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REMARKS ABOUT AXIOMATIZATIONS OF CERTAIN MODAL SYSTEMS

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In this paper I present some remarks about axiomatizations of certain modal systems investigated by several authors. Mostly, it will be shown that the axiom--systems of theories under consideration can be simplified. I shall use here a modification of Łukasiewicz's symbolism in which "*C*", "*K*", "*A*", and "*N*" possess the ordinary meaning and "*M*", "*L*", "ℭ" and "ℭ" mean "◇", "∼◇∼", "⥽" and "=" respectively. Symbol "$\vdash \alpha$" means always: formula α is provable in the system under consideration. If it will be not stated clearly to the contrary, it is always assumed tacitly that a system under consideration has Lewis' primitive terms and rules of procedure. An acquaintance with the modal systems of Lewis is presupposed. The systems often mentioned below, S1° - S4°, are defined in [3] and [9], pp. 52-53.

1. An elementary lemma presented below is used several times in this paper. Consider the following two sets, **V** and **W**, of formulas and metarules.

V

V1 ℭ*LKpqKLpLq*
V2 ℭ*KLpLqLKpq*
V3 ℭ*CpqCNqNp*
V4 ℭ*CLpMqNKLpLNq*
V5 ℭ*NLKpNqMCpq*
V6 ℭ*NKLpLNqMCpq*
V7 ℭ*MCpqNLKpNq*
V8 *If* $\vdash$ℭ$\alpha\beta$ *and* $\vdash$ℭ$\beta\gamma$, *then* $\vdash$ℭ$\alpha\gamma$
V9 *If* $\vdash\alpha$ *and* $\vdash$ℭ$\alpha\beta$, *then* $\vdash\beta$
V10 The rule of substitution ordinarily used in modal systems

W

W1 *CLKpqLKpLq*
W2 *CKLpLqLKpq*
W3 *CCpqCNqNp*
W4 *CCLpMqNKLpLNq*
W5 *CNLKpNqMCpq*
W6 *CNKLpLNqCLpMq*
W7 *CMCpqNLKpNq*
W8 *CCpqCCqrCpr*
W9 *If* $\vdash\alpha$ *and* $\vdash C\alpha\beta$, *then* $\vdash\beta$
W10 The rule of substitution ordinarily used in modal systems

LEMMA 1. For any modal system T, if either every element of **V** *or every element of* **W** *is a consequence of T, then*

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1) *in the case of* **V** *the formulas:*

V I $\mathfrak{C}CLpMqMCpq$
V II $\mathfrak{C}MCpqCLpMq$

2) *in the case of* **W** *the formulas:*

W I $CCLpMqMCpq$
W II $CMCpqCLpMq$

are provable in the system T.

The proof follows immediately by inspection of elements of **V** or of **W** respectively.

2. Modal D - systems constructed by Lemmon in [4], pp. 184-185, can be described formally as follows. Consider the following set of primitive functors, assumptions, formulas and rules:

α) Primitive functors: C, N and modal functor L.
β) **PC**, i.e. the complete classical propositional calculus with two rules of procedure; 1) ordinarily used rule of substitution, but adjusted also for functor L, 2) rule of detachment: *If* $\vdash\alpha$ *and* $\vdash C\alpha\beta$, *then* $\vdash\beta$
γ) Modal formulas:

T1	$CLCpqCLpLq$	[Lemmon's (1')]
T2	$CLCpqLCLpLq$	[(1)]
T3	$CLpNLNp$	[(2')]
T4	$CLpLLp$	[(4)]
T5	$CNLpLNLp$	[(5)]
T6	$CLpLCNLqLNLq$	[*Cf.* [16], p. 347]

δ) The special rules of procedure:

RI *If* α *and* β *contain no modal operators and* $\vdash C\alpha\beta$, *then* $\vdash CL\alpha L\beta$. [(Eb')]
RII *If* $\vdash C\alpha\beta$, *then* $\vdash CL\alpha L\beta$ [(Eb)]
RIII *If* α *is fully modalized*, *then* $\vdash CL\alpha\alpha$ [(D)]

