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SYSTEMS CLASSICALLY AXIOMATIZED AND PROPERLY CONTAINED IN LEWIS'S S3

JOHN THOMAS CANTY

It is well known that Lewis's modal systems S3, S4 and S5 can be classically axiomatized. That is, an axiomatic for those systems can be given with a finite number of axioms taking substitution for propositional variables and material detachment as the only primitive rules of inference. It will be shown in this paper that such an axiomatic is available for some systems properly contained in S3. Each section of the paper introduces new axiomatics for sub-systems of S3 and then gives new sub-systems which are classically axiomatized and in which all of Lewis's primitive rules of inference are derivable. The symbolism throughout is that of [7] and "α is a thesis" is abbreviated as "$\vdash\alpha$".

1. Lemmon in [4] gave new foundations for Lewis's systems S1-S3 of [5] analogous to a systematic for T of Feys-von Wright [2, 8, 12] due to Gödel in [3]. In this section new foundations for Lemmon's systems are described and two systems containing Lemmon's S0.5 and properly contained in S3 are classically axiomatized.

The Lemmon systems are N-C-L calculi with K and E defined in the usual way by C and N, $\mathfrak{C}$ defined as LC and $\mathfrak{E}pq$ (strict equivalence) as $K\mathfrak{C}pq\mathfrak{C}qp$. Propositional calculus (PC) is given by three rules:

(PCa) if α is a tautology, then $\vdash\alpha$;
(PCb) substitution for propositional variables;
(PCc) material detachment (that is, from α and $C\alpha\beta$ infer β);

and Lewis's systems are based on selections from the following rules and axioms:

(a) $\vdash\alpha$ only if $\vdash L\alpha$;
(a'') α is a tautology only if $\vdash L\alpha$;
(b) $\vdash LC\alpha\beta$ only if $\vdash LCL\alpha L\beta$;
(b'') $\vdash\mathfrak{E}\alpha\beta$ only if $\vdash\mathfrak{E}L\alpha L\beta$;
(1) $CLCpqLCLpLq$;
(2) $CLpp$;
(a') α is a tautology or axiom only if $\vdash L\alpha$;
(b') substitutability of strict equivalents;
(1') $CLCpqCLpLq$;
(3) $CKLCpqLCqrLCpr$.

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Lemmon's foundations for the Lewis systems, S1-S3, are then given as:

S1 = {PC; (a'); (b'); (3); (2)}
S2 = {PC; (a'); (b); (1'); (2)}
S3 = {PC; (a'); (1); (2)}

the system of Feys-von Wright as:

T = {PC; (a); (1'); (2)}

and a system introduced by Lemmon as:

S0.5 = {PC; (a"); (1'); (2)}.

Now from the following list of axioms;

A1. *LCCNppp*
A2. *LCCpqCCqrCpr*
A3. *LCpCNpq*
A4. *CLCpqCLpLq*
A5. *CLpp*
A1'. *LCLCNppp*
A2'. *LCLCpqCLCqrLCpr*
A3'. *LCLpLCNpq*
A4'. *LCLCpqLCLpLq*
A6. *CLCpqCLCqpLCLpLq*

the axiomatics for the above systems are taken as:

M0 = {PCb; PCc; *A1*; *A2*; *A3*; *A4*; *A5*}
M1 = {M0; (b'); *A1'*; *A2'*}
M2 = {M0; (b'); *A1'*; *A3'*}
M3 = {M0; *A1'*; *A4'*}
M = {M0; (a)}.

The adequacy of the revisions can be seen by observing first that all of Lemmon's systems contain M0; (i) *A1*-*A3* follow from PC and either (a); (a'), or (a"), and (ii) *A4* either follows from (1) and (2) by PC, or in S1 is provable from (3):

1.	*CKLCpqLCqrLCpr*	[(3)]
2.	*CLCpqCLCqrLCpr*	[1, PC]
3.	*CLCNqNpCLCNppLCNqp*	[2, PC]
4.	𝔊 *CpqCNpNq*	[PC, (a')]
5.	𝔊 *pCNpp*	[PC, (a')]
6.	𝔊 *CNpqCNqp*	[PC, (a')]
7.	*CLCpqCLpLCNpq*	[3, 4, 5, 6, (b')]
8.	*CLCNqNpCLCNpqLCNqq*	[2, PC]
9.	*CLCpqCLCNpqLq*	[8, 4, 5, (b')]
10.	*CLCpqCLpLq*	[7, 9, PC]

Thus, S0.5 contains M0 and T contains M.

