proves that A' is open. Suppose $w \varepsilon B'$, and U is an open, convex neighborhood of w in f which excludes y. Suppose we can find $z \varepsilon U \cap A'$; then since $\overline{wz} \subseteq U$, $w \varepsilon A'$ also, contradicting $w \varepsilon B'$. Thus A' and B' are both open subsets of f, hence are both open subsets of T, hence are both open and closed in T. Both A' and B' are non-empty, or else $T = A'$, and T would be connected. Since A' is closed in T, $A' = T \cap F$, where F is some closed subset of f. If $y \notin F$, then $A' = F$, hence A' is open and closed in f and f could not be connected, hence $y \varepsilon F$, hence $A' \cup \{y\} = F = \text{Cl}A'$. Similarly $B' \cup \{y\} = \text{Cl}B'$. Set $A = A' \cup \{y\}$ and $B = B' \cup \{y\}$. Since A' is connected, $A = \text{Cl}A'$ is connected. The proof of **2.22** is completed by the following proposition.

Prop. **2.23**: *The set* B' *described in* **2.22** *is also connected.*

Proof: If B' is connected, then it is a convex, open subset of a a locally convex space (namely, of f), hence by **2.21**, B' is polygonally connected. Therefore if B' is not connected, we can choose $w, z \varepsilon B'$ such that $y \varepsilon \overline{wz} \cap \overline{x_0 w} \cap \overline{x_0 z}$. By **2.14** we have



$\overline{wz} = \overline{x_0 w} \cup \overline{x_0 z}$, $x \varepsilon w \cap \overline{x_0 z} = \{x_0\}$ *if* $x_0 \varepsilon \overline{wz}$;
$\overline{x_0 w} = \overline{wz} \cup \overline{x_0 z}$, $\overline{wz} \cap \overline{x_0 z} = \{z\}$ *if* $z \varepsilon \overline{x_0 w}$;
$\overline{x_0 z} = \overline{wz} \cup \overline{x_0 w}$, $\overline{wz} \cap \overline{x_0 w} = \{w\}$ *if* $w \varepsilon \overline{x_0 z}$.

It follows at once that $x_0 \notin wz$, $w \notin x_0 z$, and $z \notin \overline{x_0 w}$. Consider $\overline{x_0 w} \cup \overline{x_0 z}$. By **A5** $\overline{x_0 w} \cup \overline{x_0 z} = \overline{x_0 w}$, $\overline{x_0 z}$, or $\overline{wz}$. If $\overline{x_0 w}$, then $z \varepsilon \overline{x_0 w}$; if $\overline{x_0 z}$, then $w \varepsilon \overline{x_0 z}$; and if $\overline{wz}$, then $x_0 \varepsilon \overline{wz}$, a contradiction in any case, hence B' must be connected.

Prop. **2.24**: *Any 1-flat contains at most two non-cut points.*

Proof: Suppose $x_1, x_2, x_3 \varepsilon f$, a *1*-flat, are all distinct non-cut points of f. By **2.16** then $x_i \notin \text{Int}\overline{x_j x_k}$, $i, j, k = 1,2,3$, $i \neq j$, $j \neq k$, $i \neq k$. But $\overline{x_1 x_2} \cup \overline{x_2 x_3} = \overline{x_1 x_2}$, $\overline{x_2 x_3}$, or $\overline{x_1 x_3}$, and, in any case, we obtain contradiction.

Prop. **2.25**: *If* $x, y \varepsilon f$, *a 1-flat,* $x \neq y$, *then* $\text{Int}\overline{xy}$ *is open in* f.

Proof: Let $z \varepsilon \text{Int}\overline{xy}$. Then $z \varepsilon U$, a convex open subset of X which excludes x and y. Set $V = U \cap f$. Suppose there is $w \varepsilon V - \text{Int}\overline{xy}$. If $\overline{xw} \cup \overline{wy} = \overline{xy}$, then $w \varepsilon \overline{xy}$, contrary to assumption, hence by **A5**, either $\overline{xy} \subseteq \overline{wy}$, or $\overline{xy} \subseteq \overline{wx}$. In either case, we know that $\overline{zw} \cup \overline{wy} = \overline{zw}$, $\overline{wy}$, or $\overline{zy}$. If $\overline{zw}$, then $y \varepsilon V$, contrary to assumption. If $\overline{zy}$, then $w \varepsilon \overline{zy}$, and since $\overline{xy} = \overline{xz} \cup \overline{zy}$, we would have $w \varepsilon \overline{xy}$. It must be then that $\overline{zw} \cup \overline{wy} = \overline{wy}$. We also know that $\overline{xz} \cup \overline{zw} = \overline{zw}$, $\overline{xz}$, or $\overline{xw}$. If $\overline{zw}$, then $x \varepsilon V$; if $\overline{xz}$, then $w \varepsilon \overline{xy}$, thus it must be also that $\overline{xz} \cup \overline{zw} = \overline{xw}$. We have shown therefore that $z \varepsilon \overline{xw} \cap \overline{wy}$. Suppose that $\overline{xy} \subseteq \overline{wy}$. Then $x \varepsilon \overline{wy}$, and $\overline{wy} = \overline{wx} \cup \overline{xy}$ and $\overline{wx} \cap \overline{xy} = \{x\}$(**2.14**) but $z \varepsilon \overline{xy} \cap \overline{xw}$, hence $z = x$, a contradiction. If we assume instead that $\overline{xy} \subseteq \overline{wx}$, then $y \varepsilon \overline{wx}$, hence $\overline{xy} \cup \overline{yw} = \overline{wx}$, and $\overline{xy} \cap \overline{yw} = \{y\}$, but $z \varepsilon \overline{xy} \cap \overline{wy}$, hence we again have a contradiction. It must be therefore that $V \subseteq \text{Int}\overline{xy}$, hence $\text{Int}\overline{xy}$ is open in f.

Definition 6: We now define a total ordering on a *1*-flat f as follows: Let y_0 *be a cut point of* f. *We see from* 2.22 *and* 2.23 *that* $f - \{y_0\} = A' \cup B'$, $A' \cap B' = \phi$, *and* A' *and* B' *are both connected, hence convex by* 2.2.1. *We recall that* $A = A' \cup \{y_0\}$, *and* $B = B' \cup \{y_0\}$. *Define* $>$ *by*

1) $x \varepsilon A'$ *implies* $x > y$ *for any* $y \varepsilon B$.
2) $x, y \varepsilon A'$, $x \neq y$ *implies* $x > y$ *iff* $\overline{xy_0} \supseteq \overline{yy_0}$. *We note that* $\overline{xy_0} \cup \overline{yy_0} = \overline{xy}$ *cannot occur since* A' *is convex, giving* $\overline{xy} \subseteq A'$ (2.9f)).
3) $x \varepsilon B'$ *implies* $y_0 > x$.
4) $x, y \varepsilon B'$, $x \neq y$ *implies* $x > y$ *iff* $\overline{xy_0} \subseteq \overline{yy_0}$.

We note without proof (since these facts are never needed anywhere in this paper) that 1) the order given by $>$ is independent of the choice of the cut point y_0, and 2) if A' and B' were interchanged (i.e. if A' were called B', and B' were called A'), we would obtain an ordering $\dot{>}$ such that $x \dot{>} y$ *iff* $y > x$. Define $x < y$ if $y > x$, and $y \geqslant x$ if $y > x$, or $y = x$.

Prop. 2.26: *The ordering of* f *given by* $>$ *is a total ordering.*

Proof: i) $x \neq y$ implies $x > y$, or $y > x$. The only cases of any consequence occur when $\{x,y\} \subseteq A'$, or $\{x,y\} \subseteq B'$. In either case, $\overline{xy_0} \cup \overline{yy_0} = \overline{xy_0}$, or $\overline{yy_0}$. By 2.14 $\overline{xy_0} = \overline{yy_0}$ *iff* $x = y$. ii) $x > y$, $y > z$ implies $x > z$. If $x \varepsilon A$, $y, z \varepsilon B$, or $x, y \varepsilon A$, $z \varepsilon B$, the result is trivial. Otherwise the result follows from the transitivity of set inclusion.

Prop. 2.27: *If* $x < y$, *then* $\overline{xy} = \{z \mid x \leqslant z \leqslant y\}$.

Proof: a) $\overline{xy} \subseteq \{z \mid x \leqslant z \leqslant y\}$: Suppose $w \varepsilon \overline{xy}$.
Case 1: $\{x,y\} \subseteq A$. Then $\overline{xy_0} \subseteq \overline{yy_0}$, therefore $\overline{y_0x} \cup \overline{xy} = \overline{yy_0}$ and $\overline{y_0x} \cap \overline{xy} = \{x\}$. Then $\overline{xy_0} \cup \overline{xw} = \overline{y_0w}$, since $\overline{xy_0} \cup \overline{xw} = \overline{xw}$ would give $y_0 \varepsilon A'$, and $\overline{xy_0} \cup \overline{xw} = \overline{xy_0}$ would imply that $\overline{xy_0} \cap \overline{xy} \neq \{x\}$. Since this gives $\overline{xy_0} \subseteq \overline{y_0w}$, $x \leqslant w$; similarly $\overline{y_0w} \subseteq \overline{y_0y}$, hence $w \leqslant y$.
Case 2: $\{x,y\} \subseteq B$. Entirely similar to case 1.
Case 3: $x \varepsilon B$, $y \varepsilon A$. If $w \varepsilon B$, then $w \leqslant y$, but if $w < x$, then $\overline{xy_0}$ is properly contained in $\overline{wy_0}$, thus $w \notin \overline{xy_0}$, hence w could not be in $\overline{xy}$. If $w \varepsilon A$, then $w \geqslant x$, but $w > y$ would imply that $\overline{wy_0} \supseteq \overline{yy_0}$, whence $w \notin \overline{yy_0}$, and $w \notin \overline{xy_0}$. Therefore again we have $x \leqslant w \leqslant y$.

b) $\{z \mid x \leqslant z \leqslant y\} \subseteq \overline{xy}$: Suppose $x < w < y$. By A5 $\overline{xw} \cup \overline{wy} = \overline{xw}$, $\overline{wy}$, or $\overline{xy}$. If $\overline{xw}$, then $y \varepsilon \overline{xw}$, hence by a), $x < y < w$; if $\overline{wy}$, then $x \varepsilon \overline{wy}$, hence also by a), $w < x < y$; in either case we have a contradiction of 2.26. It follows then that $\overline{xw} \cup \overline{wy} = \overline{xy}$, hence $w \varepsilon \overline{xy}$.

a) and b) together imply 2.27.

Cor 2.27.1: *If* $x < y$, *then* $\{z \mid x < z < y\}$ *is open in* f.

Proof: By 2.25.

Cor. 2.27.2: *The sets* $\{z \mid x < z\}$ *and* $\{z \mid x > z\}$ *are open in* f.

Proof: We prove $\{z \mid x < z\}$ is open in f; the proof for $\{z \mid x > z\}$ is analogous. Let $z \varepsilon \{z \mid x < z\}$. If there is $w \varepsilon f$, $w > z$, then $z \varepsilon \operatorname{Int}\overline{xw} \subseteq \{z \mid x < z\}$ by 2.27. If there is no $w > z$, let U be any convex, open neighborhood of z which excludes x. Then $U \cap \{y \mid y \leqslant x\} = \phi$, for if $y \leqslant x$, and $y \varepsilon U \cap f$, then $\overline{yz} \subseteq U \cap f$ by 2.10, which would imply that $x \varepsilon U \cap f$.

Cor. **2.27.3**: *The order topology of f is equivalent to the subspace topology and is T_2.*

Let U be an open, convex neighborhood of x in f. If there are $u,v \varepsilon U$ such that $u < x < v$, then $x \varepsilon \operatorname{Int}\overline{uv} \subseteq U$. If $x \leq w$, or $x \geq w$, for all $w \varepsilon U$, then x is contained in an open set which is contained in U of the form described in **2.27.2**. Since $\operatorname{Int}\overline{uv} = \{w | u < w < v\}$ (**2.27**), we have shown that every open, convex neighborhood of x in the subspace topology contains an open neighborhood of x in the order topology. On the other hand, sets of the form given in **2.27.1** and **2.27.2** form a basis for the order topology, and these are open in the subspace topology of f, hence the order and subspace topologies of f are equivalent. The order topology is always T_2.

Prop. **2.28**: *A 1-flat ordered as described in definition 6 satisfies the Dedekind cut axiom.*

Proof: Suppose C and D are two non-empty subsets of f such that i) $C \cup D = f$, and ii) $x \varepsilon C$, $y \varepsilon D$ implies $y < x$. Suppose neither C has a smallest element, nor D a largest. Then C is the union of open sets of the form $\{z | z > x\}$, and D is the union of open sets of the form $\{z | z < x\}$, hence C and D are non-empty, disjoint, open subsets of f whose union is f, thus f is not connected, a contradiction of **A3**.

Prop. **2.29**: *$\overline{xy}$ is compact.*

Proof: Let $\{U_\alpha\}$ be a collection of open sets (of X) covering $\overline{xy}$. We construct a Dedekind cut of $f_1(x,y)$ (ordered by $>$) as follows: $z \varepsilon C$ if 1) $z < x$ (we assume $x < y$), or if 2) $x \leq z \leq y$, and a finite number of the U_α cover $\overline{xz}$. Set $D = f_1(x,y) - C$. If $\overline{xy} \subseteq C$, we are done; otherwise C and D give a cut. Therefore by **2.28** there is w_0 largest in C, or smallest in D. Since $w_0 \varepsilon \overline{xy} - \{y\}$, $w_0 \varepsilon U_{\alpha_0}$, give one of the U_α. By **2.27.3** and the local convexity of $f_1(x,y)$ (**2.19**), there is $uv \subseteq U_{\alpha_0}$ with $x < u < w_0 < v$. Regardless of whether w_0 is in C or in D, $u \varepsilon C$, therefore a finite number of the U_α cover $\overline{xu}$, thus $v \varepsilon C$, but $v > w_0$, a contradiction.

Prop. **2.30**: *If X is metrizable, then $\overline{xy}$ is homeomorphic to $[0,1]$.*

Proof: *Cf.* Hocking and Young [5], theorem 2-27.

Cor. **2.30.1**: *If X is metrizable, then X is arc-connected.*

Prop. **2.31**: *Any segment $\overline{xy}$ is locally periphically countably compact.*

Proof: A basis for the open sets of $\overline{xy}$ consists of sets of the form $\{w | w > z\}$, $\{w | w < z\}$, or $\{w | u < w < v\}$ (**2.27.3**). In any case, the frontiers in $\overline{xy}$ of these sets consist of at most two points.

Cor. **2.31.1**: *If $\overline{xy}$ is separable, then $\overline{xy}$ is homeomorphic to $[0,1]$.*

Proof: *Cf.* Wilder [9], **11.14**, chapter I.

Cor. **2.31.2**: *If X is second countable, then X is arc-connected, and every segment in X is homeomorphic to $[0,1]$.*

Proof: If $x,y\varepsilon X$, $x \neq y$, then $\overline{xy}$ is second countable, hence separable. $\overline{xy}$ is therefore homeomorphic to $[0,1]$ by **2.31.1**, hence $\overline{xy}$ is an arc connecting x and y.

Definition **7**: *Let* $S = \{x\ \ ,\ldots,x_k\}$. *Then* $C(S)$ *is called a k-simplex iff S is linearly independent. Suppose* $C(S)$ *is a k-simplex.* $C(\{x_0,\ldots,\hat{x}_i,\ldots,x_k\})$, *denoted also by* $F^iC(S)$, *is called the* i^{th} *face of* $C(S)$; $\hat{x}_i$ *indicates* x_i *has been deleted.* $F^iC(S)$ *is a k-1-simplex.*

We define $dC(S) = \bigcup_{j=0}^{k} F^iC(S)$, and $\text{Int}C(S) = C(S) - dC(S)$, called the interior of $C(S)$. Set $\text{Ext}C(S) = [X - C(S)] \cap \mathfrak{f}_k(S)$; $\text{Ext}C(S)$ is called the exterior of $C(S)$. This chapter was primarily concerned with the properties of 1-flats. We now begin a discussion of the structure of X as a whole.

