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ON THE PROPOSITIONAL SYSTEM A OF VUČKOVIĆ AND ITS EXTENSION. II

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6.* Completeness of $\mathcal{A}$. The axioms *F1-F18* given in **5**, together with the rules of procedure **RI** and **RII** are verified by the matrices $\mathfrak{M}1$-$\mathfrak{M}4$. Therefore, in order to prove that system $\mathcal{A}$ determined by these matrices is finitely asiomatizable it has to be shown that every thesis verified by $\mathfrak{M}1$-$\mathfrak{M}4$ is a consequence of the axioms *F1-F18* taken together with the rules **RI** and **RII**. Such a proof can be obtained in several ways, and here I shall present the following one:

Let us assume that there are the theses verified by $\mathfrak{M}1$-$\mathfrak{M}4$ and which are independent from the adopted axiom-system *F1-F18*. Hence, among them there must exist the shortest independent thesis. It will be shown that such a thesis does not exist, and, therefore, that every thesis verified by $\mathfrak{M}1$-$\mathfrak{M}4$ is a consequence of *F1-F18* taken together with **RI** and **RII**.

6.1 This proof will be conducted as follows. Let us assume that there exists formula $\mathfrak{A}$ which is the shortest independent thesis. Then, it possesses a certain structural form, i.e. it belongs to a certain structural type **T**. Hence:

(i) If in the field of $\mathcal{A}$ every formula $\mathfrak{B}$ belonging to the given type **T** is inferentially equivalent to one or several such formulas that each of them either is shorter than $\mathfrak{B}$ or is a consequence of *F1-F18* or is falsified by $\mathfrak{M}1$-$\mathfrak{M}4$, then, obviously, the shortest independent thesis $\mathfrak{A}$ cannot belong to the type **T**.

(ii) On the other hand, if in the field of $\mathcal{A}$ every formula $\mathfrak{B}$ belonging to the given type **T** is inferentially equivalent to one or several such formulas that 1) at least one of these formulas belongs to certain type **T'** which is simpler in some respect than **T**, and 2) the remaining formulas are shorter than $\mathfrak{B}$, then, obviously, in the field of $\mathcal{A}$, $\mathfrak{B}$ is a consequence of the independent

*The first part of this paper appeared in *Notre Dame Journal of Formal Logic,* v. V (1964), pp. 141-153. It will be referred throughout this part as [14]. See the additional Bibliography given at the end of this part. An acquaintance with [14] is presupposed.

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thesis belonging to the simpler type **T'**. Therefore, in such a case we can assume that there exists the shortest independent thesis, say $\mathfrak{A}'$, belonging to **T'**.

(iii) Finally, after showing that in the field of $\mathcal{A}$ every formula verified by $\mathfrak{M}1$-$\mathfrak{M}4$ and allegedly independent from *F1-F18* is equivalentially reducible to one or several formulas belonging to a very simple and special structural type **T''**, it will be proved that every formula of **T''** either is falsified by $\mathfrak{M}1$-$\mathfrak{M}4$ or is a simple consequence of *F1-F18*. Since this conclusion proves that the shortest independent thesis $\mathfrak{A}$ does not exist, it proves that every thesis which is verified by $\mathfrak{M}1$-$\mathfrak{M}4$ is a consequence of the axioms *F1-F18* taken together with the rules **RI** and **RII**.

(iv) It has to be noted that in the presented below reasonings we shall constantly and tacitly make use of the fact that every well-formed propositional formula is constructed from a finite only number of capital and small Latin letters.

6.2 In order to present the proofs given below in a compact way, I shall use here the following symbols and abbreviations:

(v) The small Greek letters will denote the well-formed subformulas of the propositional formulas being under investigation.

(vi) For an arbitrary small Latin letter, say p, a small German letter, say $\mathfrak{a}$, will denote a formula which is either p or Np or NNp or $CpNp$.

(vii) If Γ represents a certain formula, then $\Gamma_{(p)}$ and $\Gamma_{(-p)}$ mean respectively that Γ contains or does not contain the variables which are equiform with the letter p.

(viii) It will be said that the certain formulas belong to the same structural type T, if after replacing their well-formed subformulas by the letters α, β, γ, etc. it can be shown that they possess a similar structural form. Thus, e.g., the axioms *F1* and *F2* belong to the type $C\alpha C\beta\gamma$, but the axioms *F3* and *F5* do not belong.

(ix) The abbreviation

$$\{\alpha\} \rightleftarrows \{\beta_1;\ \beta_2;\ \ldots\ \beta_m\}:\ [\mathbf{MR}_1;\ \ldots\ \mathbf{MR}_n; F_1;\ \ldots\ F_r],\ \text{for } m,n,r \geq 1$$

means that in virtue of the metarules $\mathbf{MR}_1 \ldots \mathbf{MR}_n$ established in **5.2** and the theses $F_1 \ldots F_r$ proved in **5.1** every formula α which belongs to the given type **T** is inferentially equivalent to the set of the formulas $\beta_1; \ldots \beta_m$.

6.3 Let us assume that formula $\mathfrak{A}$ verified by $\mathfrak{M}1$-$\mathfrak{M}4$ is the shortest thesis which is independent from the adopted axiom-system. Since for an arbitrary small Latin letter, e.g., p, the formulas p, Np and NNp are falsified by $\mathfrak{M}1$ and $\mathfrak{M}2$ and since in the field of the system $\mathcal{A}$ for any formula $NNN\alpha$

$$\{NNN\alpha\} \rightleftarrows \{N\alpha\}: [F72;\ F77]$$

holds, the shortest independent thesis, if it exists, must belong to one of the following nine structural types:

T1 $NK\alpha\beta$; **T2** $NA\alpha\beta$: **T3** $NC\alpha\beta$; **T4** $NNK\alpha\beta$; **T5** $NNA\alpha\beta$;

T6 $NNC\alpha\beta$; **T7** $K\alpha\beta$; **T8** $A\alpha\beta$; **T9** $C\alpha\beta$