Secondly, (i) each of the remaining systems, S1-S3, contain (b')(cf. p. 178 of [4]), and thus contain *A1'* by (2) and (a'); while (ii) S1 contains *A2'* by (3) and (a'), S2 contains *A3'* by *A3* and (b), and S3 contains *A4'* by (1) and (a'). Thus S1-S3 contain M1-M3 respectively.

Conversely, it can be shown that M0-M3, and M contain S0.5, S1-S3, and T. To this end the following theorems of M0 are established.

Theorem 1. *If* $\vdash\alpha$ *in* **PC**, *then* $\vdash L\alpha$.

Proof. If α is an axiom of **PC**, then $L\alpha$ is given by *A1-A3* since *CCNppp*, *CCpqCCqrCpr*, and *CpCNpq* form a set of axioms for **PC** (cf. Appendix of [7]), and if α is derived from the axioms of **PC**, then $L\alpha$ is given by:

1. $L\beta$ [Induction hypothesis]
2. $LC\beta\alpha$ [Induction hypothesis]
3. $CL\beta L\alpha$ [2,*A4*]
4. $L\alpha$ [1, 3]

Theorem 2. *If* $\vdash LC\alpha\beta$ *and* $\vdash LC\alpha\gamma$, *then* $\vdash LC\alpha K\beta\gamma$.

Proof. From hypothesis by Theorem 1 and *A4.*

Theorem 3. *If* $\vdash LC\alpha\beta$ *and* $\vdash LC\beta\gamma$, *then* $\vdash LC\alpha\gamma$.

Proof. From hypotheses by Theorem 1 and *A4.*

Theorem 4. *If* $\vdash\alpha$ *and* $\vdash\beta$, *then* $\vdash K\alpha\beta$.

Proof. From hypotheses by Theorem 1 and *A5.*

Theorem 5. *If* $\vdash\alpha$ *and* $\vdash LC\alpha\beta$, *then* $\vdash\beta$.

Proof. From hypotheses by *A5.*

Hence, **PC** is contained in each of M0-M3, and M by Theorem 1 and *A5,* and thus M0 contains S0.5 and M contains T.

Further, M3 contains (b'). The proof is given by showing that (i) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}L\alpha L\beta$, (ii) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}N\alpha N\beta$, (iii) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}C\alpha\gamma\, C\beta\gamma$, and (iv) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\mathfrak{E}C\gamma\alpha\, C\gamma\beta$. Now (i) follows by *A4'* and (ii-iv) follow from *CLCpqLCNqNp*, *CLCpqLCCqrCpr*, and *CLCpqLCCrpCrq*, each of which are obtained by Theorem 1 and *A4.*

Moreover M2 contains (b):

1. $LC\alpha\beta$ [Hypothesis]
2. $LC\alpha\alpha$ [Theorem 1]
3. $LC\alpha K\alpha\beta$ [1, 2, Theorem 2]
4. $LCK\alpha\beta\alpha$ [Theorem 1]
5. $LCL\alpha L\alpha$ [Theorem 1]
6. $LCL\alpha LK\alpha\beta$ [3, 4, 5, (b')]
7. $LCLK\alpha\beta LCNK\alpha\beta KN\alpha\beta$ [*A3'*]
8. $LCLK\alpha\beta L\beta$ [7, Theorem 1, (b')]
9. $LCL\alpha L\beta$ [6, 8, Theorem 3]

And by Theorem 1, in order to show that M1-M3 contain (a') it is sufficient to remark that (i) each system contains *LCLpp* by *A1'* and (b'), (ii) M1 contains *LCKLCpqLCqrLCpr* by *A2'* and (b'), (iii) M2 contains *LCLCpqCLpLq* by *A2'*, (b') and Theorem 3, and (iv) M3 contains *LCLCpqLCLpLq* as *A4'*.

Thus, to complete the proof that M1-M3 contain S1-S3 it need only be shown that each of the axioms of S1-S3 which is not an axiom of M1-M3 are provable in M1-M3. In M1, (3) follows from *LCKLCpqLCqrLCpr* by *A5*. And in M3, (1) follows from *A4'* by *A5*.