CHAPTER III

m-ARRANGEMENTS

Definition **1**: *A topological space with geometry G of length* $m-1 \geq 0$ *is called an m-arrangement if the following conditions are satisfied*:

3.1: $F\ \ = \{\{x\} \mid x\varepsilon X\}$.

3.2: *G is topological.*

3.3: *Any 1-flat of* G^* *is connected.*

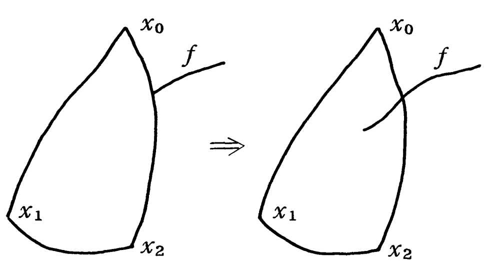
3.4: *X is locally convex (with respect to G).*

3.5: *If x,y, and z are distinct points of a 1-flat, then* $\overline{xy} \cup \overline{xy} = \overline{xy}, \overline{yz}$, *or* $\overline{xz}$.

Since **3.1** - **3.5** are nothing but **A1-A5** of chapter II, the results and definitions there are applicable in this chapter.

3.6: *If* $S = \{x_0,\ldots,x_k\}$ *is linearly independent, and* $k \geqslant 1$, *then* $C(S) = \bigcup_{x\varepsilon F^0C(S)} \overline{x_0 x}$.

3.7: *If* $C(S)$ *is a k-simplex and f is any 1-flat in* $\mathfrak{f}_k(S)$ *such that* $f \cap \text{Int}F^iC(S)$ *consists of a single point for some i, then* $f \cap \text{Int}C(S) = \phi$.



3.8: *If $C(S)$ is a k-simplex, then* $dC(S) \supseteq \mathrm{Fr}C(S)$ *in* $\mathbf{f}_k(S)$ (*in* G^*), *i.e. the topological boundary in a flat f of G^* of $C(S)$, where S is a basis of f, is contained in the "geometric boundary" of $C(S)$.*

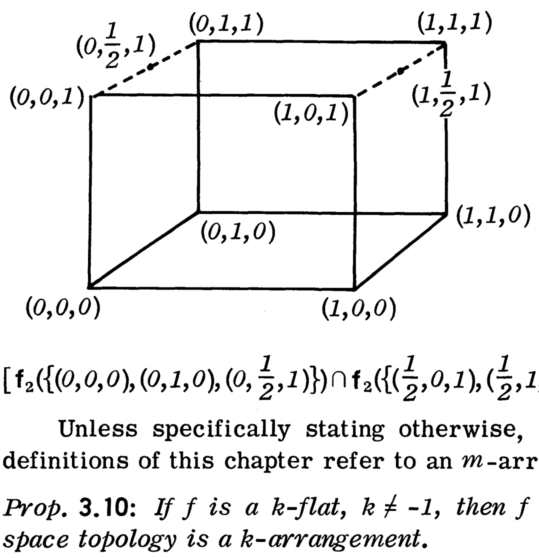
3.9: *If f and f' are flats of G^* such that $f \cap f' \neq \phi$, then* $\dim f + \dim f' = \dim(f \vee f') + (f \cap f')$. This is property **7)** (definition **4**, chapter I).

In general the independence of these postulates has not been established; however, we may make the following observations:

(i) Any topological space with geometry of length 0 which satisfies **3.1-3.4**, also satisfies **3.6-3.9.** The examples cited in chapter II to illustrate the independence of **3.4(A4)** and **3.5(A5)** also satisfy **3.6-3.9**, hence **3.4** and **3.5** are definitely independent.

(ii) Let $X = \mathbf{R}^3$ with the usual Euclidean geometry $G = \{F^{-1}, \ldots, F^2\}$. Set $G' = \{F^{-1}, F^0, F^1\}$. Then X with geometry G' satisfies all the axioms for a 2-arrangement except **3.8** and **3.7.** This example offers a clue to the role played by **3.8** and **3.7** in the sequel, namely of links between the length of G and the "dimension" of X.

(iii) Let Y be the cube in $\mathbf{R}^3$ with vertices (i,j,k), $i,j,k = 0$, *or* 1. Let $X = Y - [\overline{(0,0,1)(0,1,1)} \cup \overline{(1,0,1)(1,1,1)}] \cup \{(0,\frac{1}{2},1), (1,\frac{1}{2},1)\}$. Then X with the



subspace topology and geometry induced by the usual Euclidean geometry on $\mathbf{R}^3$ satisfies **3.1-3.8**, but not **3.9**; in particular, $\dim \mathbf{f}_2(\{(0,0,0), (0,1,0), (0,\frac{1}{2},1)\}) + \dim \mathbf{f}_2(\{(\frac{1}{2},0,1), (\frac{1}{2},1,1), (0,\frac{1}{2},1)\}) = 4 = 3 + 0 = \dim X + \dim [\mathbf{f}_2(\{(0,0,0),(0,1,0),(0,\frac{1}{2},1)\}) \cap \mathbf{f}_2(\{(\frac{1}{2},0,1),(\frac{1}{2},1,1),(0,\frac{1}{2},1)\})]$.

Unless specifically stating otherwise, all remarks, propositions, and definitions of this chapter refer to an m-arrangement, $m \geq 1$.

Prop. **3.10:** *If f is a k-flat, $k \neq -1$, then f with geometry G_f and the subspace topology is a k-arrangement.*

Proof: **3.1, 3.6, 3.7,** and **3.8** are trivially verified. **3.2** follows from **2.5**; **3.3** from **2.8** and **2.11**; **3.4** from **2.19**; **3.5** from **2.10**; and **3.9** from **1.16.**

Prop. **3.11:** *If U is a non-empty, open, convex subset of X, then U with geometry G_U and the subspace topology is an m-arrangement.*

Proof: Suppose f is a k-flat for which $f \cap U \neq \phi$. By **1.9** $\mathbf{f}_i(f \cap U) \subseteq f$, hence by **6)**, definition **1**, chapter I, $i \leq k$. But $f \cap U \subseteq \mathbf{f}_i(f \cap U)$, and since $f \cap U$ is somewhere dense in f, $\mathbf{f}_i(f \cap U)$ is somewhere dense in f. By **2.11.2** this is impossible unless $i \geq k$, hence $i = k$. Therefore by **1.4**, $\mathbf{f}_i(f \cap U) = f$. It follows at once that $f \cap U$ contains a basis for f. We have therefore shown that $\delta(U) = m - 1$ (definition **8**, chapter I), and that **3.9** holds since $\dim(f \cap U)$

that $\delta(U)=m-1$ (definition **8**, chapter I), and that **3.9** holds since $\dim(f \cap U)$ in G_U = $\dim f$ in G if $f \cap U \neq \phi$. **3.2** holds by **2.5**; **3.3** by **2.3** and **2.4**; **3.4** by **2.19**; and **3.5**, **3.6**, and **3.7** by the convexity of U. If $A \subseteq U$, then $\mathrm{Fr}A$ in $U = A-(A^0$ in $U)$, but since U is open, A^0 in $U = A^0$, thus $\mathrm{Fr}A$ in $U = \mathrm{Fr}A$, hence **3.8** also holds.

Prop. **3.12**: *If $C(S)$ is a k-simplex, $k \geqslant 1$, then* $\mathrm{Int}C(S) \neq \phi$.

Proof: **3.12** is clearly true if $k = 1$; assume **3.12** holds for $k - 1 \geqslant 1$. Suppose $S = \{x_0, \ldots, x_k\}$ is linearly independent. By the induction assumption we can select $w \,\varepsilon\, \mathrm{Int}F^0C(S)$. If $\mathbf{f}_1(x_0,w) \cap F^0C(S) \neq \{w\}$, then $F^0C(S)$ would contain a basis for $\mathbf{f}_1(x_0,w)$, hence by **1.9.1** we would have $\mathbf{f}_1(x_0,w) \subseteq \mathbf{f}_{k-1}(S-\{w\})$, and therefore $S \subseteq \mathbf{f}_{k-1}(S-\{w\})$, contradicting the linear independence of S. Therefore $\mathbf{f}_1(x_0,w) \cap \mathrm{Int}F^0C(S) = \{w\}$, and by **3.7** $\mathbf{f}_1(x_0,w) \cap \mathrm{Int}C(S) \neq \phi$, hence $\mathrm{Int}C(S) \neq \phi$.

Prop. **3.13**: *Any k-simplex $C(S)$ is closed.*

Proof: $\mathrm{Fr}\,C(S)$ in $\mathbf{f}_k(S) \subseteq dC(S) \subseteq C(S)$ by **3.8**, hence $C(S)$ is closed in $\mathbf{f}_k(S)$, a closed set, therefore $C(S)$ is closed in X.

Prop. **3.14**: *If $C(S)$ is any k-simplex, then* $\mathrm{d}C(S)$ *is closed.*

Proof: $\mathrm{Fr}C(S)$ in $\mathbf{f}_k(S) \subseteq \mathrm{d}C(S)$, therefore $\mathrm{Fr}(\mathrm{d}C(S))$ in $\mathbf{f}_k(S) \subseteq \mathrm{d}C(S)$, hence $\mathrm{d}C(S)$ is closed in $\mathbf{f}_k(S)$, hence in X.

Prop. **3.15**: *If $C(S)$ is an m-simplex, i.e. S is a maximal linearly independent subset of X, and if $\mathrm{Fr}C(S) \neq \phi$, then $\mathrm{Fr}C(S)$ disconnects X.*

Proof: From $\mathrm{Fr}(X-C(S)) = \mathrm{Fr}\,C(S) \neq \phi$, we have at once that $A = (X-C(S)) \cup \mathrm{Fr}C(S)$ is closed and $X-C(S) \neq \phi$. $A \cup C(S) = X$ and $A \cap C(S) = \mathrm{Fr}C(S)$. Since $A \nsubseteq C(S)$, we have by lemma 1 of the appendix that $\mathrm{Fr}C(S)$ disconnects X.

Cor. **3.15.1**: *If $C(S)$ is a k-simplex contained in a k-flat f, $k \geqslant 1$, and* $\mathrm{Fr}\,C(S)$ *in $f \neq \phi$, then* $\mathrm{Fr}C(S)$ *disconnects f.*

Proof: **3.10**.

Prop. **3.16**: *If $C(S)$ is an m-simplex, then* $\mathrm{Int}\,C(S)$ *is open in X.*

Proof: $\mathrm{Int}C(S) = C(S) - \mathrm{d}C(S) = (C(S)^0 \cup \mathrm{Fr}\,C(S)) - \mathrm{d}C(S) = C(S)^0 - dC(S)$. Since $\mathrm{d}C(S)$ is closed by **3.14**, $C(S)^0 - \mathrm{d}C(S)$ is open.

Cor. **3.16.1**: *If $C(S)$ is a k-simplex, then* $\mathrm{Int}C(S)$ *is open in* $\mathbf{f}_k(S)$.

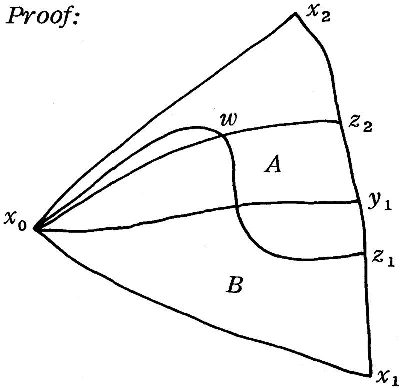
Proof: **3.10**.

Prop. **3.17**: *If $C(S)$ is an m-simplex and* $\mathrm{Ext}C(S) \neq \phi$, *then* $\mathrm{d}C(S)$ *disconnects X.*

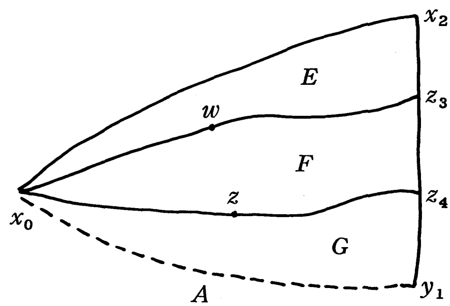
Proof: $\mathrm{Ext}C(S) = X - C(S)$ is open, hence $X - \mathrm{d}C(S) = \mathrm{Ext}\,C(S) \cup \mathrm{Int}C(S)$, the union of two disjoint, non-empty, open subsets of X.

Prop. **3.18**: *Suppose $S = \{x_0,x_1,x_2\}$ and $C(S)$ is a 2-simplex. Suppose $y_1 \varepsilon \mathrm{Int}\overline{x_1x_2}$. Then $\overline{x_0\;y_1}$ disconnects $C(S)$ into two convex components, one containing x_1, and the other containing x_2.*

Proof:



Since $C(S) \subseteq \mathbf{f}_2(S)$ by **3.10** we may regard $X = \mathbf{f}_2(S)$. Set $A = C(\{x_0, y_1, x_2\})$ and $B = C(\{x_0, y_1, x_1\})$; then $\overline{x_0 y_1} \subseteq C(S)$, $\overline{x_0 y_1} \subseteq A \cap B$. Suppose there is $w \varepsilon (A \cap B) - \overline{x_0 y_1}$. Then there must be $z_1 \varepsilon \overline{y_1 x_1}$, and $z_2 \varepsilon \overline{x_2 y_1}$ such that $w \varepsilon \overline{x_0 z_1} \cap \overline{x_0 z_2}$ (**3.6**), thus $\mathbf{f}_1(x_0, w) = \mathbf{f}_1(x_0, z_1) = \mathbf{f}_1(x_0, z_2) = \mathbf{f}_1(z_1, z_2) = \mathbf{f}_1(x_1, x_2)$ which implies that S is not linearly independent. We have therefore that $A \cap B = \overline{x_0 y_1}$, and since A and B are closed, lemma **1** of the appendix implies that $\overline{x_0 y_1}$ disconnects $A \cup B$. It is an immediate consequence of **2.14** and **3.6**, however, that $A \cup B = C(S)$, hence $\overline{x_0 y_1}$ disconnects $C(S)$.



Let w and z be arbitrary, but distinct points of $A - \overline{x_0 y_1}$. By **3.6** there are $z_3, z_4 \varepsilon \overline{x_2 y_1} - \{y_1\}$, with $w \varepsilon \overline{x_0 z_3}$ and $z \varepsilon \overline{x_0 z_4}$.
Case 1: $z_3 = z_4$. By **2.15** $\overline{x_0 z_3} - \{x_0\}$ is connected, hence it is convex by **2.2.1**. Since $\{w, z\} \subseteq \overline{x_0 z_3} - \{x_0\}$, by **2.9f)** $\overline{wz} \subset \overline{x_0 z_3} - \{x_0\}$. *But* $\overline{x_0 z_3} \cap \overline{x_0 y_1} = \{x_0\}$, hence $\overline{wz} \subseteq A - \overline{x_0 y_1}$.
Case 2: $z_3 \neq z_4$, i.e. $\{x_0, z_3, z_4\}$ is linearly independent. Set $E = C(\{x_0, z_3, x_2\})$, $F = C(\{x_0, z_3, z_4\})$, and $G = C(\{x_0, z_4, y_1\})$. Then by an argument similar to that used earlier in this proof, $A = E \cup F \cup G$, and $F \cap G = \overline{x_0 z_4}$. By **2.9f)** $\overline{wz} \subseteq F$. Since $z_3 \neq z_4$, $\overline{wz} \cap \overline{x_0 z_4} = \{z\}$, hence $\overline{wz} \cap G = \{z\}$, but $z \notin \overline{x_0 y_1}$, hence $\overline{wz} \subseteq A - \overline{x_0 y_1}$. We have therefore shown that $A - \overline{x_0 y_1}$ is convex (**2.10**), hence connected (**2.4**); similarly $B - \overline{x_0 y_1}$ is convex and connected, and **3.18** follows at once.