Lemmon's D - systems are defined as follows:

D1 = {**PC**;**RI**;*T1*;*T3*}
D1* = {**PC**;**RI**;**RIII**;*T1*;*T3*}
D2 = {**PC**;**RII**;*T1*;*T3*}
D2* = {**PC**;**RII**;**RIII**;*T1*;*T3*}
D3 = {**PC**;**RII**;**RIII**;*T2*;*T3*}
D4 = {**PC**;**RII**;**RIII**;*T1*;*T3*:*T4*}
D5 = {**PC**;**RII**;**RIII**;*T1*;*T3*;*T5*}
D5* = {**PC**;**RII**;**RIII**;*T1*;*T3*;*T6*}

Systems D1, D2, D3, D4 and D5 are given in [4], pp. 184-185. In [4], p. 184, note 11, Lemmon mentioned a possibility to construct systems D1* and D2*. According to Yonemitzu (*Cf.* [16], p. 347) S. A. Kripke puts forward system D5*. I shall show here that in each system D1*, D2*, D3-D5* axiom *T3* is superfluous. *Proof*:

(1) It is obvious that a) rule **RII** implies **RI**, and that b) **PC**, **RIII** and *T2* yield *T1* (*Cf.* [4], p, 184). Hence system D1* is contained in systems D2*, D3-D5*.

(2) *T3* follows from **PC, RI, RIII** and *T1*. Viz.:

a) We introduce, only as a pure abbreviation, the definition of *M*:

Df.1 *Mp* = *NLNp*

Hence:

Z1	*CLNNpLp*	[**PC**;**RI**]
Z2	*CLpLNNp*	[**PC**;**RI**]
Z3	*CMpNLNp*	[**PC**;*Df.1*]
Z4	*CNLNpMp*	[**PC**;*Df.1*]
Z5	*CLpNMNp*	[**PC**;*Z3*;*Z1*]
Z6	*CNMNpLp*	[**PC**;*Z4*;*Z2*]

b) In virtue of **PC, RI**, *Z3* and *Z4* we can establish without any difficulty the following metarule.

RIV *If* α *and* β *contain no modal operators and* $\vdash C\alpha\beta$, *then* $\vdash CM\alpha M\beta$

c) Since, obviously, due to point δ), **RIV** and *Z1-Z6*, every element of $\bar{\bar{\mathbf{W}}}$ follows from **PC, RI** and *T1*, lemma 1 allows us to establish that thesis.

Z7 *CCLpMqMCpq*

i.e. *W*I, holds in the axiom-system under consideration. Furthermore, we have

Z8	*CLMpMp*	[**RIII**, since formula *Mp* is fully modalized]
Z9	*MCMpp*	[*Z7,p/Mp,q/p;Z8*]
Z10	*CMKpqMp*	[**PC**;**RIV**]
Z11	*CMKpqMq*	[**PC**;**RIV**]
Z12	*CMKpqKMpMq*	[**PC**;*Z10*;*Z11*]
Z13	*CNLNKpqKMpMq*	[**PC**;*Z4*;*Z12*]
Z14	*CLNKpNqLCpq*	[**PC**;**RI**]
Z15	*CCMpLqLCpq*	[**PC**;*Z13,q/Nq;Z14;Z5,p/q*[1]]
Z16	*CNMpLCpq*	[**PC**;*Z15*]
Z17	*CNMpCLpLq*	[**PC**;*Z16*;*T1*]
Z18	*CMqCLpMp*	[**PC**;*Z17,q/Nq;Z3,p/q*]
T3	*CLpMp*	[*Z18,q/CMpp;Z9*]

Thus, axiom *T3* is superfluous in D1* and, therefore, by (1), also in D2*, D3-D5*. I was unable to obtain *T3* from the remaining axioms of D1 and D2. It is worth-while to notice that although no D-system, except D5, contains a thesis which begins with *L*, there are *M*-thesis in any D-system.[2]

3. In [2] Dammett and Lemmon analyze the systems S4.5, S4.3 and S4.2 obtained by adding to S4 the new axioms

P1 $\mathfrak{C}$*LMLpLp*

D1 *ALCLpLqLCLqLp*

and

L1 $\mathfrak{C}$*MLpLMLp*

respectively. Among other things the authors have proved metalogically that α) system S4.5 of Parry, *cf.* [8], pp. 150-151, formula 51.1, is equivalent to S5;

β) system S4.3 is weaker than S5, stronger than S4 and contains S4.2;

γ) system S4.2 is weaker than S4.3, but stronger than S4.

Since the authors gave no logical proofs that S4.5 is equivalent to S5 which implies S4.3 and that the latter system contains S4.2, I present here such, very simple, proofs. Subsequently, I shall discuss the possible simplifications of the axiom-systems of S4.3 and S4.2.

3.1 System S4.5 implies S5. Let us assume S4 and

P1	$\mathfrak{C}LMLpLp$	
P2	$\mathfrak{C}LMKMLpMqKLMLpMq$	[Provable in an elementary way in S4]
P3	$\mathfrak{C}CLMLpLqMLCMLpLq$	[*P2*,*q*/*Nq*;S1°[3]]
P4	$\mathfrak{C}\mathfrak{C}LMLpLqLMLCMLpLq$	[*P3*;S2°]
P5	$LMLCMLpLp$	[*P4*,*q*/*p*;*P1*]
P6	$LCMLpLp$	[*P1*,*p*/*CMLpLp*;*P5*]
C11	$\mathfrak{C}MpLMp$	[*P6*;S1°]

Since C11 is a proper axiom of S5 and *P1* follows from C11 at once, we have {S5} ⇆ {S4;*P1*} ⇆ {S4.5} which was already proved in a purely metalogical way in [2]. I was unable to prove the same result using the systems weaker than S4.

3.2 S5 contains S4.3. Clearly, it suffices to obtain *D1* from S5. Hence assume S5. Then:

Z1	$\mathfrak{C}MLpLp$	[S5]
Z2	$CKMLpMMNqCMLqLLp$	[*Z1*;S4°]
Z3	$CMKLpMNqCMLqLLp$	[*Z2*;S2°]
Z4	$CCMpLqLCpq$	[S1°,*cf.*[10]]
Z5	$CMKLpMNqLCLqLp$	[*Z3*;*Z4*,*p*/*Lq*,*q*/*Lp*;S1°]
D1	$ALCLpLqLCLqLp$	[*Z5*;S1°]

Thus, we have {S5} → {S4;*D1*} ⇆ {S4.3}

3.3 S4.2 is a subsystem of S4.3. In order to prove that S4.3 implies *L1* assume S4 and *D1*. Then:

Z1	$CLCpLqLCpq$	[S1]
D2	$ALCLpqLCLqp$	[*D1*;*Z1*,*p*/*Lp*;*Z1*,*p*/*Lq*,*q*/*p*;S1°]
Z2	$CLCpNpLNp$	[S1°]
Z3	$ALNLpLCLNLpp$	[*D1*,*q*/*NLp*;*Z2*,*p*/*Lp*;S1°]
Z4	$CMLpLCLNLpp$	[*Z3*;S1°]
Z5	$CMMLpLLCLNLpp$	[*Z4*;S4°]
Z6	$CCMpLqLCpq$	[S1°]
Z7	$\mathfrak{C}MLpLCLNLpp$	[*Z6*,*p*/*MLp*,*q*/*LCLNLpp*;*Z5*;S1°]
Z8	$\mathfrak{C}NLNLpMp$	[S2]
Z9	$\mathfrak{C}\mathfrak{C}Npr\mathfrak{C}\mathfrak{C}qr\mathfrak{C}LCpqLr$	[S3°]
Z10	$\mathfrak{C}LCLNLppLMp$	[*Z9*,*p*/*LNLp*,*q*/*p*,*r*/*Mp*;*Z8*;S1]
G1	$\mathfrak{C}MLpLMp$	[*Z7*;*Z10*;S1°]
L1	$\mathfrak{C}MLpLMLp$	[*G1*,*p*/*Lp*;S4°]

Thus, we obtain {S4.3} → {S4;*L1*} ⇆ {S4.2}.