On the basis of the new foundations for Lewis's systems it is now possible to classically axiomatize some systems containing S0.5 and properly contained in S3. The systems to be considered are:

R1 = {M0; *A6*; *A1'*; *A2'*}
R2 = {M0; *A6*; *A1'*; *A3'*}
R3 = {M3}.

Bull in [1] uses an equivalence relation employing schemata analogous to *A6* as Lemma 1 part III (p. 212) while for R1-R3, since *A6* yields (b''), the proof that each of the systems contains (b') can be established as was the proof above that M3 contains (b'). And thus:

R1 = {S1; *A6*}
R2 = {S2; *A6*}
R3 = {S3}.

Hence R1-R3 obviously contain S0.5. But R1 and R2 are properly contained in S3 as is shown by a variation of Parry's matrix of [6]: a regular expansion of the *C-N* matrix (as are all matrices considered in this paper) to eight values:

C	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*N*
1	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*8*
2	*1*	*1*	*3*	*3*	*5*	*5*	*7*	*7*	*7*
3	*1*	*2*	*1*	*2*	*5*	*6*	*5*	*6*	*6*
4	*1*	*1*	*1*	*1*	*5*	*5*	*5*	*5*	*5*
5	*1*	*2*	*3*	*4*	*1*	*2*	*3*	*4*	*4*
6	*1*	*2*	*3*	*3*	*1*	*2*	*3*	*3*	*3*
7	*1*	*2*	*1*	*2*	*1*	*2*	*1*	*2*	*2*
8	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*

with *L(*1*2*3*45678)* = *(26888888)*, where **n* indicates a designated value. For with this matrix all the axioms and rules of S2 are designated together with *A6* while *A4'(p/1,q/2)* = *LCLC12LCL1L2* = *LCL2LC26* = *LC6L5* = *LC68* = *L3* = *8*.

It may be noted that the independence of *A6* in R1 and R2 is shown by Parry's original matrix where only *1* and *2* are designated values. For in this case all the axioms and rules of S2 are still designated while *A6(p/1,q/2)* = *CLC12CLC21LCL1L2* = *CL2CL1LC26* = *C6C2L5* = *C6C28* = *C67* = *3*.

Moreover, the independence of *A1'* in R1-R3 is shown by a variation of Group IV of Lewis [5] in which *L(*1*234)* = *(1333)*, since *A1'(p/2)* = *LCLCN222* = *LCLC322* = *LCL22* = *LC32* = *L2* = *3*.

The independence of *A3'* in R2 is shown by Group V of Lewis [5] in which $L(*1*234) = (2434)$, since $A3'(p/3,q/2) = LCL3LCN32 = LC3LC22 = LC3L1 = LC32 = L2 = 3$.

And though Lemmon has shown that $A2'$ is derivable in S2 (cf. p. 178 of [4]), and hence R2, it remains an open question as to whether $A2'$ is derivable or independent from the remaining axioms of R1. It should be noted that Lemmon describes a proof of $A2'$ in S3 (p. 179 of [4]), but in so doing applies (a') to (1') which is not an axiom of S3. However $A2'$ is derivable in S3, and hence R3, as follows:

1.	$LCCpqCCqrCpr$	[PC, (a')]
2.	$LCLCpqLCCqrCpr$	[1, (b)]
3.	$LCLCpqLCLpLq$	[(1), (a')]
4.	$LCLpp$	[(2), (a')]
5.	$LCLCpqCLpLq$	[3, 4, Theorem 3]
6.	$LCLCpqCLCqrLCpr$	[2, 5, Theorem 3]

Indeed, this derivation points up a significant difference between Lemmon's foundations for S1-S3 and those given here. The derivation requires line 5 whereas that thesis is independent of S0.5 (given Thomas's matrix of [11] in which $L(*1*234) = (2334)$, since $LCLC34CL3L4 = LCL2C34 = LCL2C34 = LC32 = L2 = 3$). And it is the absence of this thesis and $A1'$ which gives rise to the systems to be discussed in the next section.