Prop. **3.19**: *Let $C(S)$ be a k-simplex with $S = \{x_0, \ldots, x_k\}$, and let $F^i C(S)$ and $F^j C(S)$ be distinct faces of $C(S)$. Set $S_{i,j} = S - \{x_i, x_j\}$. Then $F^i C(S) \cap F^j C(S) = C(S_{i,j})$.*

Proof: **3.19** is true for $k = 1$. Suppose it has been proved for $k - 1 \geq 1$. Set $A = F^i C(S) \cap F^j C(S)$, where $C(S)$ is a k-simplex. Certainly $C(S_{i,j}) \subseteq A$. Suppose $w \varepsilon A - C(S_{i,j})$. $C(S_{i,j}) = F^i(F^j C(S))$, and applying **3.6** to $F^j C(S)$ we can find $z \varepsilon C(S_{i,j})$ such that $w \varepsilon \overline{x_i z}$. It follows therefore that $\mathbf{f}_1(w, z) = \mathbf{f}_1(x_i, z)$. Since $\{w, z\} \subseteq F^i C(S)$, $x_i \varepsilon \mathbf{f}_{k-1}(F^i C(S))$, contradicting the linear independence of S.

We note in passing that if $S = \{x_0, \ldots, x_k\}$ is linearly independent and we define $\partial C(S) = \sum_{i=0}^{k} (-1)^i F^i C(S)$ as in simplicial homology theory, then **3.19** enables us to say that $\partial\partial = 0$. This leads to the conjecture that a meaningful homology can be defined on X using convex hulls, a subject still to be investigated.

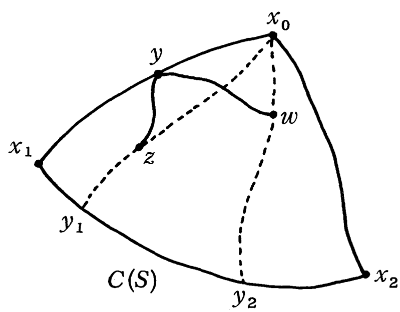
Cor. **3.19.1**: *No point of a k-simplex $C(S)$ is contained in more than k faces of $C(S)$.*

Proof: **3.19.1** is true for $k = 1$; assume it has been proved for $k - 1 \geq 1$. Suppose $w \varepsilon \bigcap_{i=0}^{k} F^i C(S)$. $dF^i C(S) = \bigcup_{\substack{j=0 \\ i \neq j}}^{k} C(S_{i,j})$, *but by* **3.19** $C(S_{i,j}) = F^i C(S) \cap F^j C(S)$, hence $w \varepsilon C(S_{i,j})$ for all $j \neq i$, hence w is in all faces of $F^i C(S)$ contradicting the induction assumption.

Prop. **3.20**: *If* $S = \{x_0, \ldots, x_k\}$ *is linearly independent, then* $\mathrm{Int}C(S)$ *is convex.*

Proof: We first prove **3.20** for $k = 1$. If $w,z \varepsilon \mathrm{Int}\overline{x_0 x_1}$, we can assume $w \leq z$, and $x_0 < x_1$ (cf. definition **6**, chapter II). Then $x_0 < w \leq z < x_1$, but $\overline{wz} = \{y | w \leq y \leq z\}$ by **2.27**, hence $wz \subseteq \mathrm{Int}\overline{x_0 x_1}$, therefore $\mathrm{Int}\overline{x_0 x_1}$ is convex by **2.10**.

Assume **3.20** is true for $k - 1 \geq 1$, but is not true for k. Then there are $z,w \varepsilon \mathrm{Int}C(S)$ such that $\overline{zw} \cap dC(S) \neq \phi$, hence we can choose a point y in this intersection. By **3.19.1** we may assume $y \notin F^0 C(S)$ (by renumbering the x_i's, if necessary), hence by **3.6** we have $y_1, y_2 \varepsilon F^0 C(S)$ with $z \varepsilon \overline{x_0 y_1}$, and $w \varepsilon \overline{x_0 y_2}$. Since $z,w \varepsilon \mathrm{Int}C(S)$, $y_1, y_2 \varepsilon \mathrm{Int}F^0 C(S)$ by a simple argument using **1.9.1**, hence by the induction assumption $\overline{y_1 y_2}$ (or just $\{y_1\}$ if $y_1 = y_2$) $\subseteq \mathrm{Int}F^0 C(S)$. Since $z,w \varepsilon C(\{x_0, y_1, y_2\})$, y is also in $C(\{x_0, y_1, y_2\})$, therefore we can find $y_3 \varepsilon \overline{y_1 y_2}$ such that $y \varepsilon \overline{x_0 y_3}$, but since $y \varepsilon dC(S) - F^0 C(S)$, y_3 must be in $dF^0 C(S)$ contradicting the induction assumption.



Cor. **3.20.1**: *A segment $\overline{wz} \subseteq C(S)$, a k-simplex, but which is not contained wholly in $dC(S)$, can touch a face of $C(S)$ only at w or at z.*

Proof: Suppose $y \varepsilon \mathrm{Int}\overline{wz} \cap dC(S)$, but $\overline{wz} \not\subseteq dC(S)$. If both w and z are in $\mathrm{Int}C(S)$, then **3.20** is contradicted; therefore suppose $w \varepsilon dC(S)$. By **2.16.2** $\mathrm{card}\overline{wy} \geq \aleph_0$, but the number of distinct faces of $C(S)$ is only $k + 1$, hence if $\overline{wy} \subseteq dC(S)$, then at least two distinct points of $\overline{wy}$ would have to be in the same face of $C(S)$, hence by **1.9.1** and **2.9f)** we would have that $\overline{wz} \subseteq dC(S)$, a contradiction. Therefore we can find $w_1 \varepsilon \mathrm{Int}\overline{wy} - dC(S)$, and, similarly, $w_2 \varepsilon \mathrm{Int}\overline{yz} - dC(S)$. Then $y \varepsilon \mathrm{Int}\overline{w_1 w_2} \subseteq \mathrm{Int}C(S)$, contradicting $y \varepsilon dC(S)$.

Cor. **3.20.2**: *If $C(S)$ is a k-simplex, $k \geq 2$, then no 1-flat can intersect three distinct faces of $C(S)$ in an interior point, i.e. in* $\mathrm{Int}F^iC(S)$.

Proof: If **3.20.2** is false, let x,y,z be distinct points of the intersection of f with the interiors of three distinct faces of $C(S)$. Either $y\,\varepsilon\,\overline{xz}$, or $z\,\varepsilon\,\overline{xy}$, or $x\,\varepsilon\,\overline{yz}$; in any case **3.20.1** is contradicted.

Cor. **3.20.3**: *If $C(S)$ is a 2-simplex, and f is a k-flat, and $C(S) \not\subseteq f$, then f cannot contain interior points from all three faces of $C(S)$.*

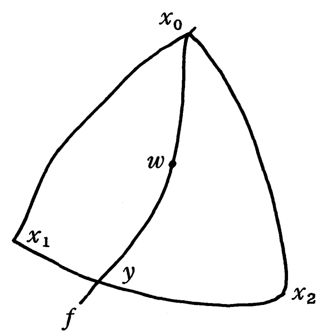
Proof: By **1.3** $\dim\,(f \cap \mathbf{f}_2(S)) \leq 1$. If $C(S) \not\subseteq f$, but f contains interior points from all three faces of $C(S)$, then $f \cap \mathbf{f}_2(S)$ is a 1-flat which contains interior points from all three faces of $C(S)$, contradicting **3.20.2.**

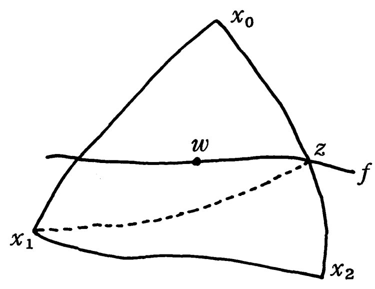
Prop. **3.21**: *Suppose $S = \{x_0,x_1,x_2\}$ is linearly independent. Suppose f is a 1-flat, $f \cap \overline{x_0x_2} \neq \phi$, and $f \cap \mathrm{Int}\,C(S) \neq \phi$. Then f intersects at least one other face of $C(S)$; moreover, if $x_0\,\varepsilon f$, then $f \cap \mathrm{Int}\overline{x_1x_2} \neq \phi$.*

Proof: Case 1: $f \cap \overline{x_0x_2} = \{x_0\}$. Select $w\,\varepsilon f \cap \mathrm{Int}C(S)$. Then $w\,\varepsilon\,\overline{x_0y}$ for some $y\,\varepsilon\,\overline{x_1x_2}$ (**3.6**), hence $f = \mathbf{f}_1(w,y) = \mathbf{f}_1(x_0, w)$ implies $f \cap \overline{x_1x_2} = \{y\}$. Since $w\,\varepsilon\,\mathrm{Int}C(S)$. $y\,\varepsilon\,\mathrm{Int}\,\overline{x_1x_2}$.

Case 2: $f \cap \overline{x_0x_2} = \{z\}$, $z\,\varepsilon\,\mathrm{Int}\overline{x_0x_2}$. Select $w\,\varepsilon f \cap \mathrm{Int}C(S)$. By **3.18** $\overline{zx_1}$ separates $C(S)$ into convex components $C(\{x_0,x_2,z\}) - \overline{x_1z}$ and $C(\{x_1,x_0,z\}) - \overline{x_1z}$. Since $\overline{zw} - \{z\}$ is connected (**2.15**), either $\overline{zw} \subseteq C(\{x_1,x_2,z\})$ or $\overline{zw} \subseteq C(\{x_1,x_0,z\})$. If $\overline{zw} \subseteq C(\{x_1,x_2,z\})$, then also by Case 1, $f \cap \overline{x_0x_1} \neq \phi$. If $\overline{zw}$ $C(\{x_1, x_0, z\})$, then also by case 1, $f \cap \overline{x_0x_1} = \phi$.

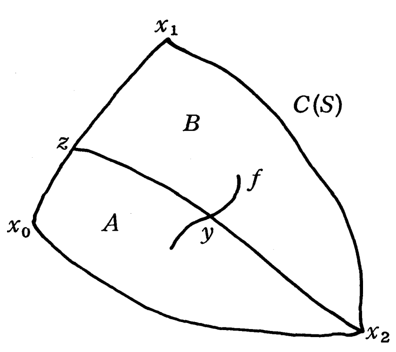
Case 3: $f \cap \overline{x_0x_2} = \{x_2\}$. As in case 1.





Cor. **3.21.1**: *If $C(S)$ is any 2-simplex, and f is any k-flat, $k \geq 1$, such that $f \cap \mathrm{d}C(S) \neq \phi$, and $f \cap \mathrm{Int}C(S) \neq \phi$, then f intersects at least two distinct faces of $C(S)$.*

Proof: If $C(S) \subseteq f$, then **3.21.1** is trivially true. If $C(S) \not\subseteq f$, then $f \cap \mathbf{f}_2(S)$ is a 1-flat, hence **3.21.1** follows at once from **3.21**.



Prop. **3.22**: *If $S = \{x_0,x_1,x_2\}$ is linearly independent, and if f is a 1-flat of $f_2(S)$ such that $f \cap \mathrm{Int}C(S) \neq \phi$, then f intersects at least two distinct faces of $C(S)$, and at least one of these faces in an interior point.*

Proof: Choose $y\,\varepsilon f \cap \mathrm{Int}C(S)$. Then $y\,\varepsilon\,\mathrm{Int}\overline{x_2z}$ for some $z\,\varepsilon\,\mathrm{Int}\overline{x_0x_1}$. Set $A = C(\{z,x_0,x_2\})$ and $B = C(\{z,x_1,x_2\})$.

Since $C(S) = A \cup B$ (**3.18** proof), and $f \subseteq \mathbf{f}_2(A) = \mathbf{f}_2(B) = \mathbf{f}_2(S)$, we have by **3.7** that $f \cap \text{Int}A \neq \phi$ and $f \cap \text{Int}B \neq \phi$. Therefore by **3.21** f intersects another face of both A and B. If both intersections occur in the same face $\overline{x_i x_j}$, then $\overline{x_i x_j} \subseteq f$ which implies that $y \,\varepsilon\, \text{d}C(S)$, a contradiction. If at least one of these intersections does not occur in the interior of some face, then clearly both intersections must occur in the same face, i.e. at x_i and x_j, hence in $\overline{x_i x_j}$, which was just shown to be impossible.

Since **3.12** could have been proved using **3.6**, **3.22** was the first proposition in this chapter which really required the use of **3.7**. The following corollary is the first time that **3.9** will have been used in this chapter.

Cor. **3.22.1**: *If $C(S)$ is a 2-simplex, and f is an m-1-flat such that $C(S) \not\subseteq f$, but $f \cap \text{Int}C(S) \neq \phi$, then f intersects at least two distinct faces of $C(S)$.*

Proof: By **3.9** $f \cap \mathbf{f}_2(S)$ is a *1*-flat in $\mathbf{f}_2(S)$. The result now follows from **3.22**.

Prop. **3.23**: *If $S = \{x_0, x_1, x_2\}$ is linearly independent, and f is an m-1-flat such that $\overline{x_0 x_1} \not\subseteq f$, but $f \cap \text{Int}\overline{x_0 x_1} \neq \phi$, then f intersects at least two distinct faces of $C(S)$.*

Proof: By **3.9** $f \cap \mathbf{f}_2(S)$ is a *1*-flat in $\mathbf{f}_2(S)$ which intersects $\overline{x_0 x_1}$ in a single point. **3.23** then follows from **3.21**.

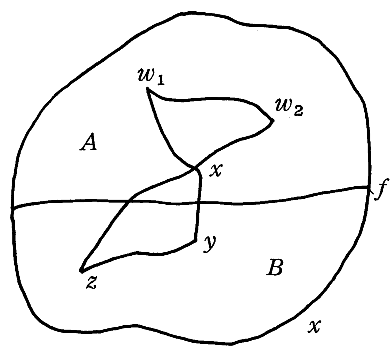
Prop. **3.24**: *If $C(S)$ is a k-simplex, and f is a 1-flat in $f_k(S)$ such that $f \cap \text{Int}C(S) \neq \phi$, then $f \cap \text{d}C(S)$ contains at least two distinct points.*

Proof: **3.24** is true for $k = 1$, and is also true for $k = 2$ by **3.22**. Assume **3.24** is true for $k - 1 \geqslant 2$. Suppose $S = \{x_0, \ldots, x_k\}$ is linearly independent, $k \geqslant 3$, and f is a 1-flat such that $f \cap \text{Int}C(S) \neq \phi$. Since $f \subseteq \mathbf{f}_k(S)$ we have that $f \cap \text{Int}C(S)$ is open in f by **3.16.1**. If $f \cap \text{Int}C(S)$ consisted of only a single point x, then $\{x\}$ would be both open and closed in f, hence f would not be connected. We therefore can find u,v distinct points in $f \cap \text{Int}C(S)$, and $z_1, z_2 \,\varepsilon\, \text{Int}F^0C(S)$ with $u \,\varepsilon\, \overline{x_0 z_1}$ and $v \,\varepsilon\, \overline{x_0 z_2}$ (**3.6**).
Case 1: $z_1 = z_2$. Then $f = \mathbf{f}_1(u,v) = \mathbf{f}_1(x_0, z_1)$, hence x_0 and z_1 are two distinct points of $f \cap \text{d}C(S)$.
Case 2: $z_1 \neq z_2$. Then $\mathbf{f}_1(z_1, z_2) \cap \text{d}F^0C(S)$ contains two distinct points w_1 and w_2 by the induction assumption. Then $\overline{x_0 z_1} \cup \overline{x_0 z_2} \subseteq C(\{x_0, w_1, w_2\})$, hence $f \cap \text{d}C(\{x_0, w_1, w_2\})$ contains at least two distinct points, but $\text{d}C(\{x_0, w_1, w_2\}) \subseteq \text{d}C(S)$.