3.4 Theses equivalent to *D1* in the field of S4. I shall show here that the theses *D1, D2* (already proved in **3.3**) and, theses *D3* and *D4*, given below, are mutually equivalent in the field of S4, and, therefore, that each of them can serve as a proper axiom of S4.3. Since in **3.3** we have {S4;*D1*} $\rightarrow$ {D2}, it remains only to show that {S4;*D2*} $\leftrightarrows$ {S4;*D3*} $\leftrightarrows$ {S4;*D4*} $\leftrightarrows$ {S4;*D1*}.

3.4.1 We assume S4 and *D2*. Then:

Z1 *ALLCLpqLLCLqp* [*D2*;S4°]
Z2 *CALpLqLApq* [S2°]
D3 *LALCLpqLCLqp* [*Z2,p/LCLpq,q/LCLqp;Z1*]

3.4.2 Assume S4 and *D3*. Whence:

D4 *LALCLpLqLCLqLp* [*D3,p/Lp,q/Lq*;S4]

Since, obviously, in virtue of S1°, *D4* implies *D1*, we have a proof that {S4.3} $\leftrightarrows$ {S4;*D1*} $\leftrightarrows$ {S4;*D2*} $\leftrightarrows$ {S4;*D3*} $\leftrightarrows$ {S4;*D4*}.

3.5 Although in the field of S4 theses *D1-D4* are mutually equivalent, they behave differently in the field of systems weaker than S4. Namely:

3.5.1 {S4°;*D2*} $\leftrightarrows$ {S4:*D1*}. Assume S4° and *D2*. Then:

Z1 *LCLpp* [*D2,q/p*;S1°]

Since S4° together with *Z1* constitues system S4 which in its turn contains S4°, the proof is complete.

3.5.2 {S3°;*D3*} $\leftrightarrows$ {S4;*D3*}. Assume S3° and *D3*. Then:

Z1 *LLCLpp* [*D3,q/p*;S1°]
Z2 *LCLpp* [*Z2*;S1°]

Since the addition of *Z2* to S3° gives S3 and in virtue of Parry's proof, cf. [8], p. 148, that an addition of *Z1* to S3 yields S4, our proof is complete.

3.5.3 {S3;*D4*} $\leftrightarrows$ {S4;*D4*}. Assume S4 and *D4*. Then:

Z1 *LLCLpLp* [*D4,q/p*;S1°]

Since, by [8], S3 together with *Z1* constitutes S4, the proof is given. Therefore, the points **3.4** and **3.5** imply at once that {S4.3} $\leftrightarrows$ {S4;*D1*} $\leftrightarrows$ {S4°;*D2*} $\leftrightarrows$ {S3°;*D3*} $\leftrightarrows$ {S3;*D4*}. I was unable to obtain S4.3 from *D1* using a weaker system than S4. It appears that {S3°;*D3*} is the simplest axiom-system of S4.3.

3.6 In [2], p. 252, it is noticed that P. T. Geach pointed out that in S4.2 axiom *L1* can be substituted by thesis *G1* which is already presented in **3.3**. Also, it was shown there that *G1* together with S4° implies *L1*. On the other hand, it is evident that in the field of S2 *L1* with the aid of ℭ*Lpp* gives *G1*. Thus, {S4.2} $\leftrightarrows$ {S4;*L1*} $\leftrightarrows$ {S4;*G1*}. We shall show here that in order to obtain S4.2 it is sufficient to add either *L1* or *G1* to S3. Viz., let us assume S3. Hence, we have:

Z1 ℭ*CMpLqLCpq* [S2°,*cf.*[10]]
Z2 ℭℭ*MpLqLLCpq* [*Z1*;S2°]

Therefore, the addition of *L1* or of *G1* to S3 generates the theses

Z3 *LLCLpMLp* [*Z2,p/Lp,q/MLp;L1*]

and

Z4 $LLCLpMp$ [*Z2*,p/Lp,q/Mp;*G1*]

respectively, each of which together with S3 yields S4, *cf.* [8], p. 148. Thus, clearly, {S4.2} ⇆ {S3;*L1*} ⇆ {S3;*G1*}.[4]

4. The effects of the addition of

C12 $\mathfrak{E}pLMp$

and of the generalized Brouwerian axioms

B_n $\mathfrak{E}pL^nMp$[5]

for any $n > 1$, to the various modal systems weaker than S4 or S3 are investigated in [1], [9], [12], [13] and [14]. I shall show here that a) *G1* is a consequence of {S1°;B_n, for any $n \geq 1$}, and that b) the addition of *L1* to {S1°;B_n, for any $n \geq 1$} implies S5. *Proof*:

α) Let us assume S1° and

B_n $\mathfrak{E}pL^nMp$

for any $n \geq 1$. Then, we have:

H1 *If* $\vdash L\alpha$, *then* $\vdash \alpha$ [S1°;*cf.*[3]]
H2 *If* $\vdash \alpha$ *and* $\vdash C\alpha\beta$, *then* $\vdash \beta$ [S1°,*cf.*[3]]
H3 $CCMpLq\mathfrak{E}pq$ [S1°,*cf.*[10]]
H4 $\mathfrak{E}M^nLpp$ [B_n;S1°]
H5 $CLpL^{n+1}Mp$ [B_n;S1°]
H6 $LLMCpp$ [*H5*,p/Cpp;S1°;*H2*;*H1*]
H7 $LMCpp$ [*H6*;*H1*]

By reasonings analogous to the deductions given by Yonemitzu in [15] it follows easily from S1°, *H6* and *H7* that

H8 *If* $\vdash \alpha$, *then* $\vdash L\alpha$

Hence our assumptions generate system T°, defined in [11], pp. 109-110. Since T° clearly contains S2°, we can establish the so-called Becker's rule, viz.

H9 *If* $\vdash \mathfrak{E}\alpha\beta$, *then* $\vdash \mathfrak{E}L\alpha L\beta$ *and* $\vdash \mathfrak{E}M\alpha M\beta$ [S2°,*cf.*[3]]

β) Therefore, we have also:

Z1 $\mathfrak{E}M^nLpL^nMp$ [*H4*;B_n;S1°]

Now, if $n = 1$, formula *Z1* is

G1 $\mathfrak{E}MLpLMp$

On the other hand, if $n > 1$, then

Z2 $\mathfrak{E}M^{n-1}LpL^{n-1}Mp$ [*H3*,$p/M^{n-1}Lp$,$q/L^{n-1}Mp$;*Z1*;*H1*;*H2*]

Since, if $n - 1 > 1$, an application of *H3*, *H2* and *H1* to *Z2* gives thesis $\mathfrak{E}M^{n-2}LpL^{n-2}Mp$, it is clear that *G1* is a consequence of {S1°;B_n; for any $n \geq 1$}.

γ) Now, let us add

L1 ℭ*MLpLMLp*

to S1° and B_n, for any $n \geq 1$. Then: If $n = 1$, formula *H4* is

Q1 ℭ*MLpp*

and, therefore,

Q2 ℭ*LMLpLp* [*Q1*;*H8*]
Q3 ℭ*MLpLp* [*L1*;*Q2*;S1°]
C11 ℭ*MpLMp* [*Q3*;S1°]

If $n > 1$, we procede as follows:

Z1 ℭ*LpMLp* [*H3*,*p*/*Lp*,*q*/*MLp*;*L1*;*H1*;*H2*]
Z2 ℭ$M^{n-1}LpM^{n}Lp$ [*Z1*;*H8* applied n - 1 times]
Z3 ℭ$M^{n-1}Lpp$ [*Z2*;*H4*;S1°]

Now, if $n - 1 = 1$, we have *Q1* and, consequently, C11. And, if $n - 1 > 1$, then an application n - 2 times of *H8* to *Z1* gives thesis ℭ$M^{n-2}LpM^{n-1}Lp$ which together with *Z3* generates ℭ$M^{n-2}Lpp$. Whence, obviously, *Q1* is provable in the system under consideration. Therefore, C11 follows from this system. But, I have proved in [11], p. 58, that the addition of C11 to S1° implies S5. Thus, the proof is complete.

5. In [6] McKinsey constructed a system which he called S4.1, but for certain reasons I prefer to call it system K1. This system is a normal extension of S4,[6] obtained by adding to S4 the formula

K1 ℭ*KLMpLMqMKpq*

McKinsey noticed that system K1 neither includes, nor is included in S5 and the only consistent system which contains both K1 and S5 is the classical propositional calculus. I shall show here that besides *K1* each of the following theses

K2 ℭ*LMpMLp*
K3 *LMCMpLp*
K4 *LMLCpLp*
K5 *LMLCMpp*

can be added to S4 as a proper axiom of the system K1. The fact that {K1} ⇆ {S4;*K1*} ⇆ {S4;*K5*} will show also that K1 and the system *S* investigated in [7], p. 9, theorem 3.10, are identical. *Proof*:

α) Since, obviously S4 satisfies the conditions of lemma 1, we have at our disposal theses *V* I and *V* II given in the conclusion of this lemma. Hence we can imply that

J1 ℭ*LCLpMqLMCpq* [*V*I; S2°]
J2 ℭ*LMCpqLCLpMq* [*V*II; S2°]

hold in the field of S4.

β) Now, let us assume S4 and *K1*. Then:

Z1 ℭℭ*pqCLMpMLq* [*K1*,*q*/*Nq*;S1°]

Z2 ℭ*L*ℭ*pq*ℭ*LMpMLq* [*Z2*;S2°]
K2 ℭ*LMpMLp* [*Z2*,*q*/*p*;S4°]

γ) Assume now S4 and *K2*. Then:

Z1 ℭℭ*rLMq*ℭ*rMLq* [*K2*,*p*/*q*;S2°]
Z2 ℭℭ*pq*ℭ*LMpLMq* [S3°]
Z3 ℭℭ*pq*ℭ*LMpMLq* [*Z2*;*Z1*,*r*/*LMp*;S1°]
Z4 ℭℭ*pqCLMpMLq* [*Z3*;S1]
K1 ℭ*KLMpLMqMKpq* [*Z4*,*q*/*Nq*;S2°]

Therefore, by β) and γ) we know that {S4;*K1*} ⇆ {S4;*K2*}

δ) Again, assume S4 and *K2*. Then:

K3 *LMCMpLp* [*J1*,*p*/*Mp*,*q*/*Lp*;*K2*]

ξ) S4 together with *K3* implies *K4*. Viz.:

Z1 *LMCMpLLp* [*K3*;S4]
Z2 ℭ*CMpLqLCpq* [*S2*,*cf.*[10]]
Z3 ℭ*LMCMpLqLMLCpq* [*Z2*;S2°]
K4 *LMLCpLp* [*Z3*,*q*/*Lp*;Z1]

ζ) Since, obviously, in the field of S4 theses *K4* and *K5* are equivalent, it remains only to prove that S4 and *K5* imply *K2*. Then, assuming S4 and *K5*, we have:

Z1 *CLMLCpqLMCLpLq* [S2°]
Z2 *LMCLMpLp* [*Z1*,*p*/*Mp*,*q*/*p*;*K5*]
Z3 *LCLLMpMLp* [*J2*,*p*/*LMp*,*q*/*Lp*;*Z2*]
K2 ℭ*LMpMLp* [*Z3*;S4]

Thus, in virtue of points α) - ζ), we have established {K1} ⇆ {S4; *K1*} ⇆ {S4;*K2*} ⇆ {S4;*K3*} ⇆ {S4;*K4*} ⇆ {S4;*K5*} and, incidentally, we have proved that {K1} ⇆ {*S*}. I like to notice that I was unable to base K1 on a modal system weaker than S4.

It is interesting to note that when the theses *K4* and *K5* are outside of a scope of S5, then the following theses in some respect akin to them, viz.

B1 *LMCpLp*
B2 *LMCMpp*

in virtue of lemma 1, are provable easily in S2.

6. In [8], pp. 152-153, point 6, Parry discussed two modal systems, say P2 and P2', obtained by adding the axiom

C16 ℭ*MLpLMp*

to S4 and S3 respectively. Since, obviously, C16 is a conjunction of *G1* and *K2*, system K2 can be considered as a normal extension of K1. On the other hand, in virtue of the deductions given in **3.6**, we have {K2'} ⇆ {S3;C16} ⇆ {S3;*G1*;*K2*} ⇆ {S4;*G1*;*K2*} ⇆ {S4;C16} ⇆ {K2}. Hence, systems K2 and K2' are equivalent.

I call a system obtained by the addition of *D1* to *K1* system K3. It is evident that K3 is a proper extension of K2. Group II of Lewis Langford, *Cf.*[5], p. 493, satisfies K1-K3, and it rejects C11. Whence even K3 neither contains, nor is contained in S5. I leave for further investigations an open question whether K1 is a proper subsystem of K2, and K2 of K3.

NOTES

1. Thesis *MCpp* holds in D1 and in D2, because in both these systems we have *T3* and *Z7*.
2. I have proved this formula in the field of S1° in [9], p. 58. Concerning this and the related formulas see, especially, [10].
3. If in the system under consideration a formula can be obtained from the formulas already given and a subsystem of the investigated system, I mentioned always the weaker system in the proper proof line. Thus, in this case I indicate S1° instead of S4.
4. Another possible axiomatization of S4.2 is given by Zeman, *cf.* [17].
5. Symbol $L^n p$ means: $L^n p = \begin{cases} L^1 p = Lp \\ L^{n+1} p = LL^n p \end{cases}$, for any natural $n \geq 1$.
6. A definition of "normal extension of S4" is given in [7], p. 7, definition 3.2.

BIBLIOGRAPHY

[1] C. W. Churchman: On finite and infinite modal systems. *The Journal of Symbolic Logic*, v. 3 (1938), pp. 77-82.

[2] M. A. Dummett and E. J. Lemmon: Modal logics between S4 and S5. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, v. 5 (1959), pp. 250-264.

[3] R. Feys: Les systèmes formalisés de modalités aristotéliciennes. *Revue Philosophique de Louvain*, v. XLVIII (1950), pp. 478-509.

[4] E. J. Lemmon: New foundations for Lewis modal systems. *The Journal of Symbolic Logic*, v. 22 (1957), pp. 176-186.

[5] C. L. Lewis and C. H. Langford: *Symbolic Logic*. Second Edition, 1959., New York, Devon Publication.

[6] J. C. C. McKinsey: On the syntactical construction of systems of modal logic. *The Journal of Symbolic Logic*, v. 10 (1945), pp. 83-94.

[7] J. C. C. McKinsey and A. Tarski: Some theorems about the sentential calculi of Lewis and Heyting. *The Journal of Symbolic Logic*, v. 13 (1948), pp. 1-15.

[8] W. T. Parry: Modalities in the Survey system of strict implication. *The Journal of Symbolic Logic*, v. 4 (1939), pp. 137-154.

[9] B. Sobociński: A contribution to the axiomatization of Lewis' system S5. *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 51-60.

[10] B. Sobociński: A note on modal systems. *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 155-157.

[11] B. Sobociński: A note on the regular and irregular modal systems of Lewis. *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 109-113.

[12] B. Sobociński: On the generalized Brouwerian axioms. *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 123-128.

[13] I. Thomas: S1° and the Brouwerian axioms. *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 151-152.

[14] I. Thomas: A final note on S1° and the Brouwerian axioms. *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 231-232.

[15] N. Yonemitzu: A note on modal systems, von Wricht and Lewis's S1. *Memoirs of the Osaka University of Liberal Arts and Education*. B. Natural Science, No. 4 (1955), p. 45.

[16] N. Yonemitzu: Review of Lemmon's New foundations for Lewis modal systems. *The Journal of Symbolic Logic*, v. 23 (1958), pp. 346-347.

[17] J. J. Zeman: Bases for S4 and S4.2 without added axioms. *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 227-230.

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