Finally, although $A6$ is useful in classically axiomatizing systems properly contained in S3, its addition to system T yields S4, and thus it is useless in attempting to axiomatize T. That T with $A6$ yields S4 clearly follows from the fact that the proper axiom of S4, $LCLpLLp$, follows from $A6$ in T^0 (= PC; (a); $A4$):

1.	$CLCqpCLCpqLCLpLq$	[$A6$, PC]
2.	$CLCqCppCLCCppqLCLCppLq$	[1, PC]
3.	$LCqCpp$	[PC, (a)]
4.	$CLCCppqLCLCppLq$	[2, 3, PC]
5.	$LCqCCppq$	[PC, (a)]
6.	$CLqLCCppq$	[5, $A4$ PC]
7.	$CLqLCLCppLq$	[4, 6, PC]
8.	$CLCppCCLCppLqLq$	[PC]
9.	$LCpp$	[PC, (a)]
10.	$CCLCppLqLq$	[8, 9, PC]
11.	$LCCLCppLqLq$	[10, (a)]
12.	$CLCLCppLqLLq$	[11, $A4$, PC]
13.	$CLqLLq$	[7, 12, PC]
14.	$LCLqLLq$	[13, (a)]

Thus this section gives a classical axiomatization for two systems R1 and R2 which contain S0.5 and are properly contained in S3.

II. Sobociński in [9] describes a system, S3*, which is properly contained in S3 and is classically axiomatized. In this section, S3* will be given a new basis and other systems properly contained in S3* will be classically axiomatized.

System S3* is an N-K-M calculus with C and E given their usual definitions, $\mathfrak{C}pq$ defined as $NMKpNq$, $\mathfrak{E}pq$ as $K\mathfrak{C}pq\mathfrak{C}qp$, and L as NMN. The rules of inference are **PCb** adjusted to N-K-M and **PCc** for N-K (that is from α and $NK\alpha N\beta$ infer β). The following are the axioms of S3*:

Z1.	$NMKpNKpp$	(i.e., $\mathfrak{C}pKpp$)
Z2.	$NMKKpqNq$	(i.e., $\mathfrak{C}Kpqq$)
Z3.	$NMKKKrpNKqrNKpNq$	(i.e., $\mathfrak{C}KKrpNKqrKpNq$)
Z4.	$NMKNMKpNqNNMKNMqNNMp$	(i.e., $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}NMqNMp$)
Z5.	$NKNMpNNp$	(i.e., $CNMpNp$)

While the new basis for S3* is the N-C-L calculus:

R3* = {M0; *A4'*}.

To see that R3* contains S3* first observe that the definitions of E, C, $\mathfrak{C}$, and L are provable in the form of strict equivalences, when K is given its usual N-C definition and $M = NLN$.

Now the proof of section **I** that M3 contains (b') in no way relies on *A1'*. Thus R3* contains (b'). And hence the axioms of S3*, are obtainable from the following theses of R3*: $LCpKpp$, $LCKpqq$, $LCKKrpNKqrKpNq$, $LCLCpqLCLpLq$, and $CLpp$, while the rules of S3* are obtainable from the rules of R3*.

Conversely, to show that S3* contains R3* it must be remarked that the definitions of K, E, $\mathfrak{C}$, and M are provable in the form of strict equivalences when C is given its usual N-K definition and $L = NMN$. Thus, once the substitutability of strict equivalents is shown for S3*, it is clear that *A4'* follows from *Z4*, *A5* from *Z5*, and *A4* from *A4'* and *A5* by **PC**.

Hence, besides showing (b') in S3* it is sufficient to show that S3* contains **PC** and that if $\vdash\alpha$ in **PC** then $\vdash NMN\alpha$ in S3*, in order to complete the proof that S3* contains R3*. For in such a case, *A1*-*A3* will be theses of S3*.

To this end, the following meta-rules and theses of S3* given passim in [9] are required.

RI. $\vdash\alpha$ *and* $\vdash NMK\alpha N\beta$ *only if* $\vdash\beta$.

RIII. $\vdash NMK\alpha N\beta$ *and* $\vdash NMK\beta N\gamma$ *only if* $\vdash NMK\alpha N\gamma$.

RIV. $\vdash NMK\alpha N\beta$ *only if* $\vdash NMKM\alpha NM\beta$.

RV. $\vdash NMK\alpha N\beta$ *and* $\vdash NMK\alpha N\gamma$ *only if* $\vdash NMK\alpha NK\beta\gamma$.

Z7.	$NMKNpp$	*Z12.*	$NMKNKNNprNNKrp$
Z8.	$NMKNMKpNqNNMKNqNNp$	*Z16.*	$NMKKpqNKqp$
Z9.	$NMKpNNNp$	*Z21.*	$NMKKpqNp$

It is now possible to derive the following meta-rules.