Cor. **3.24.1**: *f, the 1-flat described in* **3.24**, *intersects* $\text{d}C(S)$ *in exactly two distinct points from distinct faces of $C(S)$.*

Proof: If f contains two distinct points, i.e. a basis, from any one face of $C(S)$, then $f \cap C(S) \subseteq \text{d}C(S)$, which gives $f \cap \text{Int}C(S) = \phi$, a contradiction. Suppose x,y,z are distinct points of $f \cap \text{d}C(S)$. We may suppose that in the ordering of f, $x < y < z$, hence $y \,\varepsilon\, \text{Int}\overline{xz}$ by **2.27**. Then $\overline{xz} \cap \text{d}C(S) = \{x,z\}$, a contradiction of **3.20.1**.

Prop. **3.25**: *If f is an m-1-flat which disconnects X, then f disconnects X into two convex, open components.*



Proof: If f disconnects X, then we can find $x,y \varepsilon X-f$ such that $\text{Int}\overline{xy} \cap f \neq \phi$. Set $A = \{w \varepsilon X-f \mid \overline{wx} \cap f = \phi\}$, and $B = \{u \varepsilon X-f \mid \overline{uy} \cap f = \phi\}$. Choose any $z \varepsilon X-f$. If $\{x,y,z\}$ is linearly independent, then by **3.23** either $f \cap \text{Int}\overline{zx} \neq \phi$, or $f \cap \text{Int}\overline{zy} \neq \phi$. By **3.20.3** both these intersections cannot be non-empty at the same time, hence either $z \varepsilon A$, or $z \varepsilon B$. If $\{x,y,z\} \subseteq f'$, a 1-flat, and $f \cap \overline{xz}$ and $f \cap \overline{yz}$ are both non-empty, then using **3.5** we would have $f' \subseteq f$, a contradiction. Therefore again, either $z \varepsilon A$, or $z \varepsilon B$, hence $X - f = A \cup B$. Suppose $w_1, w_2 \varepsilon A$. If $\{x,w_1,w_2\}$ is linearly independent, then since f intersects neither $\overline{xw_1}$, nor $\overline{xw_2}$ by **3.24** $f \cap \overline{w_1 w_2} = \phi$. If $\{x,w_1,w_2\} \subseteq f'$, a 1-flat, then since by **3.5**, $\overline{xw_1} \cup \overline{xw_2} = \overline{w_1w_2}$, $\overline{xw_1}$, or $\overline{xw_2}$ it is clear that $f \cap \overline{w_1w_2} \neq \phi$ would imply that either $f \cap \overline{xw_1} \neq \phi$, or $f \cap \overline{xw_2} \neq \phi$, a contradiction in either case. Therefore, A and B are connected (**2.4**). If $A \cap B \neq \phi$, then $A \cup B = X - f$ is also connected contrary to assumption. Therefore A and B are disjoint. If for every convex, open neighborhood U of a point $w \varepsilon A$, there is $z \varepsilon U \cap B$, then since $\overline{zw} \subseteq U$ by **2.10**, there must also be $v \varepsilon f \cap U$. This implies that $w \varepsilon \text{Cl} f = f$, a contradiction to $w \varepsilon A$. Therefore $U \subseteq A$, hence A is open; similarly, B is open.

Cor. **3.25.1**: *If f is an m-1-flat which disconnects an open, convex subset U, then f separates U into two convex, open components.*

Proof: **3.11**; then set $X = U$ in **3.25**.

Prop. **3.26**: *If U is a convex, open subset of X, then a given m-1-flat f disconnects U iff there are points x and y in U such that f disconnects* $\overline{xy}$.

Proof: If f disconnects U, then there must be points x and y in U as described. Suppose there are points $x,y \varepsilon U$ such that f disconnects $\overline{xy}$. Using **1.3** and **2.15.1** we have that $f \cap \overline{xy}$ consists of a single point in $\text{Int}\overline{xy}$. Set $A = \{w \varepsilon U-f \mid \overline{yw} \cap f = \phi\}$ and $B = \{w \varepsilon U-f \mid \overline{xw} \cap f = \phi\}$. Arguments similar to those used in **3.25** show $A \cup B = U-f$, and A and B are open, convex subsets of U. Suppose $w \varepsilon A \cap B$. If $\{x,y,w\}$ is linearly independent, then since $f \cap \overline{xw} \neq \phi$, either $f \cap \overline{xw} \neq \phi$, or $f \cap \overline{yw} \neq \phi$, a contradiction. Suppose $\{x,y,w\} \subseteq f'$, a 1-flat. By **3.5** $\overline{xy} \cup \overline{yw} = \overline{xy}$, $\overline{yw}$, or $\overline{yw}$ but $\overline{xw}$ and $\overline{xw}$ are impossible since these segments would then contain $\overline{xy}$, hence a point of f. If $\overline{xy} \cup \overline{yw} = \overline{xy}$, then $w \varepsilon \text{Int}\overline{xy}$, hence by **2.14**, $\overline{xy} = \overline{xw} \cup \overline{yw}$ and $\overline{xw} \cap \overline{yw} = \{w\}$, hence again either $f \cap \overline{wx} \neq \phi$, or $f \cap \overline{yw} \neq \phi$, another contradiction. Therefore $A \cap B = \phi$, hence $U-f$ is disconnected.

Cor. **3.26.1**: *An m-1-flat disconnects X iff it disconnects some segment* $\overline{xy} \subseteq X$. *Moreover, if f disconnects* $\overline{xy}$, *then x and y are in different components of X-f.*

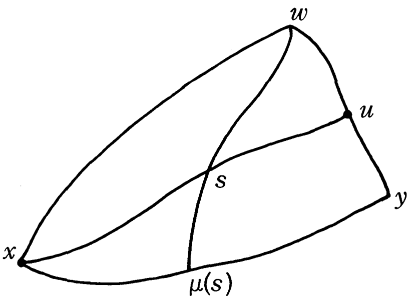
Proof: **3.26** with $U = X$; $x \varepsilon A$ and $y \varepsilon B$.

Cor. **3.26.2**: *If f is a k-1-flat contained in k-flat f', $k \geqslant 1$, then f disconnects f' iff f disconnects some segment $\overline{xy}$ in f'. If f disconnects $\overline{xy} \subseteq f'$, then x and y are in different components of f'-f.*

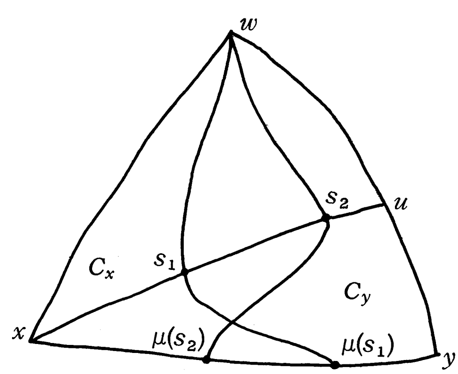
Proof: **3.10** and **3.26.1**.

Prop. **3.27**: *Suppose $l(G) \geqslant 1$. Then if $\overline{xw}$ and $\overline{zy}$ are any two segments in X, then $\overline{xw}$ and $\overline{zy}$ are homeomorphic.*

Proof: Case 1: $x = z$, $w \neq y$, and $\{x,w,y\}$ linearly independent. Choose $u \varepsilon \mathrm{Int}\overline{wy}$; then $\overline{xu} \subseteq C(\{x,w,y\})$. For each $s \varepsilon \overline{xu}$, $\mathbf{f}_1(w,s) \cap \overline{xy}$ consists of a single point by **3.21**; denote this point by $\mu(s)$ to obtain $\mu{:}\overline{xu} \to \overline{xy}$. μ is 1-1 since if $\mu(s_1) = \mu(s_2)$, then $\overline{w\mu(s_1)} = \overline{w\mu(s_2)}$ therefore $\overline{w\mu(s_1)} \cap \overline{xu} = \{s_1\} = \overline{w\mu(s_2)} \cap \overline{xu} = \{s_2\}$. μ is onto: Choose $v \varepsilon \overline{xy}$. Then $\overline{wv} \cap \overline{xu} \neq \phi$ if $v \neq x$ since $\overline{xu}$ disconnects $C(\{x,y,w\})$ in a manner which puts w and v in different components (**3.18** proof). If $v = x$, then $v \varepsilon \overline{xu}$. $\overline{xu}$ and $\overline{xy}$ can both be totally ordered so as to have the order topology (cf. **2.26** and **2.27.3**) and, without loss of generality, we may assume $x < u$ (in $\overline{xu}$) and $x < y$ (in $\overline{xy}$). Suppose



u is not order-preserving. Then we can find $s_1, s_2 \varepsilon \overline{xu}$ with $s_1 < s_2$ such that $\mu(s_2) < \mu(s_1)$. It is impossible that $\mu(u) = x$, for then w would have to be in $\overline{xu}$, hence we may assume at least one of the $\mu(s_i)$, say $\mu(s_1)$, is in $\mathrm{Int}\overline{xy}$. Then $\overline{w\mu(s_1)}$ disconnects $C(\{w,x,y\})$ into two convex components (**3.18**), C_x containing x, and C_y containing u and y. $C_x \cap \overline{xy} = \{v \varepsilon \overline{xy} \mid v < \mu(s_1)\}$ by **2.14** and **2.27**, therefore $\mu(s_2) \not\varepsilon C_x$. But $C_x \cap \overline{xu} = \{s \varepsilon \overline{xu} \mid s < s_1\}$, hence $s_2 \varepsilon C_x$, therefore $\overline{w\mu(s_1)} \cap \mathrm{Int}\overline{w\mu(s_2)} \neq \phi$ since $\overline{w\mu(s_1)}$ must disconnect $\overline{w\mu(s_2)}$. $\overline{w\mu(s_1)}$ and $\overline{w\mu(s_2)}$ therefore contain two points in common, hence $\mathbf{f}_1(w,s_2) = \mathbf{f}_1(w,s_1)$, and it is impossible that $\mathbf{f}_1(w,s_2)$ intersect $\overline{xu}$ in more than one point, hence $s_1 = s_2$, a contradiction. Since $u{:}\overline{xu} \to \overline{xy}$ is 1-1, onto, and order-preserving it is a homeomorphism. A similar argument shows $\overline{xu}$ is homeomorphic to $\overline{xw}$, hence $\overline{xw}$ is homeomorphic to $\overline{xy}$.



Case 2: x,w,z,y are all contained in the same 1-flat f. Choose $u \varepsilon X$-f. Then $\overline{xw}$ is homeomorphic to $\overline{xu}$ and $\overline{zy}$ is homeomorphic to $\overline{zu}$, both by case 1. Either $\overline{xu} = \overline{zu}$, or at least $\overline{xu}$ is homeomorphic to $\overline{zu}$ by case 1, hence $\overline{xw}$ is homeomorphic to $\overline{zy}$.

Case 3: x,y,z,w are all distinct, but are not all contained in the same 1-flat. By case 1 $\overline{xw}$ is homeomorphic to $\overline{wy}$ which is homeomorphic to $\overline{zy}$, hence $\overline{xw}$ is homeomorphic to $\overline{zy}$.

Note that all 1-flats of X are not necessarily homeomorphic to each other, e.g. let $X = \{(x,y)\varepsilon \mathbf{R}^2 | x^2 + y^2 < 1\} \cup \{(x,y)\varepsilon \mathbf{R}^2 | x > 0,\ x^2 + y^2 = 1\}$, and give x the induced geometry and topology from $\mathbf{R}^2$. Then some 1-flats of X have two non-cut points, some one, and others none at all.

CHAPTER IV

OPEN m-ARRANGEMENTS

4.0: In this chapter and the next we shall deal with m-arrangements which have special properties. Using the theory developed in chapter II (cf. **2.26**, etc.), we shall consider that all 1-flats have been totally ordered by $>$ so as to have the order topology.

Suppose $\gamma: f \to f'$ is an order-reversing map from a 1-flat f into another 1-flat f'. If we define a new order $\circledcirc$ on f' by $x \circledcirc y$ iff $y > x$ for all $x,y \varepsilon f'$, then f' with the order $\circledcirc$ still satisfies **2.26** and the subsequent propositions of chapter II; γ then becomes order-preserving. We see then that in dealing with a map from one 1-flat into another which is either order-preserving or order-reserving, we lose no generality in assuming the map is order-preserving.

Definition 1: An open m-arrangement is an m-arrangement in which every point is a cut point of every 1-flat (in G^*) which contains it.

Unless specifically stating otherwise, the propositions of this chapter will refer to a space X with geometry G such that X and G form an open m-arrangement. Since a 0-arrangement consists of a space which contains only one point and the geometry $G = \{F^{-1}\}$, we shall also assume that $m \geqslant 1$.

Prop. **4.1:** *If f is a k-flat of G, $k \neq -1$, then f with geometry G_f and the subspace topology is an open k-arrangement.*

Proof: Since g is a 1-flat of G_f^* iff g is a 1-flat of G, the proposition follows at once from **3.10**.

Prop. **4.2:** *If X is the space of an open 1-arrangement, then $\mathcal{B} = \{\text{Int}\overline{xy} | x,y \varepsilon X,\ x \neq y\}$ is a basis for the topology of X.*

Proof: Since X has the order topology, and $\text{Int}\overline{xy} = \{z | x < z < y\}$ if $x < y$ (**2.27**), any open set of X is the union of elements of $\mathcal{B}$ and sets of the form $\{z | z > x\}$, or $\{z | z < x\}$. Let $w \varepsilon \{z | z > x\}$. Since w disconnects X, $w \varepsilon \text{Int}\overline{uv}$ for some $u,v \varepsilon X$, and we may assume $v > w$, i.e. $u < w < v$. Then $x < w < v$, hence $w \varepsilon \text{Int}\overline{xv} \subseteq \{z | z > x\}$. Similarly, if $w \varepsilon \{z | z < x\}$, then there is $v \varepsilon X$ such that $w \varepsilon \text{Int}\overline{xv} \subseteq \{z | z < x\}$. We therefore have that each set of the form $\{z | z < x\}$ or $\{z | z > x\}$ is the union of elements of $\mathcal{B}$, hence $\mathcal{B}$ is a basis for the topology of X.

Prop. **4.3**: *If U is a non-empty, open, convex subset of X, then U with geometry G_U and the subspace topology is an open m-arrangement.*

Proof: By **3.11** U and G_U form an m-arrangement. Let g be a 1-flat of G_U^*; then $g = U \cap f$ where f is a 1-flat in G^*. Let $w \varepsilon g$. Since g is an open neighborhood of w in f, by **4.2** we can find $u,v \varepsilon g$ such that $w \varepsilon \mathrm{Int}\overline{uv} \subseteq U$. Then w disconnects g by **2.16**.

Cor. **4.3.1**: *If U is a non-empty, convex subset of some k-flat f, and U is open in f, then U with geometry G_U and the subspace topology is an open k-arrangement.*

Proof: **3.12, 3.16.1,** and **4.3.1.**

Prop. **4.4**: *A necessary and sufficient condition that some m-1-flat disconnect an open set $U \subseteq X$ is that $f \cap U \neq \phi$.*

Proof: The condition is clearly necessary. Suppose $f \cap U \neq \phi$, and choose $y \varepsilon f \cap U$. By **2.11** f is nowhere dense in U, hence $U - f \neq \phi$. Choose $w \varepsilon U - f$. Then $\mathbf{f}_1(y,w) \cap f = \{y\}$. Since $\mathbf{f}_1(y,w) \cap U$ is an open neighborhood of y in $\mathbf{f}_1(y,w)$, by **4.2** we can find $u,v \varepsilon \mathbf{f}_1(y,w) \cap U$ such that $y \varepsilon \mathrm{Int}\overline{uv} \subseteq U$. Therefore f disconnects $\overline{uv}$ (**2.13**), consequently f disconnects X into two convex, open components A and B such that $u \varepsilon A$ and $v \varepsilon B$ (if $m > 1$, by **3.26.1**; if $m = 1$, by **2.22** proof and **2.23**). Therefore $(U-f) \cap A \neq \phi$, and $(U-f) \cap B \neq \phi$, but since f is closed, $U-f$ is open, hence $(U-f) \cap A$ and $(U-f) \cap B$ are disjoint, open subsets of $U-f$ whose union is $U-f$, hence $U-f$ is disconnected.