6.4 But, since in the field of $\mathcal{A}$ for any formula belonging to the types **T1-T8** the following deductions:

(1) *Ad* **T1**: $\{NK\alpha\beta\} \rightleftarrows \{CNN\alpha N\beta\}$: [*F96*; *F95*]
(2) *Ad* **T2**: $\{NA\alpha\beta\} \rightleftarrows \{CN\alpha N\beta; CN\beta N\alpha; C\alpha N\beta; C\beta N\beta\}$: [*F15*;*F16*;*F103*;*F104*; *F123*]
(3) *Ad* **T3**: $\{NC\alpha\beta\} \rightleftarrows \{\alpha; N\beta\}$: [*F70*; *F5*; *F4*]
(4) *Ad* **T4**: $\{NNK\alpha\beta\} \rightleftarrows \{NN\alpha; NN\beta\}$: [*F97*; *F98*; *F11*]
(5) *Ad* **T5**: $\{NNA\alpha\beta\} \rightleftarrows \{CN\beta NN\alpha; CNN\alpha CNN\beta CC\alpha N\alpha\beta\}$: [*F118*;*F121*; *F120*]
(6) *Ad* **T6**: $\{NNC\alpha\beta\} \rightleftarrows \{C\alpha NN\beta\}$: [*F84*; *F86*]
(7) *Ad* **T7**: $\{K\alpha\beta\} \rightleftarrows \{\alpha; \beta\}$: [*F6*; *F7*; *F8*]
(8) *Ad* **T8**: $\{A\alpha\beta\} \rightleftarrows \{CC\alpha\beta\beta\}$: [*F109*; *F106*]

hold, we know that $\mathfrak{A}$, if it exists, is such that

(a) it cannot belong to the types **T3, T4** and **T7,**
and
(b) If it belongs to one of the types **T1, T2, T5, T6** or **T8,** then it is inferentially equivalent to one or several theses belonging to **T9.**

Hence, it is sufficient to analyse the case when there are independent theses of **T9.**

6.4 Every propositional formula $\mathfrak{K}$ belonging to **T9** possesses, obviously, the following structural form

(α) $C\alpha_1 C\alpha_2 C \ldots C\alpha_n \beta$ for $n \geq 1$, and where β is not an implication

If, for $1 \leq i \leq n$, α_i has a form $NNN\gamma$ or if β has a form $NNN\delta$, then in virtue of **MRI, MRII,** *F72, F77* and *F24* we can always eliminate equivalently the first two negations from such α_i or β. Moreover, if in formula $\mathfrak{K}$, which satisfies (α), for $1 \leq i < j \leq n$, $\alpha_i = \alpha_j$, then due to **MRI,** *F38* and *F20* we are able to drop equivalently α_j from formula $\mathfrak{K}$. Therefore, for obvious reasons in our further considerations we shall analyse only such formulas of **T9** which have the structural form

(β) $C\alpha_1 C\alpha_2 C \ldots C\alpha_n \beta$, for $n \geq 1$, where β is not an implication, and where, for $1 \leq i < j \leq n$, $\alpha_i \neq \alpha_j$, and neither α_i nor β have a form $NNN\gamma$.

Since every formula of **T9** satisfying (β) possesses at least one antecedent α_1, it can be presented conveniently as a formula belonging to the type

$$\textbf{T10} \quad C\alpha_1\Gamma$$

where formula $C\alpha_1\Gamma$ belongs to **T9** satisfying condition (β), and Γ is an abreviation of the formula $C\alpha_2 C \ldots C_{\alpha n} \beta$.

6.4.1 Since thesis $\mathfrak{A}$, if it exists, is reducible equivalently to one or several independent theses of **T9,** it is sufficient now for our purpose to assume that there is thesis $\mathfrak{B}$ which is the shortest independent thesis belonging to the type **T9.** Since, by assumption, $\mathfrak{B}$ is the shortest independent thesis of **T9**, it must belong to **T10**, and, therefore, it must belong to one of the following subtypes of **T10**:

T11 $CNK\alpha\beta\Gamma$; **T12** $CNA\alpha\beta\Gamma$; **T13** $CNC\alpha\beta\Gamma$; **T14** $CNNK\alpha\beta\Gamma$

T15 $CNNA\alpha\beta\Gamma$; **T16** $CNNC\alpha\beta\Gamma$; **T17** $CK\alpha\beta\Gamma$; **T18** $CA\alpha\beta\Gamma$

T19 $CC\alpha\beta\Gamma$, where for an arbitrary small Latin letter, say p, $C\alpha\beta \neq CpNp$

T20 $C\alpha\Gamma$, where for an arbitrary small Latin letter, say p, α is either p or Np or NNp or $CpNp$.

Since in the field of $\mathcal{A}$ for any formula belonging to the types **T11-T18** the following equivalent transformations:

(9) *Ad* **T11**: $\{CNK\alpha\beta\Gamma\} \rightleftarrows \{CCNN\alpha N\beta\Gamma\}$: [*F24*; *F86*; *F84*]
(10) *Ad* **T12**: $\{CNA\alpha\beta\Gamma\} \rightleftarrows \{CC\alpha N\alpha CC\beta N\beta CCN\alpha N\beta CCN\beta N\alpha\Gamma\}$: [*F35*; *F123*; *F34*; *F102*; *F104*; *F15*; *F16*]
(11) *Ad* **T13**: $\{CNC\alpha\beta\Gamma\} \rightleftarrows \{C\alpha\Gamma; CN\beta\Gamma\}$: [*F28*; *F4*; *F30*; *F70*; *F5*]
(12) *Ad* **T14**: $\{CNNK\alpha\beta\Gamma\} \rightleftarrows \{CNN\alpha CNN\beta\Gamma\}$: [*F28*; *F11*; *F30*; *F37*; *F38*]
(13) *Ad* **T15**: $\{CNNA\alpha\beta\Gamma\} \rightleftarrows \{CCNN\alpha CNN\beta CC\alpha N\alpha\beta CCN\beta NN\alpha\Gamma\}$: [*F28*; *F120*; *F30*; *F121*; *F18*]
(14) *Ad* **T16**: $\{CNNC\alpha\beta\Gamma\} \rightleftarrows \{CC\alpha NN\beta\Gamma\}$: [*F24*; *F86*; *F84*]
(15) *Ad* **T17**: $\{CK\alpha\beta\Gamma\} \rightleftarrows \{C\alpha C\beta\Gamma\}$: [*F28*; *F8*; *F30*; *F6*; *F7*]
(16) *Ad* **T18**: $\{CA\alpha\beta\Gamma\} \rightleftarrows \{C\alpha\Gamma; C\beta\Gamma\}$: [*F24*; *F109*; *F106*]

hold, we know that $\mathfrak{B}$, if it exists, is such that
(c) it cannot belong to the types **T13** and **T18**
(d) if it belongs to **T11**, **T12**, **T15** or **T16**, it is reducible equivalently to a thesis of **T19** and
(e) if it belongs to **T14** or **T17**, then it is inferenially equivalent to a thesis which instead of one antecedent of the form $NNK\alpha\gamma$ or $K\alpha\gamma$ possesses two antecedentes each of which is shorter than that from which they are generated.

Therefore, if the case (e) occurs, then it means that in the field of $\mathcal{A}$ thesis $\mathfrak{B}$ which belongs to **T10** and has a form $C\alpha_1\Gamma$ is inferentially equivalent to a thesis, say $\mathfrak{B}'$, which belongs to **T9** and has such form $C\gamma_1 C\gamma_2\Gamma$ that the formulas γ_1 and γ_2 are shorter than α_1. Hence we can now distinguish the following three cases
(f) thesis $\mathfrak{B}'$ belongs to **T9**, but it has no form of **T10**
(g) thesis $\mathfrak{B}'$ belongs to one of the types **T11-T18**
(h) thesis $\mathfrak{B}'$ belongs to **T19** or **T20**.