RVII. $\vdash NMK\alpha N\beta$ *only if* $\vdash NMKN\beta NN\alpha$.

Proof: From hypothesis by *Z8* and **RI**.

RVIII. $\vdash NMK\alpha N\beta$ *only if* $\vdash NMKK\alpha\gamma NK\beta\gamma$.

Proof:

1.	$NMK\alpha N\beta$	[Hypothesis]
2.	$NMKK\alpha\gamma N\alpha$	[*Z21*]
3.	$NMKK\alpha\gamma N\beta$	[1, 2, **RIII**]
4.	$NMKK\alpha\gamma N\gamma$	[*Z2*]
5.	$NMKK\alpha\gamma NK\beta\gamma$	[3, 4, **RV**]

RIX. $\vdash \mathfrak{E}\alpha\beta$ *and* $\vdash\gamma$ *only if* $\vdash\delta$ *where* δ *results from* γ *by replacing* α *by* β (β *by* α) *in one or more places.*

Proof: The meta-rule follows immediately from (i) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}M\alpha M\beta$, (ii) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}N\alpha N\beta$, (iii) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}K\alpha\gamma K\beta\gamma$, *and* (iv) $\vdash\mathfrak{E}\alpha\beta$ *only if* $\vdash\mathfrak{E}K\gamma\alpha K\gamma\beta$, which are obtained from **RIV**, **RVII**, **RVIII**, and **RVIII** with *Z16*, respectively.

RX. *If* $\vdash\alpha$ *in* **PC**, *then* $\vdash\alpha$.

Proof: The meta-rule is established by deriving a sufficient set of axioms for **PC.** *B1-B4,* below, is such a set, given by Sobociński in [10].

B1.	$NKpNKpp$	[*Z1, Z5*]
B2.	$NKKpqNq$	[*Z2, Z5*]
B3.	$NKKKrpNKqrNKpNq$	[*Z3, Z5*]
B4.	$NKNKpNqNNKNqNNp$	[*Z12, Z7, Z9,* **RIX**]

RXI. *If* $\vdash\alpha$ *then* $\vdash NMN\alpha$.

Proof: In case α is an axiom of **PC,** $NMN\alpha$ is given by:

$NMNNKpNKpp$	[*Z1, Z7, Z9,* **RIX**]
$NMNNKKpqNq$	[*Z2, Z7, Z9,* **RIX**]
$NMNNKKKrpNKqrNKpNq$	[*Z3, Z7, Z9,* **RIX**]
$NMNNKNKpNqNNKNqNNp$	[*Z12, Z7, Z9,* **RIX**]

and in case α is derived from the axioms of **PC,** the proof is completed by the following derivation.

1.	$NMN\beta$	[Induction hypothesis]
2.	$NMNNK\beta N\alpha$	[Induction hypothesis]
3.	$NMK\beta N\alpha$	[2, *Z7, Z9,* **RIX**]
4.	$NMKN\alpha NN\beta$	[3, **RVII**]
5.	$NMKNMN\beta NNMN\alpha$	[4, *Z4,* **RI**]
6.	$NMN\alpha$	[1, 5, **RI**]

Thus **RIX-RXI** complete the proof that S3* contains R3*.

With S3* given a basis analogous to those of section **I,** the following systems are now defined:

S2* = {M0; (b'); *A3'*}; R2* = {M0; *A6, A3'*};
S1* = {M0; (b')}; R1* = {M0; *A6* }.

The independence of *A2'* from all the systems under consideration in this section is given by the Thomas matrix, for $A2'(p/1,q/2,r/4) = CCLC12CLC24LC14 = LCL2CL3L4 = LC3C34 = LC32 = L2 = 3$. Thus

S3 properly contains R3*, while proofs of the previous section establish that

R2* = {S2*; *A6*};
R1* = {S1*; *A6*};

as well as showing that each of the following systems properly contains its predecessor: S0.5, R1*, R2*, and R3*.

Thus this section gives a classical axiomatization for three systems, R1*, R2*, and R3*, which properly contain S0.5 and are properly contained in S3.

III. In [9] Sobociński has also described a system, S3^0, analogous to S1^0 and S2^0 of Feys [2], which is properly contained in S3. In this section, analogues, R1^0-R3^0, of these systems will be classically axiomatized.