Cor. **4.4.1**: *Every m-1-flat disconnects X.*

Cor. **4.4.2**: *If $k \geqslant 1$, and f is a k-1-flat contained in a k-flat f', then f disconnects f'.*

Cor. **4.4.3**: *If f is a k-flat, then no flat of dimension less than k-1 can disconnect f.*

Proof: If f' is any i-flat in f, $i < k-1$, then f' is properly contained in g where g is some k-1-flat in f. By **4.4.2** and **2.12** g is a minimal disconnecting subset of f, hence $f - f'$ is connected.

Together **4.42** and **4.43** give

Cor. **4.4.4**: *A non-empty flat f disconnects a flat f' iff* i) $f \subseteq f'$, *and* ii) $\mathit{dimf} = \mathit{dimf}' - 1$.

Prop. **4.5**: *If a space X with geometry G is an m-arrangement, then a point x_0 is a cut point of every 1-flat which contains it iff $x_0 \varepsilon \mathrm{Int}C(S)$ where $C(S)$ is some m-simplex.*

Proof: Suppose x_0 is a cut point of every 1-flat which contains it. Let $S' = \{x_0, y_1, \ldots, y_m\}$ be any maximal linearly independent set which contains x_0. **4.5** is true for $m = 1$ by **2.16,** hence we may assume $m \geqslant 2$. Choose $w \varepsilon \mathrm{Int}F^0C(S')$ ($\neq \phi$ by **3.12**). Then by hypothesis x_0 is a cut point of $\mathbf{f}_1(x_0,w)$, hence there is $u \varepsilon \mathbf{f}_1(x_0,w)$ such that $x_0 \varepsilon \mathrm{Int}\overline{wu}$. It follows at once by **3.20.1** that $x_0 \varepsilon \mathrm{Int}C(\{u, y\ , \ldots, y_m\})$. Suppose $x_0 \varepsilon \mathrm{Int}C(S)$ where $C(S)$ is some m-simplex. Let f be any 1-flat which contains x_0. By **4.3.2**, x_0 disconnects $f \cap \text{Int}\, C(S)$, a 1-flat in $G_{\text{Int}C(S)}$, hence by **2.16** there are $u,v \varepsilon \text{Int}\, C(S) \cap f$ with $x_0 \varepsilon \text{Int}\, \overline{uv}$. Then, again by **2.16** x_0 disconnects f.

Cor. **4.5.1**: *If a space X with geometry G is an open m-arrangement, then for any* $x \varepsilon X$ *and for any* k, $1 \leq k \leq m$, *there is a k-simplex* $C(S)$ *such that* $x \varepsilon \text{Int}\, C(S)$.

Proof: **4.1** and **4.5** together with **1.11**.

Prop. **4.6**: *Set* $\mathcal{B} = \{\text{Int}\, C(S) | C(S) \text{ is an } m\text{-simplex in } X\}$. *Then* $\mathcal{B}$ *is the basis for a topology on X which is equivalent to the original topology on X.*

Proof: If $C(S)$ and $C(T)$ are arbitrary m-simplices, then $\text{Int}\, C(S)$ and $\text{Int}\, C(T)$ are convex, open subsets of X (**3.16** and **3.20**), hence $\text{Int}\, C(S) \cap \text{Int}\, C(T)$ is convex and open. Any $x \varepsilon X$ is in the interior of some m-simplex by **4.5.1**. To complete the proof of **4.6** (including that $\mathcal{B}$ is a basis for a topology on X), it merely remains to be shown that if U is an open, convex neighborhood of an arbitrary point x, then there is some m-simplex $C(S) \subseteq U$ such that $x \varepsilon \text{Int}\, C(S)$. This, however, follows at once from **4.3**, **4.5.1**, and the trivial fact that an m-simplex in U is also an m-simplex in X.

Following E. Kamke [6], by a border element of a totally ordered set, we shall mean a first, or last element. A totally ordered set is called unbordered if it is non-empty and has no border elements. An ordered set is said to be dense if given any $s_1, s_2 \varepsilon S$, $s_1 < s_2$, then there is $s_3 \varepsilon S$ such that $s_1 < s_3 < s_2$.

Prop. **4.7**: *If X is the space of an open 1-arrangement, then, if Y is dense in X, then Y is unbordered and dense.*

Proof: Y is dense: Since $X \neq \phi$, $Y \neq \phi$. Choose $y_1, y_2 \varepsilon Y$, $y_1 < y_2$ (cf. **4.0**). By **2.25** $X\text{-Int}\, \overline{y_1 y_2}$ is closed in X. If there were no $y_3 \varepsilon Y \cap \text{Int}\, \overline{y_1 y_2}$, then $\text{Cl}\, Y \subseteq X\text{-Int}\, \overline{y_1 y_2} \neq X$, and Y would not be dense in X, hence we can find $y_3 \varepsilon Y \cap \text{Int}\, \overline{y_1 y_2}$. If $y_3 \varepsilon \text{Int}\, \overline{y_1 y_2}$ and $y_1 < y_2$, then by **2.27**, $y_1 < y_3 < y_2$.

Y is unbordered: : Suppose there is $y_0 \varepsilon Y$ such that $y_0 < y$ for all $y \varepsilon Y$. Since y_0 is a cut point of X, by **2.16** $y_0 \varepsilon \text{Int}\, \overline{wz}$, $w < y_0 < z$ for suitable $w, z \varepsilon X$. Then $\text{Int}\, \overline{wy_0} \cap Y \neq \phi$, contradicting Y dense in X. Similarly, there can be no $y_0 \varepsilon Y$ such that $y > y_0$ for all $y \varepsilon Y$.

Cor. **4.7.1**: *If X is the space of an open 1-arrangement, and Y is a countable dense subset of X, then Y is of the same order type as the rational numbers.*

Proof: By **4.7**, Y is dense and unbordered, but all unbordered, dense, countable sets are of the same order type (*cf.* Kamke [7], p. 71).

Prop. **4.8**: *If X is the space of an open 1-arrangement, and X is second countable, then X is homeomorphic to* **R**, *the real line.*

Proof: Since X is second countable, X contains a countable, dense subset Y. By **4.7.1**, Y is of the same order type as the rationals $\mathbf{Q} \subseteq \mathbf{R}$. Let $u: Y \to \mathbf{Q}$ be a 1-1, onto, order-preserving map. Define that $\mu(y_i) = i$, for all $i \varepsilon Z$, the integers. $\overline{y_{i-1}y_i}$ is also second countable, hence by **2.31.1** $\overline{y_{i-1}y_i}$ is homeomorphic to $[i\text{-}1,i] = \{r \varepsilon \mathbf{R} \mid i\text{-}1 \leqslant r \leqslant i\}$ by an order-preserving map ν_i (not necessarily an extension of $\mu|\overline{y_{i-1}y_i}$) with $\nu_i(y_{i-1}) = i\text{-}1$ and $\nu_i(y_i) = i$. Since $\overline{y_{i-1}y_i} \cap \overline{y_{k-1}y_k}$ can consist of at most either y_{i-1}, or y_i, if $k = 1$, we can define $\nu: \bigcup_{i\varepsilon Z} \overline{y_{i-1}y_i} \to \mathbf{R}$ by $\nu(x) = \nu_i(x)$ for $x \varepsilon \overline{y_{i-1}y_i}$. Since ν is 1-1, onto, and order-preserving, it is a homeomorphism; we must now show that $\bigcup_{i\varepsilon Z} \overline{y_{i-1}y_i} = X$. Choose $w \varepsilon X$. Then $w \varepsilon \mathrm{Int}\overline{uv}$ for suitable $u,v \varepsilon X$ by **2.16.** Therefore since Y is dense in X, there are $a_1 \varepsilon \overline{uw} \cap Y$ *and* $a_2 \varepsilon \overline{wv} \cap Y$ such that $w \varepsilon \mathrm{Int}\overline{a_1a_2}$. Since $Y \bigcup_{i\varepsilon Z} \overline{y_{i-1}y_i}$, it follows that $w \varepsilon \overline{a_1a_2} \subseteq \bigcup_{i\varepsilon Z} \overline{y_{i-1}y_i}$.

Cor. **4.8.1**: *If X is the space of a 1-arrangement (not necessarily open), and if X is second countable, then X is homeomorphic to $(0,1)$, $[0,1)$, or $[0,1]$, depending on whether X has no, one, or two non-cut points respectively.*

Proof: Case 1: X has no non-cut points. By **4.8**, X is homeomorphic to $\mathbf{R}$, hence to $(0,1)$.

Case 2: X has one non-cut point x_0. Then clearly x_0 must be a border element of X. Assume x_0 is a first element. Then by case 1, $X\text{-}\{x_0\}$ is homeomorphic to $(0,1)$ by an order-preserving map ν. Define $\nu': X \to [0,1)$ by $\nu'(x_0) = 0$, and $\nu'(x) = \nu(x)$ for $x \neq x_0$. Then ν' is clearly a homeomorphism onto. If x_0 is a last element, then we can similarly show that X is homeomorphic to $(0,1]$, hence to $[0,1]$.

Case 3: X has two non-cut points x_0 and x_1. Then by **2.27**, $X = \overline{x_0x_1}$, hence by **2.31.1**, X is homeomorphic to $[0,1]$.

Definition **2**: *Let X with geometry G form an m-arrangement. Set* $M(X) = \{C(S) \mid C(S)$ *is an m-simplex in* $X\}$. *Define* $\mathrm{Int}X = \bigcup_{M(X)} \mathrm{Int}C(S)$, *and* $\mathrm{Bd}X = X\text{-}\mathrm{Int}X$. *We shall call* $\mathrm{Int}X$, *the interior of* X, *and* $\mathrm{Bd}X$, *the border of* X.

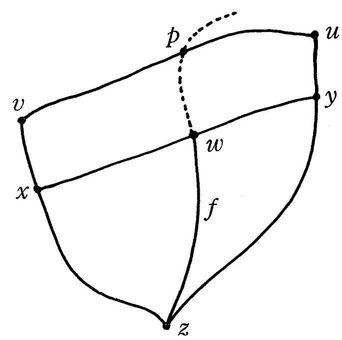
Because of **4.5** we have equivalently $\mathrm{Int}X = \{x \varepsilon X \mid x$ *is a cut point of every 1-flat which contains it*$\}$.

Prop. **4.9**: $\mathrm{Int}X$ *is open, and non-empty.*

Proof: $\mathrm{Int}X$ is the union of non-empty, open sets (**3.16** and **3.12**). Since by **1.6** X has at least one basis of $m+1$ points, $M(X)$, hence $\mathrm{Int}X$, is non-empty.

Prop. **4.10**: $\mathrm{Int}X$ *is convex.*

Proof: Let x and y be arbitrary points in $\mathrm{Int}X$. Suppose $w \varepsilon \mathrm{Int}\overline{xy}$. Then by **2.16**, w is a cut point of $\mathbf{f}_1(x,y)$. Let f be any other 1-flat which contains w. Choose $z \varepsilon f\text{-}\{w\}$. Then since x and y are cut points of



$f_1(z,x)$ and $f_1(z,y)$ respectively, by **2.16** we can find $u \varepsilon f_1(z,y)$ and $v \varepsilon f_1(z,x)$ such that $x \varepsilon \mathrm{Int}\overline{vz}$ and $y \varepsilon \mathrm{Int}\overline{uv}$. Then $\overline{xy} \subseteq C(\{z,u,v\})$, hence $f \cap \overline{uv} \neq \phi$ by **3.21**. But if $p \varepsilon \overline{uv} \cap f$, and $w \varepsilon \mathrm{Int}C(\{z,u,v\})$ by **3.20.1**, then we also have by **3.20.1** that $w \varepsilon \mathrm{Int}\overline{pz} \subseteq f$, hence by **2.16** w is a cut point of f. Since f was an arbitrary 1-flat which contained w, w is a cut point of any 1-flat which contains it, hence $w \varepsilon \mathrm{Int}X$.

Cor. **4.10.1**: *Let X be a space with geometry G such that X and G form an m-arrangement. Then* Int$X$ *with geometry* $G_{\mathrm{Int}X}$ *and the subspace topology is an open m-arrangement.*

Proof: Int$X$ is open and convex by **4.9** and **4.10**, hence by **3.11**, Int$X$ with geometry $G_{\mathrm{Int}X}$ is an m-arrangement. Since every point of Int$X$ is in the interior of some m-simplex, by **4.5** $x \varepsilon \mathrm{Int}X$ is a cut point of every 1-flat in $G_{\mathrm{Int}X}$ which contains it, hence the arrangement is open.

Prop. **4.11**: BdX *is closed and nowhere dense.*

Proof: Since $\mathrm{Bd}X = X - \mathrm{Int}X$, Bd$X$ is closed. Suppose some non-empty, open set $U \subseteq \mathrm{Bd}X$. Choose $x \varepsilon U$, and let V be an open, convex neighborhood of x with $V \subseteq U$. Then by **3.11**, y contains a basis S of X, and by **2.9f**), $C(S) \subseteq V$. But since $\mathrm{Int}C(S) \subseteq \mathrm{Int}X$ we have the contradiction that $U \cap \mathrm{Int}X \neq \phi$.

Prop. **4.12**: *Let X with geometry G form an m-arrangement. Then an m-1-flat f in G disconnects X iff f* $\cap$ Int$X \neq \phi$.

Proof: Suppose $f \cap \mathrm{Int}X \neq \phi$. Then $f \cap \mathrm{Int}X$ is an m-1-flat in $G_{\mathrm{Int}X}$ (**3.11**), hence $f \cap \mathrm{Int}X$ disconnects Int$X$ by **4.10.1** and **4.4.1**. By **3.26.1**, therefore, f disconnects a segment in $\mathrm{Int}X \subseteq X$, hence by **3.26.1** f disconnects X. Suppose f disconnects X. Then by **3.25** f disconnects X into two convex, open components A and B. Since A is open, by **3.11** it contains a basis $S' = \{x_0, \ldots, x_m\}$ of X. Since B is open and $f_{m-1}(S\text{-}\{x_0\})$ is nowhere dense in X (**2.11**), we can find $y_0 \varepsilon B\text{-}f_{m-1}(S\text{-}\{x_0\})$. Then $S = \{y_0, x_1, \ldots, x_m\}$ is linearly independent. By **2.9f**) $F^0C(S) \subseteq A$, but since $y_0 \varepsilon B$, f must disconnect any segment of the form $\overline{y_0 w}$, where $w \varepsilon F^0C(S)$. But then $f \cap \mathrm{Int}C(S) \neq \phi$ (choose $w \varepsilon \mathrm{Int}F^0C(S)$ and apply **3.20.1**), hence $f \cap \mathrm{Int}X \neq \phi$.

Cor. **4.12.1**: *Any k-flat f, k* $\geqslant 1$, *contains at least one k-1-flat which disconnects it.*

Proof: By **3.10** and **4.9**, Int$f \neq \phi$. Choose $x \varepsilon \mathrm{Int}f$, and let f' be any k-1-flat which contains x; f' exists by **1.11**. Then f' disconnects f by **4.12**.

Cor. **4.12.2**: *If a space X and geometry G form an m-arrangement, then no flat of dimension less than m-1 disconnects X.*

Proof: Let f be an i-flat in X, $i < m$-1, and let $S = \{x_0, \ldots, x_i\}$ be a basis for f. Since f is nowhere dense in X by **2.11**, and Int$X$ is non-empty and open by **4.9**, $\mathrm{Int}X - f \neq \phi$. Choose $x_{i+1} \varepsilon \mathrm{Int}X - f$, and extend $S \cup \{x_{i+1}\}$ to a basis $T = \{x_0, \ldots, x_i, x_{i+1}, \ldots, x_{m-1}\}$ of some m-1-flat $f_{m-1}(T)$. Then by **4.12** and **2.12**, $f_{m-1}(T)$ is a minimal disconnecting subset of X. By **1.9.1**, $f \subseteq f_{m-1}(T)$, and since $i < m$-1, $f_{m-1}(T)\text{-}f \neq \phi$, hence X-f is connected.

In this chapter we have concentrated on results useful in chapter V, even though there are many interesting propositions and conjectures pertaining to open m-arrangements and Int$X$ and BdX which might have been mentioned herein. The author hopes to discuss these topics more at length in a future paper.

CHAPTER V

AFFINE m-ARRANGEMENTS

Definition 1: An affine m-arrangement is an m-arrangement which is open if $m = 1$, or in which the geometry is affine if $m \neq 1$ (cf. definition **6** chapter I).

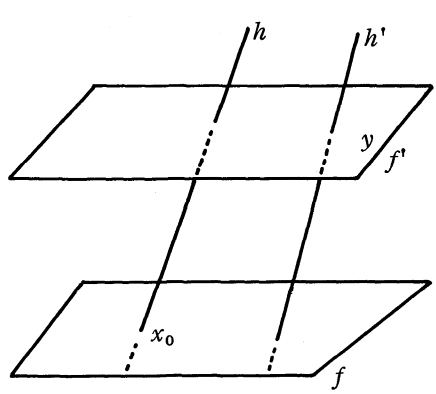
Unless specifically stating otherwise, all propositions in this chapter will refer to a space X and geometry G such that X and G form an affine m-arrangement. In order to avoid trivial cases, we again assume $m \geqslant 1$. The remarks of **4.0** apply also in this chapter.

Prop. **5.1**: *Suppose f is an $m-1$-flat and h is a 1-flat such that $h \cap f$ consists of a single point. Then if h' is a 1-flat parallel to h, then $h' \cap f$ consists of a single point.*

Proof: If $h = h'$, we are done. If $h \neq h'$, then by **3.9**, $(h \vee h') \cap f$ is a 1-flat. Since G is affine, $(h \vee h') \cap f$ intersects both h and h', but is equal to neither, hence $h' \cap f = h' \cap [(h \vee h') \cap f]$ consists of exactly one point.

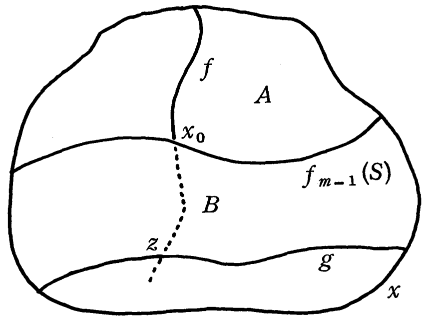
Cor. **5.1.1**: *Suppose f is an $m-1$-flat and h is a 1-flat such that $h \cap f = \{x_0\}$. Suppose f' is any $m-1$-flat parallel to f. Then $f' \cap h$ consists of exactly one point.*

Proof: Should $h \cap f'$ consists of more than one point, then by **1.9.1**, $h \subseteq f'$ an impossibility regardless of whether $f' = f$, or $f' \cap f \neq \phi$, hence $h \cap f'$ consists of at most one point. Select $y \varepsilon f'$. If not, let h' be the unique 1-flat which contains y and is parallel to h. By **5.1**, $h' \cap f$ consists of exactly one point. Since if $f = f'$, the proposition is trivial, assume $f \cap f' \neq \phi$. Then since $h' \subseteq f'$, $h' \cap f'$ consists of y alone, but then $f' \cap h$ consists of exactly one point by **5.1**.



Prop. **5.2**: *Any affine m-arrangement is open.*

Proof: **5.2** is true for $m = 1$ by definition. Suppose **5.2** has been proved for $m - 1 \geqslant 1$, and let X and G form an affine m-arrangement, $m \geq 2$. If **5.2** is false, we can find some 1-flat f and $x_0 \varepsilon f$ such that x_0 is not a cut point of f. It is clear from **2.16** that x_0 must be a border point of f. Select $y_1 \varepsilon \operatorname{Int} X - f$ ($\neq \phi$ by **2.11** and **4.9**). By **3**), definition **1**, chapter I, we may select $y_2, \ldots, y_{m-1}$ such that $\{x_0, y_1, y_2, \ldots, y_{m-1}\} = S$ is linearly independent, and $y_i \varepsilon \mathbf{f}_i(f \cup \mathbf{f}_{i-1}(\{x_0, y_1, \ldots, y_{i-1}\}))$, $2 \leq i \leq m-1$. Then $\mathbf{f}_{m-1}(S) \cap f = \{x_0\}$. *By* **4.12**, $\mathbf{f}_{m-1}(S)$ disconnects X into two components A and B (**3.25**). Since $f - \mathbf{f}_{m-1}(S) = f - \{x_0\}$



is connected, it must be either in A or in B; we may assume it is in A. Choose $z \varepsilon B$ and let g be the unique m-1-flat which contains z and is parallel to $\mathbf{f}_{m-1}(S)$. Since $g - \mathbf{f}_{m-1}(S) = g$ is connected, $g \subseteq B$. Then by **5.1.1**, $f \cap g \neq \phi$, hence $f \cap B \neq \phi$, a contradiction.

Prop. **5.3**: *If k is any k-flat of G, $k \neq -1$, then f with geometry G_f and the subspace topology is an affine k-arrangement.*

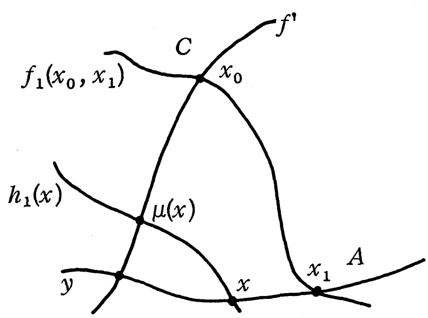
Proof: If $k = 1$, **5.3** follows from **5.2** and **4.1**. For $k \neq 1$, then **5.3** follows from **1.18** and **3.10**.

Prop. **5.4**: *Suppose f is a k-flat contained in a k + 1-flat f' and h is a 1-flat in f' which intersects f in exactly one point. Then if h' is a 1-flat in f' which is parallel to h, then $f \cap h'$ consists of exactly one point.*

Proof: **5.3** and **5.1**.

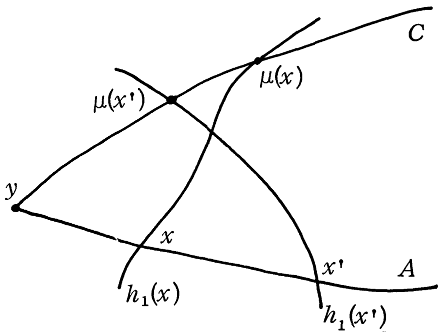
Prop. **5.5**: *Suppose $m \geq 2$, $y \varepsilon f$, a 1-flat, and $\{y\} = A \cap B$, $A \cup B = f$ as in* **2.22**. *Then A and B are homeomorphic.*

Proof: Let f' be any 1-flat which contains y, but is distinct from f. Then $f \cap f' = \{y\}$. y is a cut point of f' by **5.2**, and therefore determines sets C and D such that $C \cap D = \{y\}$, and $C \cup D = f'$ as in **2.22**. Choose $x_1 \varepsilon A$, and $x_0 \varepsilon C$. We first define $\mu: A \rightarrow C$ as follows: Set $\mu(y) = y$, and $\mu(x_1) = x_0$. For any $x \varepsilon A - \{y, x_1\}$, consider $h_1(x)$, the unique 1-flat



which contains x and is parallel to $\mathbf{f}_1(x_0, x_1)$. By **5.4**, since f', $h_1(x)$, and $\mathbf{f}_1(x_0, x_1)$ are all in the same 2-flat, $h_1(x) \cap f'$ consists of a single point; define this point to be $\mu(x)$. If $x \varepsilon \operatorname{Int} \overline{yx_1}$, then $\mu(x) \varepsilon C$ by **3.7** and **3.21**. Suppose $x \varepsilon A - \overline{yx_1}$. Then by **3.26.2**, $\mathbf{f}_1(x_0, x_1)$ disconnects $\mathbf{f}_2(\{y, x_0; x_1\})$ such that x and y are in different components. Since $h_1(x) \cap \mathbf{f}_1(x_0, x_1) = \phi$, a simple argument reveals that $h_1(x) \cap f' \subseteq C - \overline{yx_0}$. Therefore $\mu: A \rightarrow C$. μ is 1-1: Suppose $x \neq x'$, $x, x' \varepsilon A$, but $\mu(x) = \mu(x')$. Then $h_1(x)$ and $h_1(x')$ are both parallel to $\mathbf{f}_1(x_0,x_1)$ and both contain $\mu(x)$, a contradiction since G is affine. μ is onto: Suppose $w \varepsilon C$. Let $h_1(w)$ be the unique *1*-flat which contains w and is parallel to $\mathbf{f}_1(x_0,x_1)$. Then by **5.4**, $h_1(w) \cap f$ consists of a single point x. Then $h_1(x) = h_1(w)$, hence $\mu(x) = w$. μ is therefore 1-1 and onto.

μ is order-preserving (cf. **4.0**): Suppose $x < x'$, $x,x' \varepsilon A$. We suppose

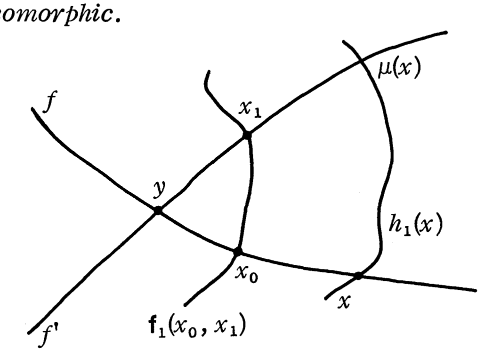


$y < w$ for all w in either A or C. If $x = y$, then since μ is 1-1, $\mu(x') \neq y$, hence $\{y,x,\mu(x)\}$ is linearly independent. Suppose $y < \mu(x') < \mu(x)$. Then by **2.27**, $\mu(x') \varepsilon \text{Int}\,\overline{y\mu(x)}$, hence by **3.7** and **3.21**, either $h_1(x') \cap \overline{yx} \neq \phi$, or $h_1(x') \cap \overline{x\mu(x)} \neq \phi$. If the latter is true, then $h_1(x)$ and $h_1(x')$ are both parallel to $\mathbf{f}_1(x_0,x_1)$ and have a point in common, contradicting G affine.

Since $x' \notin \overline{yx}$, if $h_1(x') \cap \overline{yx} \neq \phi$, then $h_1(x') = f$, hence $h_1(x')$ could not be parallel to $\mathbf{f}_1(x_0,w)$. Therefore μ is order-preserving.

Since μ is 1-1, onto, and preserves order, μ is a homeomorphism between A and C. An entirely similar argument shows that B and C are homeomorphic, hence A and B are homeomorphic.

Prop. **5.6**: *Any two 1-flats are homeomorphic.*



Proof: Let f and f' be arbitrary *1*-flats. If $f = f'$, **5.6** is trivially true. Assume $f \neq f'$, hence $m \geqslant 2$. Case 1: $f \cap f'$ consists of a single point y. Choose $x_0 \varepsilon f - \{y\}$ and x_1 $f' - \{y\}$. We define $\mu: f \rightarrow f'$ as follows: Set $\mu(x_0) = x_1$ and $\mu(y) = y$. For $x \varepsilon f - \{x_0,y\}$, let $h_1(x)$ be the unique *1*-flat which contains x and is parallel to $\mathbf{f}_1(x_0,x_1)$. f, f', and $h_1(x)$ are all contained in the *2*-flat $\mathbf{f}_2(\{y,x_0,x_1\})$, therefore $h_1(x) \cap f'$ consists of a single point by **5.4**. Let this point be $\mu(x)$. By arguments similar to those used in **5.5**, μ is 1-1, and onto. We may assume $y < x_1$, and $y < x_0$. Then the methods of **5.5** show that μ is also order-preserving, hence is a homeomorphism.

Case 2: $f \cap f' = \phi$. Choose $w \varepsilon f$ and $z \varepsilon f'$. Then f and f' are both homeomorphic to $\mathbf{f}_1(w,z)$, hence to each other.

5.7: We now coordinatize X. Fix $x_0 \varepsilon X$. Let $S = \{x_0, y_1, \ldots, y_m\}$ be a maximal linearly independent set which contains x_0. The $\mathbf{g}_1(x_0, y_i)$, $i = 1, \ldots, m$, are m distinct *1*-flats, no $k+1$ of which are contained in the same k-flat. Set $S_i = S - \{y_i\}$, $1 \leqslant i \leqslant m$. We define the i^{th} coordinate of a point $x \varepsilon X$, $1 \leqslant i \leqslant m$, as follows: Let $f^i_{m-1}(x)$ be the unique m-*1*-flat which contains x and is parallel to $\mathbf{f}_{m-1}(S_i)$. Since $\mathbf{g}_i(x_0, y_i) \nsubseteq \mathbf{f}_{m-1}(S_i)$, $\mathbf{g}_1(x_0, y_i) \cap \mathbf{f}_{m-1}(S_i) = \{x_0\}$, hence by **5.1.1**, $f^i_{m-1}(x) \cap \mathbf{g}_i(x_0,y_i)$ consists of a single point

x_i. Assign x_i as the i^{th} coordinate of x. The 1-flats $\mathbf{g}_1(x_0,y_i)$ are called the coordinate axes, the x_0 is called the origin. Each point in X is thus assigned an ordered set of m coordinates. The notation of this section (**5.7**) will be retained for the rest of this chapter.

Prop. **5.8:** $\dim\left(\bigcap_{i=1}^{k} f^{i}_{m-1}(x)\right) = \dim\left(\bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i)\right) = m-k$; *moreover,* $\bigcap_{i=1}^{k} f^{i}_{m-1}(x) \,\|\, \bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i)$.

Proof: $\dim \mathbf{f}_{m-1}(S_1) = m-1$. Suppose we have shown that $\dim\left(\bigcap_{i=1}^{k-1} \mathbf{f}_{m-1}(S_i)\right)$ $m-k+1$, $k-1 \geq 1$. $S \subseteq \mathbf{f}_{m-1}(S_k) \vee \left(\bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i)\right)$, hence $\dim\left(\mathbf{f}_{m-1}(S_k) \vee \left(\bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i)\right)\right) = m$. Applying **3.9**, we have $\dim \bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i) = m-k$.

Denote by $g^{i}_1(x)$ the unique 1-flat which contains x and is parallel to $\mathbf{g}_1(x_0,y_i)$, $1 \leq i \leq m$. We first show that $g^{i}_1(x) \subseteq f^{j}_{m-1}(x)$ for $j \neq i$.
Case 1: $x \varepsilon \mathbf{f}_{m-1}(S_j)$. By **1.9.1**, $\mathbf{g}_1(x_0,y_i) \subseteq \mathbf{f}_{m-1}(S_j)$ if $j \neq i$, hence a basis for $g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)$ is contained in $\mathbf{f}_{m-1}(S_j)$, hence $g^{i}_1(x) \subseteq g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i) \subseteq \mathbf{f}_{m-1}(S_j)$, if $j \neq i$.
Case 2: $x \not\varepsilon f_{m-1}(S_j)$. Then $\dim(g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)) = 2$, hence by **3.9**, $\dim((g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)) \cap f^{j}_{m-1}(x)) = 1$. *If* $j \neq i$, *and* $((g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)) \cap f^{j}_{m-1}(x)) \cap \mathbf{g}_1(x_0,y_i) \neq \phi$, then $f^{j}_{m-1}(x) \cap \mathbf{g}_1(x_0;y_i) \neq \phi$, hence since $\mathbf{g}_1(x_0,y_i) \subseteq \mathbf{f}_{m-1}(S_j)$ for $i \neq j$, $f^{j}_{m-1}(x) \cap \mathbf{f}_{m-1}(S_j) \neq \phi$. But $f^{j}_{m-1}(x) \,\|\, \mathbf{f}_{m-1}(S_j)$, hence $f^{j}_{m-1}(x) = \mathbf{f}_{m-1}(S_j)$, which implies $x \varepsilon \mathbf{f}_{m-1}(S_j)$, a contradiction. This gives $((g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)) \cap f^{j}_{m-1}(x)$ and $g^{i}_1(x)$ both containing x and parallel to $\mathbf{g}_1(x_0,y_i)$, hence $g^{i}_1(x) = ((g^{i}_1(x) \vee \mathbf{g}_1(x_0,y_i)) \cap f^{j}_{m-1}(x)$, hence $g^{i}_1(x) \subseteq f^{j}_{m-1}(x)$ for $i \neq j$.

For each $1 \leq i \leq m$, choose $y'_i \varepsilon g^{i}_1(x) - \{x\}$. If $S' = \{x, y'_1, \ldots, y'_m\}$ is not linearly independent, then $\bigcup_{j=1}^{k} g^{j}_1(x) \subseteq f^{i}_{m-1}(x)$ for some i. But $\mathbf{f}_{m-1}(S_i)$ intersects $\mathbf{g}_1(x_0,y_i)$ in a single point, hence by **5.1**, $\mathbf{f}_{m-1}(S_i)$ also intersects $g^{i}_1(x)$ is a single point, but this is impossible if $g^{i}_1(x) \subseteq f^{i}_{m-1}(x)$, which is parallel to $\mathbf{f}_{m-1}(S_i)$. Therefore S' is linearly independent. Setting $S'_i = S' - \{y'_i\}$, we see that S' is a basis for $f_{m-1}(x)$. An argument similar to that used to show $\dim \bigcap_{i=1}^{k} \mathbf{f}_{m-1}(S_i) = m-k$ also shows that $\dim \bigcap_{i=1}^{k} f^{i}_{m-1}(x) = m-k$.

$f_{m-1}(x) \,\|\, \mathbf{f}_{m-1}(S_1)$. Assume $\bigcap_{q=1}^{k-1} f^{q}_{m-1}(x) \,\|\, \bigcap_{q=1}^{k-1} \mathbf{f}_{m-1}(S_q)$. Suppose h is the unique m-k-flat which contains x and is parallel to $\bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)$.

Case 1: $h = \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)$. Then $x \varepsilon \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)$, hence $f_{m-1}(x) = \mathbf{f}_{m-1}(S_q)$, $1 \leq q \leq k$, hence $h = \bigcap_{q=1}^{k} f^{q}_{m-1}(x)$.

Case 2: $h \neq \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)$. Then $\dim (h \vee \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)) = m-k+1$, hence by **3.9**, $\dim (f_{m-1}(x) \cap (h \vee \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)) = m-k$. Since $f^{i}_{m-1}(x) \,||\, \mathbf{f}_{m-1}(S_i)$, $1 \leq i \leq m$, we have $f^{i}_{m-1}(x) \cap (h \vee \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)) \,||\, \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q)$, and hence $h = f^{i}_{m-1}(x) \cap (h \vee \bigcap_{q=1}^{k} \mathbf{f}_{m-1}(S_q))$. Therefore $h \subseteq f^{q}_{m-1}(x)$, $1 \leq q \leq k$, hence by **1.4**, $h = \bigcap_{q=1}^{k} f^{q}_{m-1}(x)$.

Cor. **5.8.1**: $\bigcap_{i=1}^{m} f_{m-1}(x) = \{x\}$, *and* $\bigcap_{i=1}^{m} \mathbf{f}_{m-1}(S_i) = \{x_0\}$.

Proof: Both these intersections are 0-flats which contain x and x_0 respectively.

While proving **5.8**, we also proved

Cor. **5.8.2**: $g^{i}_{1}(x) \subseteq f^{j}_{m-1}(x)$, $j \neq i$.

Prop. **5.9**: *If x has i^{th} coordinate x_i, then* $f^{i}_{m-1}(x) = \{w \varepsilon x \mid$ *the i^{th} coordinate of w is x_i*$\}$.

Proof: Set $T = \{w \varepsilon X \mid$ *the i^{th} coordinate of w is x_i*$\}$. Clearly $f^{i}_{m-1}(x) \subseteq T$ by the manner in which x_i was defined. Suppose $w \varepsilon T$. Since $x_i \varepsilon f^{i}_{m-1}(w) \cap f^{i}_{m-1}(x)$ and $f^{i}_{m-1}(w)$ and $f^{i}_{m-1}(x)$ are both parallel to $\mathbf{f}_{m-1}(S_i)$, $f^{i}_{m-1}(x) = f^{i}_{m-1}(w)$, hence $w \varepsilon f^{i}_{m-1}(x)$.

Cor. **5.9.1**: *Two distinct points cannot have the same coordinates.*

Proof: By **5.9**, all points which have the same coordinates as x are in $\bigcap_{i=1}^{m} f^{i}_{m-1}(x) = \{x\}$ (**5.8.1**).

Cor. **5.9.2**: $g^{i}_{1}(x)$, *the unique 1-flat which contains x and is parallel to* $\mathbf{g}^{i}_{1}(x_0,y_i)$ *consists of* $\{w \varepsilon X \mid j^{\text{th}}$ *coordinates of w is the j^{th} coordinate of x, $i \neq j$*$\}$.

Proof: By **5.8.2**, $g^{i}_{1}(x) \subseteq f_{m-1}(x)$, $i \neq j$, hence $g^{i}_{1}(x) = \bigcap_{\substack{j=1 \\ i \neq j}}^{m} f_{m-1}(x)$. **5.9.2** then follows at once from **5.9**.

Cor. **5.9.3**: *Given* $\{x_1, \ldots, x_m\}$ *with* $x_i \varepsilon \mathbf{g}_1(x_0,y_i)$, $1 \leq i \leq m$, *there is some point x whose coordinates are* $x_1, \ldots, x_m$.

Proof: x_1 has first coordinate x_1. Suppose we have found z having coordinates $x_1, \ldots, x_{k-1}, w_k, \ldots, w_m$. If $w_k = x_k$, we are done. If not, we have $g_1^k(z)$, the unique 1-flat which contains z and is parallel $g_1(x_0, y_k)$. By **5.1**, $g_1^k(z) \cap f_{m-1}^k(x_k)$ consists of a single point. By **5.9** and **2.9.2**, the point has the first k coordinates of x.

5.10: $(x_1, \ldots, x_m)$ shall denote the point having coordinates $x_1, \ldots, x_m$. Define $\gamma: X \to \bigtimes_{i=1}^{m} g_1(x_0, y_i)$ by $\gamma(x) = (x_1, \ldots, x_m)$ where $x_1, \ldots, x_m$ are the coordinates of x. By **5.9.1** and **5.9.3**, γ is 1-1 and onto. We now show that γ is a homeomorphism.

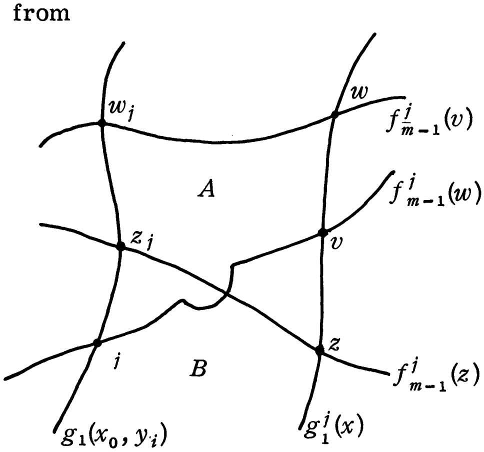
Definition **2**: *Let $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$ be points of X such that $a_i \leqslant b_i$, $1 \leqslant i \leqslant m$. Define* $I(a,b) = \{x \varepsilon X | x = (x_1, \ldots, x_m),\ a_i < x_i < b_i \text{ if } a_i \neq b_i;\ x_i = a_i \text{ if } a_i = b_i\}$;

or equivalently by **2.27**,

$I(a,b) = \{x \varepsilon X | x = (x_1, \ldots, x_m),\ x_i \varepsilon \operatorname{Int}\overline{a_i b_i} \text{ if } a_i \neq b_i;\ x_i = a_i \text{ if } a_i = b_i\}$.

$I(a,b)$ is said to be k-degenerate if $a_i = b_i$ for k coordinates of a and b. Suppose $I(a,b)$ is k-degenerate, but non-degenerate in the j^{th} coordinate, i.e. $a_j \neq b_j$. We form a j-cross section of $I(a,b)$, denoted by $K_j I(a,b)$, as follows: Choose $c_j \varepsilon \operatorname{Int}\overline{a_j b_j}$. Then $K_j I(a,b) = I(a', b')$, where $a' = (a_1, \ldots, a_{j-1}, c_j, a_{j+1}, \ldots, a_m)$ and $b' = (b_1, \ldots, b_{j-1}, c_j, b_{j+1}, \ldots, b_m)$. $K_j I(a,b)$ is $k+1$-degenerate. $K_j I(a,b)$ is not unique, but depends on the choice of c_j. $I(a,b)$ is called a k-degenerate open box.

5.11: Let $I(a,b)$ and $K_j I(a,b)$ (with c_j fixed) be as given in definition **2**. Then $I(a,b) = \{x = (z_1, \ldots, z_{j-1}, w_j, z_{j+1}, \ldots, z_m) \varepsilon X | (z_1, \ldots, z_{j-1}, c_j, \ldots, z_m) \varepsilon K_j I(a,b),\ a_j < w_j < b_j\}$. *Let* $x = (z_1, \ldots, z_{j-1}, c_j, \ldots, z_m) \varepsilon K_j I(a,b)$. If we hold the z_i fixed and allow the j^{th} coordinate to vary over all points in $\operatorname{Int}\overline{a_j b_j}$, by **5.9.2** we obtain a subset T_x of $g_1^j(x)$, the unique 1-flat which contains x and is parallel to $g_1(x_0, y_j)$. It is clear that i) $I(a,b) = \bigcup_{x \varepsilon K_j I(a,b)} T_x$, and ii) $T_x \cap T_{x'} = \phi$, if $x \neq x'$. We now show that $T_x = \operatorname{Int}\overline{(z_1, \ldots, z_{j-1}, a_j, z_{j+1}, \ldots, z_m)(z_1, \ldots, z_{j-1}, b_j, z_{j+1}, \ldots, z_m)}$. This follows at once from



Prop. **5.12:** *Let w,v,z be distinct points of $g_1(x)$, and let w_j, v_j, and z_j be their respective j^{th} coordinates. Then $v \varepsilon \overline{wz}$ iff $v_j \varepsilon \overline{w_j z_j}$.*

Proof: Suppose $v \varepsilon \operatorname{Int}\overline{wz}$, but $v_j \not\varepsilon \overline{w_j z_j}$. By **3.26.1**, $f^j_{m-1}(v)$ disconnects X such that w and x are in different components of $X - f^j_{m-1}(v)$. Since w,v, and z are distinct, we have $f^j_{m-1}(w) \cap f^j_{m-1}(v) = \phi$ and $f^j_{m-1}(z) \cap f^j_{m-1}(v) =$

ϕ. Since $\overline{w_j w} \subseteq f^j_{m-1}(w)$ and $\overline{z_j z} \subseteq f^j_{m-1}(z)$ (**2.9f**)) and $\overline{w_j z_j} \cap f^j_{m-1}(v) = \phi$, it follows that $\overline{w_j w} \cup \overline{w_j z_j} \cup \overline{z_j z}) \cap f^j_{m-1}(v) = \phi$, but then $(\overline{w_j w} \cup \overline{w_j z} \cup \overline{z_j z})$ - $f^j_{m-1}(v)$ is connected, thus w and z are in the same component of X-$f^j_{m-1}(v)$. a contradiction. If $v_j \varepsilon \operatorname{Int}\overline{w_j z_j}$, an entirely similar argument yields $v \varepsilon \overline{wz}$.

Prop. **5.13**: *Any open box* I(a,b) *is convex, and if* I(a,b) *is 0-degenerate, then* I(a,b) *is also open.*

Proof: Suppose $w \varepsilon \mathbf{g}_1(x_0, y_i)$. Then $f^i_{m-1}(w)$ disconnects X into open, convex subsets $A(w)$ and $B(w)$ (**5.2**, **4.4.1**, and **3.25**). By **2.22** (proof) and **2.23**, w disconnects $\mathbf{g}_1(x_0, y_i)$ into connected sets α and β. We may assume $\alpha \subseteq A(w)$, $\beta \subseteq B(w)$. We may also suppose without loss of generality that for some $u_0 \varepsilon \alpha$, $u_0 > w$. If there were $u \varepsilon \alpha$ such that $u < w$, then $w \varepsilon \operatorname{Int}\overline{u_0 u}$, but since by **2.2.1**, α is convex, $\overline{u_0 u} \subseteq \alpha$ by **2.9f**), hence $w \varepsilon \operatorname{Int}\overline{u_0 u}$. Then $\alpha = \{u \varepsilon \mathbf{g}_1(x_0, y_i) | u > w\}$; and, similarly, $\beta = \{v \varepsilon \mathbf{g}_1(x_0, y_i) | v < w\}$. Suppose $x = (x_1, \ldots, x_m) \varepsilon A(w)$. Since G is affine and $f^i_{m-1}(x_i) = f^i_{m-1}(x)$, $f^i_{m-1}(x) \cap f^i_{m-1}(w) = \phi$. Therefore if $v \varepsilon f^i_{m-1}(x)$, then $\overline{xv} \subseteq f^i_{m-1}(x)$-$f^i_{m-1}(w)$, hence $v \varepsilon A(w)$. Therefore $f^i_{m-1}(x) \subseteq A(w)$, hence $x_i \varepsilon \alpha$, hence $x_i > w$. Suppose $x = (x_1, \ldots, x_m) \varepsilon X$ and $x_i > w$. Then again $f^i_{m-1}(x) = f^i_{m-1}(x_i) \subseteq A(w)$ since $w \varepsilon \alpha \subseteq A(w)$, hence $x \varepsilon A(w)$. We have thus shown that $A(w) = \{x = (x_1, \ldots, x_m) \varepsilon X | x_i > w\}$. Similarly, we may show that $B(w) = \{x = (x_1, \ldots, x_m) \varepsilon X | x_i < w\}$.

$$\text{Since } a_j \leqslant b_j,\ 1 \leqslant j \leqslant m,\ \mathrm{I}(a,b) = \bigcap_{a_i = b_i} [A(a_i) \cap B(b_i)] \cap \bigcap_{a_k = b_k} f_{m-1}(a_k).$$

This is convex since each set of intersection is convex, and is open if $a_j \neq b_j$ for $1 \leqslant j \leqslant m$.

Prop. **5.14**: *Set* $\mathscr{B}$ = $\{$I$(a,b)|$I(a,b) *is 0-degenerate*$\}$. *Then* $\mathscr{B}$ *is the basis for a topology on X.*

Proof: Elementary considerations show that the intersection of the interiors of two segments contained in the same *1*-flat is either ϕ, or the interior of a segment. Using this fact that the definition of I(a,b), it is clear that the intersection of any two *0*-degenerate open boxes is either empty, or a *0*-degenerate open box. Suppose $x = (x_1, \ldots, x_m) \varepsilon X$. By **5.2**, x_i disconnects $g_1(x_0, y_i)$, $1 \leqslant i \leqslant m$, hence by **2.16**, $x_i \varepsilon \operatorname{Int}\overleftrightarrow{a_i b_i}$ for some $a_i, b_i \varepsilon \mathbf{g}_1(x_0, y_i)$, $a_i < b_i$, $1 \leqslant i \leqslant m$. Then $x \varepsilon$I(a,b) where $a = (a_1, \ldots, a_m)$ and $b = (b_1, \ldots, b_m)$.

Prop. **5.15**: *Let* $x = (x_1, \ldots, x_m) \varepsilon X$, *and let U be any open, convex neighborhood of x. Then there is a 0-degenerate open box* I(a,b) *such that* $x \varepsilon$I$(a,b) \subseteq U$.

Proof: **5.15** is true for $m = 1$ by **5.2** and **4.2**. Assume **5.15** is true for $m - 1 \leqslant 1$. $f^1_{m-1}(x) \cap U$ is a convex, open neighborhood of x in $f^1_{m-1}(x)$; therefore, by **5.3** and the induction assumption, there is an open box I$(a',b') \subseteq f_{m-1}(x) \cap U$, $a' = (x_1, a_2, \ldots, a_m)$, $b' = (x_1, b_2, \ldots, b_m)$ such that $x \varepsilon$I(a',b'), and I(a',b') is degenerate only in the first coordinate by **5.9**.

Let $\{a^\lambda_1\}$, $\{b^\lambda_1\}$, $\lambda \varepsilon \Lambda$ be nets of points of $\mathbf{g}_1(x_0, y_1)$ such i) the ordering $\textcircled{>}$ of Λ is total; ii) $a^\lambda_1 < x_1 < b^\lambda_1$ for all $\lambda \varepsilon \Lambda$, hence by **2.27**, $x_1 \varepsilon \operatorname{Int} a^\lambda_1 b^\lambda_1$, for all λ; iii) $a_1^\lambda \to x_1$, $b_1^\lambda \to x_1$; and iv) $\lambda_1 \bigcirc\!\!\!\!> \lambda_2$ implies $\overline{a_1^{\lambda_1} b_1^{\lambda_1}}$ is properly contained in $a_1^{\lambda_2} b_2^{\lambda_2}$. We may use **5.5** to verify that such a net exists. For each $\lambda \varepsilon \Lambda$ set $a^\lambda = (a_1^\lambda, a_2, \ldots, a_m)$ and $b^\lambda = (b_1^\lambda, b_2, \ldots, b_m)$. Then for each $\lambda \varepsilon \Lambda$ we have the 0-degenerate open box $I(a^\lambda, b^\lambda)$; moreover, $I(a', b')$ is a 1-cross section of $I(a^\lambda, b^\lambda)$ for each λ. For each $z = (x_1, z_2, \ldots, z_m) \varepsilon I(a', b')$ and each $\lambda \varepsilon \Lambda$, set $T_z(\lambda) = \text{Int}\overline{(a_1^\lambda, z_2, \ldots, z_m)(b_1^\lambda, z_2, \ldots, z_m)}$.

Then $I(a^\lambda, b^\lambda) = \bigcup_{z \varepsilon I(a', b')} T_z(\lambda)$ by **5.11** and **5.12.** Because of iv) and **5.12,** we also have $\lambda_1 \bigcirc\!\!\!\!> \lambda_2$ implying that $T_z(\lambda)$ is properly contained in $T_z(\lambda_2)$. It follows at once that if $T_z(\lambda_0) \subseteq U$ then $\lambda \bigcirc\!\!\!\!> \lambda_0$ implies $T_z(\lambda) \subseteq U$. If $I(a^\lambda, b^\lambda) \subseteq U$ for all λ, then for each $\lambda \varepsilon \Lambda$, we can find $u_\lambda \varepsilon I(a^\lambda, b^\lambda) - U$. By the remarks at the end of the preceding paragraph, it follows that for some $z' = (x_1, x_2', \ldots, z_m') \varepsilon I(a', b')$, for each $\lambda \varepsilon \Lambda$, u_λ may be selected from $T_z(\lambda)$. Assume this has been done.

Let V be any open neighborhood of z_0. Let $g_1^1(z')$ be the unique 1-flat which contains z' and is parallel to $\mathbf{g}_1^1(x_0, y_1)$. Then $T_{z'}(\lambda) \subseteq g_1^1(z')$ for all $\lambda \varepsilon \Lambda$ (cf. **5.11**). $V \cap g_1^1(z')$ is an open neighborhood of z' in $g_1^1(z')$ hence by **4.2** and **5.9.2,** there are $u = (c_1, z_2, \ldots, z_{m'})$, $v = (d_1, z_2, \ldots, z_m) \varepsilon g_1^1(z')$ such that $\text{Int}\overline{uv} \supseteq V$. By iii), there are $a^{\lambda'}, b^{\lambda'}$ such that $\text{Int}\overline{a^{\lambda'} b^{\lambda'}} \supseteq \text{Int}\overline{c_1 d_1}$. Therefore by **5.12,** $T_{z'}(\lambda') \subseteq \text{Int}\overline{uv}$, hence $u_{\lambda'} \varepsilon V$. No u_λ is in U, but at least a subnet of $\{u_\lambda\}_{\lambda \varepsilon \Lambda}$ converges to $z' \varepsilon U$, hence $z' \varepsilon X - U$, a contradiction since U is open. We have therefore shown that $x \varepsilon I(a^\lambda, b^\lambda) \subseteq U$ for some $\lambda \varepsilon \Lambda$.

Cor. **5.15.1:** The *topology for which* $\mathcal{B}$ *of* **5.14** *is a basis is equivalent to the original topology on X.*

Cor. **5.15.2:** $\gamma: X \to \bigtimes_{i=1}^{m} \mathbf{g}_1(x_0, y_i)$ *(cf.* **5.10***) is a homeomorphism.*

Proof: γ is 1-1, onto, and takes a basis for X into a basis for $\bigtimes_{i=1}^{m} \mathbf{g}_1(x_0, y_i)$.

Cor. **5.15.3:** *X is homeomorphic to* Γ^m *where* Γ *is any topological space homeomorphic to some 1-flat in X.*

Proof: By **5.6,** all 1-flats in X are homeomorphic to Γ, hence $\bigtimes_{i=1}^{m} \mathbf{g}_i(x_0, y_i)$ is homeomorphic to Γ^m, hence by **5.15.2,** X is homemorphic to Γ^m.

Cor. **5.15.4:** *If* Γ *is as in* **5.15.3** *and f is any k-flat of X, then f is homeomorphic to* Γ^k, *where we set* $\Gamma^{-1} = \phi$ *and* $\Gamma^0 = \{x\}$.

Proof: **5.3** and **5.15.3** together with g is a 1-flat of G_f iff $g \subseteq f$ and g is a 1-flat of G.

Cor. **5.15.5:** *If some 1-flat is homeomorphic to R, the real line, then X is homeomorphic to* $\mathbf{R}^m$ *and any k-flat is homeomorphic to* $\mathbf{R}^k$.

Proof: **5.15.3** and **5.15.4.**

Cor. **5.15.6**: *If X is second countable, then X is homeomorphic to* $\mathbf{R}^m$.

Proof: If f is a *1*-flat of X, then f is second countable as a subspace. By **5.2** and **4.8**, then f is homeomorphic to **R**. **5.15.6** then follows from **5.15.5**.

Cor. **5.15.7**: $\mathbf{R}^m$ *is the only second countable topological space which admits a geometry G such that the space and G form an affine m-arrangement.*

Proof: $\mathbf{R}^m$ with the usual Euclidean geometry forms an affine m-arrangement, and $\mathbf{R}^m$ is second countable. This together with **5.15.6** gives **5.15.7**.

APPENDIX

This appendix contains material of the following types: 1) the notation used in this paper for elementary topological concepts when no universally accepted notation exists; 2) any lemmas needed in this paper for which no references can be cited; 3) some concepts and terminology used in this paper which may not be familiar to the reader because of their limited use elsewhere. The appendix consists of two sections: A. Topological; and B. Lattice Theoretic.

A. Topological. ϕ shall always denote the empty set. Let X be a topological space. Suppose $A \subseteq X$. The interior of A, i.e. the union of all open sets contained in A, is denoted by A^0. The closure of A, i.e. the intersection of all closed subsets of X which contain A, is denoted by $\mathrm{Cl}A$. The frontier of A, i.e. $\mathrm{Cl}A\text{-}A^0$, is denoted by $\mathrm{Fr}A$. $A \subseteq X$ is said to be countably compact if every infinite subset of A has a limit point in A. X is called locally peripherally countably compact if given any $x \varepsilon X$ and any neighborhood U of x, then there exists a neighborhood V of x such that $U \subseteq V$ and $\mathrm{Fr}\, V$ is a closed, countably compact subset of X.

Suppose X is connected. Then a subset A of X is said to disconnect X if $X\text{-}A$ is not connected. A point $x \varepsilon X$ is a cut point of X if $\{x\}$ disconnects X; otherwise, x is called a non-cut point. $A \subseteq X$ is a minimal disconnecting subset of X if A disconnects X, but no proper subset of A disconnects X. If $A \subseteq X$ such that no proper connected subset of X contains A, then X is said to be irreducibly connected to A.

The following lemma is necessary for certain propositions in this paper. We assume that X is a connected topological space.

Lemma 1: Suppose A and B are subsets of X such that i) A is not a subset of B, nor B of A; ii) A and B are both closed; and iii) $X = A \cup B$. Then $A \cap B = C$ disconnects X.

Proof: $A\text{-}C = X\text{-}B$ is open, as is $B\text{-}C$. Because of i), $A\text{-}C$ and $B\text{-}C$ are non-empty, but $(A\text{-}C) \cap (B\text{-}C) = \phi$.

B. Lattice Theoretic. Let S be a set with partial ordering $\leqslant$, i.e. i) $\leqslant$ is reflexive; ii) $\leqslant$ is antisymmetric; and iii) $\leqslant$ is transitive. $C \subseteq S$ is called a chain if for every pair a,b of elements of C, either $a \leq b$, or $b \leq a$. If $s,t \varepsilon S$, and $s \leq t$, then the interval $[s,t]$ is defined as $\{x \varepsilon S | s \leq x \leq t\}$. If $[s,t] = \{x,t\}$, then we say that t covers s.

If $A \subseteq S$, then $b \varepsilon S$ is called an upper bound (u.b.) of A if $a \leq b$ for all a in A. b is a least upper bound (l.u.b.) of A if b is an upper bound of A, and if b' is any other upper bound of A, then $b \leq b'$. Correspondingly, we may define what is meant by a lower bound and greatest lower bound (g.l.b.) for A. S (with partial ordering $\leq$) is called a lattice of every two element subset of S has a l.u.b. and a g.l.b. For each pair $s,t \varepsilon S$, define $s \vee t =$ l.u.b. $\{s,t\}$ and $s \wedge t =$ g.l.b. $\{s,t\}$. We assume henceforth that S is a lattice. If $A \subseteq S$, then A is a sublattice of S if A is closed under the operations $\vee$ and $\wedge$ as defined in S. S is said to be complete if given any subset B of S, then l.u.b. B and g.l.b. B exist.

Suppose S is complete, and set $0 =$ g.l.b. S. For each $s \varepsilon S$, set $K\{s\} = \{C \subseteq S | C$ *is a chain of distinct elements of* S *with* $C = \{a_0 = 0, a_1, \ldots, a_n = s\}, n < \infty, a_i \leq a_{i+1}\}$. For any $C \varepsilon K(s)$, set $d(C) = n$. Define $d(s) = \underset{K(s)}{\text{l.u.b.}}\ d(C)$. S is said to be finite dimensional if $d(s) < \infty$ for all $s \varepsilon S$.

S is called upper semi-modular if s and t covering $s \wedge t$ implies $s \vee t$ covers s and t for all s,t in S. S is called lower semi-modular if $s \vee t$ covering s and t implies s and t cover $s \wedge t$ for all s,t in S. S is said to be modular if $s,t \varepsilon S$ and $s \leq t$ implies $s \vee (t \wedge x) = t \wedge (s \vee x)$ for all x in S. S is modular iff it is upper and lower semi-modular. If S is finite dimensional and modular, then for any $s,t \varepsilon S$, $d(s) + d(t) = d(s \vee t) + d(s \wedge t)$. If S is finite dimensional and upper semi-modular, then $d(s) + d(t) \geq d(s \vee t) + d(s \wedge t)$.

For a more complete discussion of lattices and the proofs of propositions stated here, the reader is referred to Garrett Birkhoff, *Lattice Theory*, American Mathematical Society (Colloquium Publications, Vol. 25), New York, 1948.

EPILOGUE

Given below are selected topics for possible future research in the theory developed in this paper.

1. Independence of the axioms **3.1-3.9** (cf. chapter III, definition **1**).
2. Necessary and sufficient conditions for a given space X to admit a geometry G such that X and G form an m-arrangement.
3. A characterization of $\mathbf{S}^m$ in terms of the usual Riemannian geometry.
4. Homology and homotopy theory for m-arrangements (cf. remark following **3.19**).
5. Let $\gamma: X \to Y$ be 1-1 and onto. Then if G is a geometry on X, then defining f' to be a k-flat in Y iff $f' = \gamma(f)$ where f is a k-flat in X, we obtain a geometry, denoted by $\gamma(G)$, on Y. Two geometries G and G' on X are said to be homeomorphically equivalent if there is a homeomorphism γ of X such that $G' = \gamma(G)$. Are all affine geometries on $\mathbf{R}^m$ which satisfy **3.1-3.9**

homeomorphically equivalent? The entire question of homeomorphically equivalent geometries seems to provide a rich and very difficult area for future study.

6. What axioms in additions to **3.1-3.9** would allow us to characterize Euclidean geometry on $\mathbf{R}^m$?

7. It is known (though not proved in this paper) that every second countable space which can serve as the space of an open m-arrangement is an m-manifold Is it true that any second countable space which can serve as the space for a closed m-arrangement, i.e. an m-arrangement in which every 1-flat has two non-cut points, is an m-manifold with boundary? Is every space of a closed m-arrangement compact? Is BdX for the space of a closed m-arrangement compact, and also connected if $m \geqslant 2$?

8. A semi-projective geometry of length n on a set X is defined to be projective if $F^0 = \{\{x\} | x \varepsilon X\}$ and spherical if each 0-flat consists precisely of two points. A discussion of projective, spherical, affine, and semi-projective geometries and their relations to one another can either follow classical lines, or branch into subjects hitherto unexplored, such as finite spherical geometries. We note here the following known propositions:

a) If G is semi-projective, the G_{X-f} *is affine where f is any n-flat.*

b) If G is affine and satisfies **3.1** *and* **3.9**, *then all 1-flats of G have the same cardinality and X can be coordinatized as in* **5.7**. *In particular, if each 1-flat contains q points, then* card $X = q^n$.

9. In definition **1**, chapter III, replace **3.1** and **3.5** by

3.1': *Each 0-flat consists of a finite set of n points.*

3.5': *If* $S = \{x,y,z\}$ *is such that S is contained in some 1-flat but no pair of distinct points of S is in the same 0-flat, then* $\overline{xy} \cup \overline{yz} = \overline{xy}$, $\overline{yz}$ *and/or* $\overline{xz}$.

and add

*: *Every linearly independent set has a convex hull.*

The structure thus determined has yet to be investigated; in particular, the following conjecture is believed to be true: For $x \varepsilon X$, let $M(x)$ be any maximal convex set which contains x. Then there are $x_1, \ldots, x_n \varepsilon X$ and $M(x_1), \ldots, M(x_n)$ such that $X = \bigcup_{i=1} M(x_i)$ and $M(x_i) \cap M(x_j) = \phi$ for $i \neq j$.

10. Assume that X has geometry G such that X and G form an m-arrangement. Define $D_k(X) \subseteq X^k$ *by* $D_k(X) = \{(x_1, \ldots, x_k) \varepsilon X^k | \{x_1, \ldots, x_k\}$ *is linearly dependent in* $X\}$. Let $\{f_\nu\}_{\nu \varepsilon N}$ be a net of flats of G. Define $\overline{\lim} f_\nu = \{x | x$ *is a limit point for some* $\{x_\nu\}_{\nu \varepsilon N}, x_\nu \varepsilon f_\nu\}$, $\underline{\lim} f_\nu = \{x |$ *there is* $\{x_\nu\}_{\nu \varepsilon N}, x_\nu \varepsilon f_\nu, x_\nu \to x\}$. We say that $f_\nu \to f$ if $\overline{\lim} f_\nu = \underline{\lim} f_\nu = f$. *If* $w = (w_1, \ldots, w_k) \varepsilon X^k$, let w^* denote the set $\{w_1, \ldots, w_k\} \subseteq X$. A set $Q \subseteq X$ is said to have a k-1-tangent h at $y \varepsilon Q$ if for each net $\{w_\nu\} \subseteq D_k\{x\}$ such that $w_\nu \to (y, \ldots, y)$ *in* X^k, $\mathfrak{f}_k(w_\nu^*) \to h$. Using these concepts, we may be able to develop a geometric theory of differential calculus.

Note: Problems 3 and 7 have been solved. The author has made his findings the subject of future papers.

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