If $\mathfrak{B}'$ belongs to (f), then, by **MRI**, **MRII**, *F20*, *F24*, *F38*, *F72* and *F77*, it is obviously equivalent to a thesis belonging to the cases (g) or (h). Since the antecedent α_1 of $\mathfrak{B}'$ is shorter than α_1 of $\mathfrak{B}$, in the case (g) a finite number of applications of the reasonings (9)-(16) presented above and, eventually, of **MRI**, **MRII**, *F20*, *F24*, *F38*, *F72* and *F77* will reduce equivalently thesis $\mathfrak{B}'$, and, therefore, also thesis $\mathfrak{B}$ to one or several theses belonging to **T19** or **T20**. Thus, we obtained the proof that in the field of $\mathcal{A}$ thesis $\mathfrak{B}$, if it belongs to **T11-T18**, is equivalent to one or several theses belonging to **T19** or **T20**. Since $\mathfrak{B}$ is an independent thesis, there must, therefore, exist the independent theses belonging to **T19** or **T20**.

6.4.2 Hence, let us assume that there are the independent theses of **T19** and among them thesis $\mathfrak{C}$ is the shortest one. Then, $\mathfrak{C}$ belongs to one of the following subtypes of **T19**:

T21 $CC\alpha C\gamma\delta\Gamma$; **T22** $CC\alpha\gamma\Gamma$, where γ has no form $C\delta\xi$.

Since in the field of $\mathcal{A}$:

(17) *Ad* **T21**: $\{CC\alpha C\gamma\delta\Gamma\} \rightleftarrows \{C\delta\Gamma; CC\gamma\delta\Gamma; CC\alpha\delta\Gamma\}$: $[F58; F55;^{3} F59; F37]$

holds, thesis $\mathfrak{C}$ either is reducible equivalently to the theses belonging to **T20** or cannot belong to **T21**. On the other hand, if $\mathfrak{C}$ belongs to **T23**, it belongs to one of the following subtypes of **T22**:

T23 $CCN\alpha\gamma\Gamma$; **T24** $CCK\alpha\gamma\delta\Gamma$; **T25** $CCA\alpha\gamma\delta\Gamma$; **T26** $CCC\alpha\gamma\delta\Gamma$;
T27 $CCp\gamma\Gamma$, where p is a variable and γ has no form $C\delta\xi$.

Since in the field of $\mathcal{A}$ the following equivalent transformations:

(18) *Ad* **T23**: $\{CCN\alpha\gamma\Gamma\} \rightleftarrows \{C\gamma\Gamma; CNN\alpha\Gamma\}$: $[F55; F69; F79]$
(19) *Ad* **T24**: $\{CCK\alpha\gamma\delta\Gamma\} \rightleftarrows \{CC\alpha C\gamma\delta\Gamma\}$: $[F90; F91]$
(20) *Ad* **T25**: $\{CCA\alpha\gamma\delta\Gamma\} \rightleftarrows \{CC\alpha\delta CC\beta\delta\Gamma\}$: $[F111; F107]$
(21) *Ad* **T26**: $\{CCC\alpha\gamma\delta\Gamma\} \rightleftarrows \{C\delta\Gamma; CC\gamma\delta C\alpha\Gamma\}$: $[F55; F61; F31]$
(22) *Ad* **T27**: $\{CCp\gamma\Gamma\} \rightleftarrows \{C\gamma\Gamma; CCpNp\Gamma\}$: $[F55; F66; F57]$

hold, we know that thesis $\mathfrak{C}$, if it exists, is such that
(i) in virtue of (17) it cannot belong **T24**
(j) if $\mathfrak{C}$ belongs to one of the types **T23, T25, T26** or **T27**, then in the field of $\mathcal{A}$ $\mathfrak{C}$ is inferetially equivalent to one or two such theses that either
(α) are shorter than $\mathfrak{C}$ and belong to **T9** (*cf.* (18), (21) and (22)) or
(β) belong to **T19**, but instead of one antecedent of the form $CK\alpha\gamma\delta$ or $CA\alpha\gamma\delta$ or $CC\alpha\gamma\delta$ which, eventually, $\mathfrak{C}$ can have they possess two antecedents each of which is shorter than that from which they are generated (*cf.* (20) and (21))
or
(γ) belong to **T20**.

If the first case occurs, then, obviously, the application of the reasonings presented above to them will reduce equivalently each of the theses under discussion to one or several theses belonging to the cases (β) and (γ). On the other hand the application of the deductions presented in (17)-(22) to the theses of (β) transform these theses gradually and in an equivalent way into one or several theses belonging to the type **T20**. It is evident that it always take place, because the deductions indicated in (17)-(22) give the formulas which either are shorter than the initial formula or have instead of one, two antecedents each of which is shorter than that from which they are generated.

Thus, it is proved that in the field of $\mathcal{A}$ thesis $\mathfrak{C}$ is inferentially equivalent to one or several theses belonging to **T20**. Therefore, since $\mathfrak{C}$ is an independent thesis, there must exist independent theses belonging to **T20**.

6.4.3 Hence, let us assume that $\mathfrak{D}$ is the shortest independent thesis belonging to the type **T20**. Obviously, every formula of **T20** has the following form

$$C\mathfrak{a}_1 C\alpha_2 C\alpha_3 C \ldots C\alpha_n \beta$$

where β is not an implication and, for any $2 \leq i \leq n$, α_i either is small German letter, say $\mathfrak{a}_i$, or has a more complicated form. If the latter case occurs, in virtue of **MRI** such formula is equivalent to

$$C\alpha_i C\alpha_2 C \ldots C\alpha_n C\mathfrak{a}_1 \beta$$

i.e. to a formula which belongs to one of the types **T11-T19**. Hence the application of the reasonings which are presented in **6.4.1** and **6.4.2** so many times as needed allows us to establish without any difficulty that a formula of **T20** either is not an independent thesis or is equivalent to one or several such formulas of **T20** that each of them belongs to the type

T28 $C\mathfrak{a}_1 C\mathfrak{a}_2 C \ldots C\mathfrak{a}_n \beta$, where β is not an implication and, for $1 \leq i \leq n$, and for a given small Latin letter, say p, $\mathfrak{a}_i$ is either p or Np or NNp or $CpNp$.

It is clear that it always take place, because the operations proved in **6.4.1** and **6.4.2** allow us to eliminate gradually and equivalentially functors K and A from each antecedent of any formula of **T20**, and to split each implicational antecedent belonging to such a formula and having no form $CpNp$ into two shorter antecedents than the previous one. Thus, if there is the shortest independent thesis $\mathfrak{D}$ of **T20**, then there must exist independent theses belonging to **T28**.

6.5 Therefore, let us assume that $\mathfrak{E}$ is the shortest independent thesis of **T28**. Then, accepting a convenient abbreviation $\Upsilon = \mathfrak{a}_1 C\mathfrak{a}_2 C \ldots C\mathfrak{a}_n$ (obviously, this abbreviation does not present a well-formed formula) we can say that $\mathfrak{E}$ possesses a form

$$C\Upsilon\beta$$

where β is not an implication and, clearly, has no form $NNN\gamma$. Hence, $\mathfrak{E}$, if it exists, must belong to one of the following subtypes of **T28**

T29 $C\Upsilon NK\alpha\beta$; **T30** $C\Upsilon NA\alpha\beta$; **T31** $C\Upsilon NC\alpha\beta$; **T32** $C\Upsilon NNK\alpha\beta$;
T33 $C\Upsilon NNA\alpha\beta$; **T34** $C\Upsilon NNC\alpha\beta$; **T35** $C\Upsilon K\alpha\beta$; **T36** $C\Upsilon A\alpha\beta$;
T37 $C\Upsilon\mathfrak{b}$, where, for any small Latin letter, say p, $\mathfrak{b}$ is either p or Np or NNp

Since in the field of $\mathcal{A}$ we have **MRII-MRIV** at our disposal, the following deductions

(23) *Ad* **T29**: $\{C\Upsilon NK\alpha\beta\} \rightleftarrows \{C\Upsilon CNN\alpha N\beta\}$: [**MRII**; *F96*; *F95*]
(24) *Ad* **T30**: $\{C\Upsilon NA\alpha\beta\} \rightleftarrows \{C\Upsilon CN\alpha N\beta; C\Upsilon CN\beta N\alpha; C\Upsilon C\alpha N\alpha; C\Upsilon C\beta N\beta\}$: [**MRIV**; *F15*; *F16*; *F102*; *F104*; *F123*]
(25) *Ad* **T31**: $\{C\Upsilon NC\alpha\beta\} \rightleftarrows \{C\Upsilon\alpha; C\Upsilon N\beta\}$: [**MRIII**; *F79*; *F5*; *F4*]
(26) *Ad* **T32**: $\{C\Upsilon NNK\alpha\beta\} \rightleftarrows \{C\Upsilon NN\alpha; C\Upsilon NN\beta\}$: [**MRII**; *F99*; *F100*]
(27) *Ad* **T33**: $\{C\Upsilon NNA\alpha\beta\} \rightleftarrows \{C\Upsilon CNN\alpha CNN\beta CC\alpha N\alpha\beta; C\Upsilon CN\beta NN\alpha\}$: [**MRII**; *F121*; *F118*; *F120*]
(28) *Ad* **T34**: $\{C\Upsilon NNC\alpha\beta\} \rightleftarrows \{C\Upsilon C\alpha NN\beta\}$: [**MRII**; *F84*; *F86*]
(29) *Ad* **T35**: $\{C\Upsilon K\alpha\beta\} \rightleftarrows \{C\Upsilon\alpha; C\Upsilon\beta\}$: [**MRIII**; *F92*; *F93*; *F94*]
(30) *Ad* **T36**: $\{C\Upsilon A\alpha\beta\} \rightleftarrows \{C\Upsilon CC\alpha\beta\beta\}$: [**MRII**; *F109*; *F106*]

hold. It shows that $\mathfrak{G}$, if it exists, is such that in the field of $\mathcal{A}$ it is equivalent to one or several theses of **T20**, but such that each of them has the last consequence shorter than $\mathfrak{G}$ possesses. Since, as we know, any formula of **T20** is equivalent to one or several formulas of **T29**, in each of which the last consequence has the same length as in a formula of **T20** under consideration, an application of the deductions given in **6.4** and presented above in **6.5** and, if needed, of **MRI**, **MRII**, *F24*, *F72* and *F73* so many times as required, implies that thesis $\mathfrak{G}$ is equivalent to one or several formulas belonging to **T37**. Hence, if $\mathfrak{G}$ does not belong to **T37**, there are the independent theses of **T37**, and among them there is the shortest one.

6.6 Any formula of **T37** obviously has the form

$C\alpha_1 C\alpha_2 C\ldots C\alpha_n \mathfrak{b}$, where, for $1 \leq i \leq n$, and for a given small Latin letter, say p, α_i is either p or Np or NNp or $CpNp$, and for a given small Latin letter, say q, b is either q or Nq or NNq

If in such formula, for $1 \leq i < j \leq n$, $\alpha_i = \alpha_j$, then in virtue of **MRI**, *F38* and *F20* antecedent α_j can be eliminated from this formula. Hence, assuming that $\mathfrak{F}$ is the shortest independent thesis of **T37**, we know that $\mathfrak{F}$ must belong to a certain subtype of **T37**, viz.

T38 $C\alpha_1 C\alpha_2 C\ldots C\alpha_n \mathfrak{b}$, where, for $1 \leq i < j \leq n$, $\alpha_i \neq \alpha_j$

6.6.1 An analysis of the forms of the formulas of **T38** allows us to distinguish the following cases.

I. Certain small Latin letter, say p, occurs in one only α_i, for $1 \leq i \leq n$.
III. Certain small Latin letter, say p, occurs in two only α_i and α_j, for $1 \leq i < j \leq n$.
III. Certain small Latin letter, say p, occurs in three only α_i, α_j, and α_k, for $1 \leq i < j < k \leq n$.
IV. Certain small Latin letter, say p, occurs in four only α_i, α_j, α_k and α_m, for $1 \leq i < j < k < m \leq n$.

Hence the cases I-IV divide all formulas belonging to **T38** into fifteen subtypes which due to **MRI** can be presented as follows

T39 $CpC\alpha_2 C\ldots C\alpha_n \mathfrak{b}$
T40 $CNpC\alpha_2 C\ldots C\alpha_n \mathfrak{b}$
T41 $CNNpC\alpha_2 C\ldots C\alpha_n \mathfrak{b}$
T42 $CCpNpC\alpha_2 C\ldots C\alpha_n \mathfrak{b}$
} where, for $2 \leq i \leq n$, no p occurs in α_i

T43 $CpCNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
T44 $CpCNNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
T45 $CpCCpNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
T46 $CNpCNNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
T47 $CNpCCpNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
T48 $CNNpCCpNpC\alpha_3 C\ldots C\alpha_n \mathfrak{b}$
} where, for $3 \leq i \leq n$, no p occurs in α_i

T49 $CpCNpCNNpC\alpha_4 C\ldots C\alpha_n \mathfrak{b}$
T50 $CpCNpCCpNpC\alpha_4 C\ldots C\alpha_n \mathfrak{b}$
T51 $CpCNNpCCpNpC\alpha_4 C\ldots C\alpha_n \mathfrak{b}$
T52 $CNpCNNpCCpNpC\alpha_4 C\ldots C\alpha_n \mathfrak{b}$
} where, for $4 \leq i \leq n$, no p occurs in α_i

T53 $CpCNpCNNpCCpNpC\mathfrak{a}_5C\ldots C\mathfrak{a}_n\mathfrak{b}$, where, for $5 \leq i \leq n$, no p occurs in $\mathfrak{a}_i$

Since in the field of $\mathcal{A}$ every formula belonging to one of the types **T43, T45, T46, T49, T50, T51, T52** and **T53** is an obvious consequence of *F20, F39, F2, F36* and **MRI**, the independent thesis $\mathfrak{F}$ of **T38**, if it exists, must belong to one of the remaining types, viz. **T39, T40, T41, T42, T44, T47** and **T48**. In each of the latter seven types we can distinguish two subtypes according to the following cases:
I. letter p does not occur in $\mathfrak{b}$
and
II. Letter p occurs in $\mathfrak{b}$

6.7 If the types **T39, T40, T41, T42, T44, T47** and **T48** satisfy the condition of case I, then using Γ as an abbreviation of formula $C\mathfrak{a}_mC\ldots C\mathfrak{a}_n\mathfrak{b}$, for $m = 2$ or 3, we can present the formulas belonging to these types as follows

T54 $Cp\Gamma_{(-p)}$ (for **T39**); **T55** $CNp\Gamma_{(-p)}$ (for **T40**); **T56** $CNNp\Gamma_{(-p)}$ (for **T41**)
T57 $CCpNp\Gamma_{(-p)}$ (for **T42**); **T58** $CpCNNp\Gamma_{(-p)}$ (for **T44**);
T59 $CNpCCpNp\Gamma_{(-p)}$ (for **T47**); **T60** $CNNpCCpNp\Gamma_{(-p)}$ (for **T48**)

Since, obviously, Γ represents always a formula of **T38** and since in the field of $\mathcal{A}$ the deductions

(31) *Ad* **T54**: $\{Cp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F19; F20*]
(32) *Ad* **T55**: $\{CNp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F73; F20*]
(33) *Ad* **T56**: $\{CNNp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F73; F20*]
(34) *Ad* **T57**: $\{CCpNp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F74; F20*]
(35) *Ad* **T58**: $\{CpCNNp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F19; F73; F22*]
(36) *Ad* **T59**: $\{CNpCCpNp\Gamma_{(-p)}\} \rightleftarrows \{\Gamma_{(-p)}\}$: [*F73; F74; F22*]

hold, thesis $\mathfrak{F}$, being the shortest independent thesis of **T38**, cannot belong to the types **T54-T59**. Also, it cannot belong to **T60**. Namely, let us assume to the contrary that $\mathfrak{F}$ belongs to **T60**. Then $\mathfrak{F}$ possesses the form

$$CNNpCCpNp\Gamma_{(-p)}$$

Since, by assumption, $\mathfrak{M}1$ and $\mathfrak{M}2$ verify $\mathfrak{F}$, the value of $\mathfrak{F}$ is always 3. Hence we have

$$3 = \mathfrak{F} = CNNpCCpNp\Gamma_{(-p)} = CNN1CC1N1\Gamma_{(-p)} = CN2CC12\Gamma_{(-p)} = C3C3\Gamma_{(-p)}$$

regardless of the value which formula $\Gamma_{(-p)}$ takes. But, according to $\mathfrak{M}1$, any formula $C3C3\alpha = 3$ if and only if $\alpha = 3$. Hence, $\mathfrak{F}$ is verified by $\mathfrak{M}1$ if and only if $\Gamma_{(-p)}$ is a thesis. Therefore, since $\Gamma_{(-p)}$ belongs to **T38**, is shorter than $\mathfrak{F}$ and, besides, by *F22*, implies $\mathfrak{F}$, the latter thesis cannot be the shortest independent thesis of **T38**. Thus, $\mathfrak{F}$ does not belong to **T60**.

6.8 Whence it remains to analyse the second possibility, viz. that the types **T39, T40, T41, T42, T44, T47** and **T48** satisfy the condition of case II. But, clearly, if an arbitrary formula $\mathfrak{L}$ belongs to one of the types now considered, then in virtue of **MRI** and the reasonings given in **6.7** we can dismiss in an equivalential way from $\mathfrak{L}$ any such $\mathfrak{a}$ which contains a small Latin letter not occuring in $\mathfrak{b}$. Hence, if $\mathfrak{F}$ belongs to one of the types which are now discussed, it must possesses one of the following forms

T61 $Cp\mathfrak{b}_{(p)}$ (for **T39**); **T62** $CNp\mathfrak{b}_{(p)}$ (for **T40**); **T63** $CNNp\mathfrak{b}_{(p)}$ (for **T41**)
T64 $CCpNp\mathfrak{b}_{(p)}$ (for **T42**); **T65** $CpCNNp\mathfrak{b}_{(p)}$ (for **T44**);
T66 $CNpCCpNp\mathfrak{b}_{(p)}$ (for **T47**); **T67** $CNNpCCpNp\mathfrak{b}_{(p)}$ (for **T48**)

Since $\mathfrak{b}_{(p)}$ is either p or Np or NNp, each of the types **T61-T67** represents only three formulas, namely:

1) In the case of **T61**: *Z1 Cpp*; *Z2 CpNp*; *Z3 CpNNp*
2) In the case of **T62**: *Z4 CNpp*; *Z5 CNpNp*; *Z6 CNpNNp*
3) In the case of **T63**: *Z7 CNNpp*; *Z8 CNNpNp*; *Z9 CNNpNNp*
4) In the case of **T64**: *Z10 CCpNpp*; *Z11 CCpNpNp*; *Z12 CCpNpNNp*
5) In the case of **T65**: *Z13 CpCNNpp*; *Z14 CpCNNpNp*; *Z15 CpCNNpNNp*
6) In the case of **T66**: *Z16 CNpCCpNpp*; *Z17 CNpCCpNpNp*; *Z18 CNpCCpNpNNp*
7) In the case of **T67**: *Z19 CNNpCCpNpp*; *Z20 CNNpCCpNpNp*; *Z21 CNNpCCpNpNNp*

But, formulas *Z1*, *Z3*, *Z5*, *Z9*, *Z13*, *Z15*, *Z17* and *Z21* are either the theses *F19*, *F72*, *F20* and *F21* or the substitutions of these theses. Hence, $\mathfrak{F}$, as an independent thesis, cannot be one of them.

On the other hand, the formulas *Z2*, *Z4*, *Z6*, *Z7*, *Z8*, *Z10*, *Z11*, *Z12*, *Z14*, *Z16*, *Z18*, *Z19* and *Z20* are falsified by matices $\mathfrak{M}1$ and $\mathfrak{M}2$. Namely, *Z2*, for *p/3*: *C3N3 = C32 = 2*; *Z4*, for *p/2*: *CN22 = C32 = 2*; *Z6*, for *p/2*: *CN2NN2 = C3N3 = C32 = 2*; *Z7*, for *p/1*: *CNN11 = CN21 = C31 = 1*; *Z8*, for *p/1*: *CNN1N1 = CN22 = C32 = 2*; *Z10*, for *p/2*: *CC2N22 = CC232 = C32 = 2*; *Z11*, for *p/1*: *CC1N1N1 = CC122 = C32 = 2*; *Z12*, for *p/2*: *CC2N2NN2 = CC23N3 = C32 = 2*; *Z14*, for *p/3*: *C3CNN3N3 = C3CN22 = C3C32 = C32 = 2*; *Z16*, for *p/2*: *CN2CC2N22 = C3CC232 = C3C32 = C32 = 2*; *Z18*, for *p/2*: *CN2CC2N2NN2 = C3CC23N3 = C3C32 = C32 = 2*; *Z19*; *for p/1*: *CNN1CC1N11 = CN2CC121 = C3C31 = C31 = 1*; *Z20*, for *p/1*: *CNN1CC1N1N1 = CN2CC122 = C3C32 = C32 = 2*. Since, by assumption, $\mathfrak{F}$ is verified by $\mathfrak{M}1$ and $\mathfrak{M}2$, $\mathfrak{F}$ cannot be one of these formulas.

6.9 Thus, the shortest independent thesis $\mathfrak{F}$ of **T38** does not exist, and, therefore, since it has been proved before that if there exists the shortest thesis which is independent from the axiom-system *F1-F18*, then in the field of $\mathcal{A}$ this thesis is inferentially equivalent to one or several independent theses belonging to **T38**, it completes the proof that every formula verified by the matrices $\mathfrak{M}1$-$\mathfrak{M}4$ is a consequence of the axioms *F1-F18* taken together with the rules **RI** and **RII**.

7 Mutual independency of the axioms *F1-F18*. The following twenty-five matrices

𝔐1

C	1	2	3
1	3	3	3
2	3	3	3
* 3	1	2	3

𝔐2

p	Np
1	2
2	3
* 3	2

𝔐3

K	1	2	3
1	1	2	1
2	2	2	2
* 3	1	2	3

𝔐4

A	1	2	3
1	2	1	3
2	1	2	3
* 3	3	3	3

𝔐5

p	Np
1	3
2	3
* 3	3

𝔐6

p	Np
1	1
2	1
* 3	1

𝔐7

C	1	2	3
1	3	3	3
2	3	3	3
* 3	1	1	3

𝔐8

C	1	2	3
1	3	3	3
2	3	3	3
* 3	2	2	3

𝔐9

K	1	2	3
1	1	2	3
2	2	2	2
* 3	1	2	3

𝔐10

K	1	2	3
1	1	2	1
2	2	2	2
* 3	3	2	3

𝔐11

K	1	2	3
1	1	2	1
2	2	2	2
* 3	1	2	1

𝔐12

K	1	2	3
1	1	2	1
2	1	2	2
* 3	1	2	3

𝔐13

K	1	2	3
1	1	2	1
2	2	2	2
* 3	1	1	3

𝔐14

K	1	2	3
1	2	2	1
2	2	2	2
* 3	1	2	3

𝔐15

A	1	2	3
1	2	1	3
2	1	2	3
* 3	1	3	3

𝔐16

A	1	2	3
1	2	1	1
2	1	2	3
* 3	3	3	3

𝔐17

A	1	2	3
1	2	3	3
2	3	2	3
* 3	3	3	3

𝔐18

A	1	2	3
1	2	1	3
2	2	2	3
* 3	3	3	3

𝔐19

A	1	2	3
1	2	2	3
2	1	2	3
* 3	3	3	3

𝔐20

A	1	2	3
1	2	1	3
2	1	1	3
* 3	3	3	3

𝔐21

A	1	2	3
1	1	1	3
2	1	2	3
* 3	3	3	3

𝔐22

C	0	1	2	3
0	1	1	3	3
* 1	0	1	2	3
2	1	1	1	1
3	0	1	1	1

𝔐23

p	Np
0	3
* 1	2
2	1
3	0

𝔐24

K	0	1	2	3
0	0	0	2	2
* 1	0	1	2	3
2	2	2	2	2
3	2	2	2	3

𝔐25

A	0	1	2	3
0	0	1	1	1
* 1	1	1	1	1
2	0	1	2	2
3	1	1	3	3

in which an asterisk indicates the designated value, establish the mutual independency of *F1-F18*. We have to note that 𝔐1-𝔐4 are the characteristic matrices of system 𝒜, and that 𝔐1-𝔐25 satisfy the rules **RI** and **RII**. Then:

1) 𝔐22-𝔐25 verify *F2-F18*, and they falsify *F1* for *p/0*, *q/2*, *r/2* and *s/1*: *CCC022CC20C10 = CC32C10 = C10 = 0*

2) $\mathfrak{M}1$, $\mathfrak{M}5$, $\mathfrak{M}3$ and $\mathfrak{M}4$ verify *F1* and *F3-F18*, and they falsify *F2* for *p/3* and *q/1*: *CN3C31 = C31 = 1*
3) $\mathfrak{M}1$, $\mathfrak{M}6$, $\mathfrak{M}3$ and $\mathfrak{M}4$ verify *F1*, *F2* and *F4-F18*, and they falsify *F3* for *p/1*: *CCN11NN1 = CC11N1 = C31 = 1*
4) $\mathfrak{M}7$ and $\mathfrak{M}2$-$\mathfrak{M}4$ verify *F1-F3* and *F5-F18*, and they falsify *F4* for *p/3* and *q/2*: *C3CN2NC32 = C3C3N1 = C3C32 = C32 = 1*
5) $\mathfrak{M}8$ and $\mathfrak{M}2$-$\mathfrak{M}4$ verify *F1-F4* and *F6-F18*, and they falsify *F5* for *p/3* and *q/1*: *CNC31N1 = CN22 = C32 = 2*
6) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}9$ verify *F1-F5* and *F7-F18*, and they falsify *F6* for *p/1* and *q/3*: *CK131 = C31 = 1*
7) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}10$ verify *F1-F6*, and *F8-F18*, and they falsify *F7* for *p/3* and *q/1*: *CK311 = C31 = 1*
8) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}11$ verify *F1-F7* and *F9-F18*, and they falsify *F8* for *p/3* and *q/3*: *C3C3K33 = C3C31 = C31 = 1*
9) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}12$ verify *F1-F8* and *F10-F18*, and they falsify *F9* for *p/2* and *q/1*: *CN2NK21 = C3N1 = C32 = 2*
10) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}13$ verify *F1-F9* and *F11-F18*, and they falsify *F10* for *p/3* and *q/2*: *CN2NK32 = C3N1 = C32 = 2*
11) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}4$ and $\mathfrak{M}14$ verify *F1-F10* and *F12-F18*, and they falsify *F11* for *p/1* and *q/1*: *CNN1CNN1NNK11 = CN2CN2NN2 = C3C3N3 = C3C32 = C32 = 2*
12) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}15$ verify *F1-F11* and *F13-F18*, and they falsify *F12* for *p/3* and *q/1*: *C3A31 = C31 = 1*
13) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}16$ verify *F1-F12* and *F14-F18*, and they falsify *F13* for *p/1* and *q/3*: *C3A13 = C31 = 1*
14) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}17$ verify *F1-F13* and *F15-F18*, and they falsify *F14* for *p/2*, *q/1* and *r/1*: *CA21CC21CC111 = C3C3C31 = C3C31 = C31 = 1*
15) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}18$ verify *F1-F14* and *F16-F18*, and they falsify *F15* for *p/2* and *q/1*: *CNA21CN2N1 = CN2C32 = C32 = 2*
16) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}19$ verify *F1-F15* and *F17* and *F18*, and they falsify *F16* for *p/1* and *q/2*: *CNA12CN2N1 = CN2C32 = C32 = 2*
17) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}20$ verify *F1-F16* and *F18*, and they falsify *F17* for *p/1* and *q/2*: *CN2CN2NA22 = C3C3N1 = C3C32 = C32 = 2*
18) $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}21$ verify *F1-F17*, and they falsify *F18* for *p/1* and *q/1*: *CC1N1CC1N1CNN1CNN1NA11 = CC12CC12CN2CN2N1 = C3C3C3C32 = C3C3C32 = C3C32 = C32 = 2*

8 Degree of completeness of system $\mathcal{A}$. Let us assume that a well-formed $\{C;N;K;A\}$-formula

$$\Phi(p, q, r, \ldots)$$

is not a thesis of system $\mathcal{A}$, i.e. that there is at least one such substitution of the values *1*, *2* and *3* for the variables occuring in Φ such that Φ is falsified by $\mathfrak{M}1$-$\mathfrak{M}4$. Since, by assumption, such substitution exists, we can assume that we substitute *p*, *NCpp* and *Cpp* for variables occuring in Φ in the same way as it would be done with the substitution of *1*, *2* and *3* for the same variables respectively in order to falsify Φ by $\mathfrak{M}1$-$\mathfrak{M}4$. Thus, there is a formula

$$\Phi(p, NCpp, Cpp)$$

which, by **RI**, we can obtain from Φ and which is such that, since the formulas *NCpp* and *Cpp* have the constant values *2* and *3* respectively, it is falsified by 𝔐1-𝔐4, if we substitute *1* for *p*. Hence, for *p/1*, Φ (*p*, *NCpp*, *Cpp*) has value *1* or *2*.

On the other hand, since formula *CCNppp* is such that, for *p/2* or *p/3*, its value is *3*, and only for *p/1* we have *CCN111* = *CC211* = *C31* = *1*, the following formula

$$C\Phi(p, NCpp, Cpp)CCNppp$$

is, obviously, satisfied by 𝔐1-𝔐4, and, therefore, is a consequence of *F1-F18*. Whence, the addition of an arbitrary well-formed {*C*;*N*;*K*;*A*}-formula which is falsified by 𝔐1-𝔐4 to *F1-F18* as a new axiom allows as to deduce *CCNppp*, and, therefore, to obtain at least the complete bi-valued propositional calculus. It proves that the degree of completeness of system 𝒜 is 3.

9 Mutual independency of the functors *C*, *N*, *K* and *A* in system 𝒜. Using the reasoning similar to the deductions given by McKinsey in [15] we can easily prove that in the field of 𝒜 no one of these functors is definable in terms of the other three. For this end let us accept as a model such subsystem ℬ of the nine-valued propositional calculus with one designated value that ℬ satisfies the following conditions:

(a) the rules **RI** and **RII** hold in ℬ,

(b) system ℬ contains every thesis which is verified by the following five matrices

𝔐26

	C	1	2	3	4	5	6	7	8	9
	1	9	9	9	9	9	9	9	9	9
	2	9	9	9	9	9	9	9	9	9
	3	7	8	9	7	8	9	7	8	9
	4	9	9	9	9	9	9	9	9	9
	5	9	9	9	9	9	9	9	9	9
	6	7	8	9	7	8	9	7	8	9
	7	3	3	3	6	6	6	9	9	9
	8	3	3	3	6	6	6	9	9	9
*	9	1	2	3	4	5	6	7	8	9

𝔐27

	p	Np
	1	6
	2	6
	3	5
	4	9
	5	9
	6	8
	7	6
	8	6
*	9	5

𝔐28

	K	1	2	3	4	5	6	7	8	9
	1	1	2	2	4	5	5	1	2	2
	2	2	2	2	5	5	5	2	2	2
	3	2	2	3	5	5	6	2	2	3
	4	4	5	5	4	5	5	4	5	5
	5	5	5	5	5	5	5	5	5	5
	6	5	5	6	5	5	6	5	5	6
	7	1	2	2	4	5	5	7	8	9
	8	2	2	2	5	5	5	8	8	9
*	9	2	2	3	5	5	6	8	8	9

𝔐29

	A	1	2	3	4	5	6	7	8	9
	1	4	5	6	1	2	3	7	8	9
	2	5	5	6	2	2	3	8	8	9
	3	6	6	6	3	3	3	9	9	9
	4	1	2	3	4	5	6	7	8	9
	5	2	2	3	5	5	6	8	8	9
	6	3	3	3	6	6	6	9	9	9
	7	7	8	9	7	8	9	7	8	9
	8	8	8	9	8	8	9	8	8	9
*	9	9	9	9	9	9	9	9	9	9

𝔐30

	E	1	2	3	4	5	6	7	8	9
	1	9	8	7	6	5	4	3	2	1
	2	8	9	7	5	6	4	3	2	1
	3	7	9	9	4	5	6	1	2	3
	4	6	5	4	9	8	7	3	2	1
	5	5	6	4	8	9	7	2	3	1
	6	4	5	6	7	8	9	1	2	3
	7	3	2	1	6	5	4	9	8	7
	8	2	3	1	5	6	4	8	9	7
*	9	1	2	3	4	5	6	7	8	9

in which an asterisk indicates the designated value, and
(c) system 𝔅 does not contain a $\{C;N;K;A;E\}$ - formula which is falsified by 𝔐26-𝔐30.

I omit here an easy proof that the matrices 𝔐26-𝔐29 verify only such theses which are the consequences of *F1-F18*. Now, let us suppose that there are such well-formed formulas $\Phi_1\{N;K;A\}(pq)$, $\Phi_2\{C;K;A\}(pq)$, $\Phi_3\{C;N;A\}(pq)$ and $\Phi_4\{C;N;K\}(pq)$ (where $\Phi_1\{N;K;A\}(pq)$ etc mean that in Φ_1 only the functors N, K and A and the variables p and q occur, etc) that the following formulas

(1) $ECpq\,\Phi_1\{N;K;A\}(pq)$
(2) $ENp\,\Phi_2\{C;K;A\}(pp)$
(3) $EKpq\,\Phi_3\{C;N;A\}(pq)$
(4) $EApq\,\Phi_4\{C;N;K\}(pq)$

are the theses of 𝔅. But, an inspection of the matrices 𝔐26-𝔐29 shows that the operations C, N, K and A are defined there in such a way that
𝔞) for the values *4, 5, 6, 8* and *9* functors N, K and A are closed in the subset of the values $\{5, 6, 8, 9\}$,
𝔟) for the value *9* C, K and A are closed in the subset $\{9\}$,
𝔠) for the values *3, 5, 6, 8* and *9* C, N and A are closed in the subset $\{3, 5, 6, 8, 9\}$, and
𝔡) for the values *1, 2, 3, 5, 6, 8,* and *9* C, N and K are closed in the subset $\{1,2,3,5,6,8,9\}$,
It proves that the formulas (1)-(4) cannot be true in 𝔅. Namely:

Ad (1): By 𝔐26, $C64 = 7$, by 𝔞) $\Phi_1\{N;K;A\}(64)$ cannot have the value *7*, and, by 𝔐30, $E7p = 9$ if and only if $p = 7$.
Ad (2): By 𝔐27, $N9 = 5$, by 𝔟) $\Phi_2\{C;K;A\}(99)$ cannot have the value *5*, and, by 𝔐30, $E5p = 9$ if and only if $p = 5$
Ad (3): By 𝔐28, $K83 = 2$, by 𝔠), $\Phi_3\{C;N;A\}(83)$ cannot have the value *2*, and, by 𝔐30, $E2p = 9$ if and only if $p = 2$
Ad (4): By 𝔐29, $A11 = 4$, by 𝔡), $\Phi_4\{C;N;K\}(11)$ cannot have the value *4*, and, by 𝔐30, $E4p = 9$ if and only if $p = 4$.

Thus, we obtain a proof that in the field of 𝒜 the primitive functors C, N, K and A are not mutually definible. Since system **A** of Vučković is a subsystem of 𝒜, it also holds for that system.

10 Functional incompleteness of the system $\mathcal{A}$. An inspection of the matrices $\mathfrak{M}1$-$\mathfrak{M}4$ shows at once that in the field of $\mathcal{A}$ the following, e.g., three-valued functor

p	*Jp*
1	*1*
2	*1*
3	*1*

cannot be defined in terms of the primitive functors *C*, *N*, *K* and *A*, because for the values *2* and *3* these functors are closed in the subset of the values $\{2,3\}$. Hence, no well-formed formula $\Phi_5\{C; N; K; A\}(pq)$ is such that for $p = 3$ and $q = 3$ a value of these formula could be *1*. Thus, in the field of $\mathcal{A}$ it is impossible to define functor *J*, and, therefore, system $\mathcal{A}$ is functionally incomplete.

11 Final remarks. It was established in **2.5.3** (*cf.* [14], p. 145) that the following theses *F1*(**C**), *F2* (*A11*), *F3* (*B10*), *F6* (*B1*), *F7* (*B2*), *F8* (*B3*), *F12* (*B4*), *F13* (*B5*), *F14* (*B6*), *F128* (*A14*), *F126* (*A15*) and *F99* (*A16*) constitute the axiom-system **B4** of Vučković's system **A**. Since the theses *F1*, *F2*, *F3*, *F6*, *F7*, *F8*, *F12*, *F13* and *F14* are the axioms of $\mathcal{A}$ and *F128*, *F126* and *F99* are proved in **5.1**, and, besides, since not only in the field of $\mathcal{A}$, but even in **A** it can be proved without any difficulty that
a) *F128* is equivalent to *F11*,
b) *F126* is equivalent to *F17*
and
c) *F99* is equivalent to the conjunction of *F9* and *F10*,
it is clear that system $\mathcal{A}$ is a proper extension of **A** obtained by the addition to **B4** of the new axioms *F4*, *F5*, *F15*, *F16* and *F18* whose independency from **B4** is given in **7**.

In [13] Vučković notices that he was unable to prove in the field of **A** the formulas *W1* and *W2* although they are verified by his recursive model. An analysis of the proofs of these theses, *cf.* [14], pp. 150 and 151, formulas *F127* and *F87*, explains this situation fully. Namely, in the field of $\mathcal{A}$ theses *F127* and *F87* are obtained in virtue of the axioms *F16* and *F5* respectively. In fact, the matrices $\mathfrak{M}1$-$\mathfrak{M}3$ and $\mathfrak{M}19$ which in **7** falsify *F16* reject also *F127* for $p/1$: $NNA1N\bar{1} = N\bar{N}A12 = \bar{N}N2 = N3 = 2$, and the matrices $\mathfrak{M}8$ and $\mathfrak{M}2$-$\mathfrak{M}4$ used in order to prove the independency of *F5* falsify also *F87* for $p/1$: $NNCNN11 = NNCN21 = NNC31 = NN2 = N3 = 2$. Moreover, we have to note that in the field of $\mathcal{A}$ thesis

W1' *NNANpp*

which, obviously, is akin to *W1* and which is easily provable by *F120*, *F2* and *F72* is not connected with *F16*, but with the axiom *F15*. The matrices $\mathfrak{M}1$-$\mathfrak{M}3$ and $\mathfrak{M}18$ which falsify *F15* reject also *W1'* for $p/1$: $NNAN11 = NNA21 = NN2 = N3 = 2$.

These considerations and the fact that axiom *A16* (*F99*) of Vučković is not verified by his recursive model (*cf.* [14], p. 141) show clearly that system **A** is an entirely unadequate axiomatization of the propositional calculus which is defined by the recursive model given in [13].

It is known that Glivenko has proved that if $\{C;N;K;A$ -formula of the form $N\alpha$ is a thesis of the bi-valued propositional calculus, then it is also provable in the intuitionistic logic of Heyting.[4] It is worth while to note that this theorem of Glivenko is not valid for system $\mathcal{A}$. Viz., e.g., the bi-valued thesis $NNCCpNpNp$ is falsified by $\mathfrak{M}1$-$\mathfrak{M}4$ for $p/1$: $NNCC1N1N1 = NNCC122 = NNC32 = NN2 = N3 = 2$.

NOTES

3. In [14], p. 149, line 16, there is an obvious typographical error. Viz., instead of "*F55 *CCCpqrCpr*" must be "*F55 *CCCpqrCqr*."

4. *Cf.* [14], p. 152, note 2.

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