Systems S1^0-S3^0 have the same primitive basis as S3*, while their rules of inference are the four given by Lewis [5]: substitution for propositional variables adjusted to *N-K-M;* substitutability of strict equivalents; adjunction (that is, $\vdash\alpha$ and $\vdash\beta$ only if $\vdash K\alpha\beta$); and strict detachment (that is, $\vdash\alpha$ and $\vdash NMK\alpha N\beta$ only if $\vdash\beta$). Their axioms are drawn from:

F1. $NMKKpqNp$ (i.e., $\mathfrak{C}Kpqp$)
F2. $NMKKpqNKqp$ (i.e., $\mathfrak{C}KpqKqp$)
F3. $NMKKKpqrNKpKqr$ (i.e., $\mathfrak{C}KKpqrKpKqr$)
F4. $NMKpNKpp$ (i.e., $\mathfrak{C}pKpp$)
F5. $NMKKNMKpNqNMKqNrNNMKpNr$ (i.e., $\mathfrak{C}K\mathfrak{C}pq\mathfrak{C}qr\mathfrak{C}pr$)
K1. $NMKMKpqNMp$ (i.e., $\mathfrak{C}MKpqMp$)
L1. $NMKNMKpNqNNMKMpNMq$ (i.e., $\mathfrak{C}\mathfrak{C}pq\mathfrak{C}MpMq$)

so that, with the above rules of procedure:

S1° = {*F1*; *F2*; *F3*; *F4*; *F5*};
S2° = {S1°; *K1*};
S3° = {S1°; *L1*},

The R-systems to be considered are:

R1° = {M0; *A6*; *A2'*};
R2° = {M0; *A6*; *A2'*; *A3'*};
R3° = {M0; *A2'*; *A4'*};

and it will be shown that

R1° = {S1°; *A5*; *A6*};
R2° = {S2°; *A5*; *A6*};
R3° = {S3°; *A5*}.

As in section **II** the required definitions are provable in both the R- and S-systems as strict equivalences, as is the substitutability of strict equivalences. Thus R1°-R3° contain: *F1-F4* by Theorem 1 and *A5; F5* by *A2'*; and the remaining rules of inference by Theorems 4 and 5. Moreover, R2° contains *K1:*

1. $LCNMNpNMNCNpq$ [*A3'*, (b'), M0]
2. $LCMNCNpqMNp$ [1, (b'), M0]

3. *LCMNCNNpNqMNNp* [2, PC]
4. *NMKMKpqNMp* [3, (b'), M0]

and finally, *L1* follows in R3° by *A4'*.

Conversely, the required containments are obvious, since S1° together with *A5* yields M0 as is clear from Feys [2], and thus the adequacy of R1°-R3° is established.

It should be noted that the presence of *A5* with S3° (S1°, or S2°) defines a system which properly contains S3° (S1°, or S2°) and is properly contained in S3 (S1, or S2). *A5* is shown to be independent from S1°-S3° by interpreting *L* as *verum,* and the independence of *A1'* from R1°-R3° was given in section I. (But whether or not it is possible to classically axiomatize S3° remains an open question.)

Finally, the proofs of independence given in the two previous sections show that (i) R1-R3 properly contain R1°-R3° respectively, just as these systems properly contain R1*-R3* respectively, while (ii) each of the following systems properly contains its predecessor: S0.5, R1°, R2°, R3° and S3.

Thus this section gives a classical axiomatization for three systems R1°-R3° which contain S0.5 and are properly contained in S3.

IV. In conclusion the appended table shows the containment relations between the systems discussed in this paper.

$$\begin{tikzcd}
 & & & \mathrm{R1} \arrow[dl] & \mathrm{R2} \arrow[l] \arrow[dl] & \mathrm{R3}\ (=\mathrm{S3}) \arrow[l] \arrow[dl] \\
 & & \mathrm{R1}^\circ \arrow[dl] & \mathrm{R2}^\circ \arrow[l] \arrow[dl] & \mathrm{R3}^\circ\ (=\mathrm{S3}^\circ;\ CLpp) \arrow[l] \arrow[dl] & \\
 & \mathrm{S0.5} & \mathrm{R1}^* \arrow[l] & \mathrm{R2}^* \arrow[l] & \mathrm{R3}^*\ (=\mathrm{S3}^*) \arrow[l] &
\end{tikzcd}$$

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Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana