24
Notre Dame Journal of Formal Logic
Volume V, Number 1, January 1964

# A NOTE ON PSEUDO DOUBLY CREATIVE PAIRS

THOMAS G. McLAUGHLIN

1. In [2], Smullyan has called attention to a drawback in the definition of "doubly productive pair" as given in [1]. He has suggested[1] the term "pseudo doubly productive pair" for the concept defined in [1]; in this note, we adopt the suggested terminology and say, in particular, that a pair $(\alpha, \beta)$ of sets of natural numbers is *pseudo doubly creative* just in case $\alpha$ and $\beta$ are r.e. sets and the pair $(\tilde{\alpha}, \tilde{\beta})$ is pseudo doubly productive. The writer has given[2] an example of a class of pseudo doubly creative pairs which are not doubly creative according to Smullyan's revised definition[3]; namely, the class of pairs $(\zeta,\zeta)$ such that $\zeta$ is a creative set. In fact, each such pair $(\zeta,\zeta)$, $\zeta$ creative, is even "pseudo **D.C.**$^+$", i.e., $(\tilde{\zeta},\tilde{\zeta})$ is **D.P.**$^+$ in the sense of [1]. In the present note, we shall look at a few other pairs $(\alpha,\beta)$ which are pseudo **D.C.**$^+$, and comment on whether their members "differ nicely," in that various of $\alpha - \beta$, $\beta - \alpha$, $\alpha \triangle \beta$[4] are recursive or at least r.e. We begin by establishing a simple "chaining" lemma.

Lemma. *Suppose $(\alpha,\beta)$ is pseudo doubly creative under $f(x,y)$, and $\gamma$ is an r.e. set such that there exists an index, $i_0$, of the empty set $\phi$ for which $\gamma \cap \{f(i_0,y) | \omega_y \subseteqq \tilde{\beta}\} = \phi$. Then, the pair $(\alpha \cup \beta, \beta \cup \gamma)$ is pseudo* **D.C.**$^+$.

Proof. (i) $\alpha \cup \beta$, $\beta \cup \gamma$ will, of course, be r.e. whenever $\alpha,\beta,\gamma$ are all r.e.

(ii) The operation $\omega_x \cup \omega_y$ is effective in the sense of [1, Chapter IV]; hence let $\phi(x,y)$ be a recursive function such that $\omega_i \cup \omega_j = \omega_{\phi(i,j)}$ for all $i,j$. Suppose $\omega_i \subseteqq (\alpha \cup \beta)^{\sim} = \tilde{\alpha} \cap \tilde{\beta}$, $\omega_j \subseteqq (\beta \cup \gamma)^{\sim} = \tilde{\beta} \cap \tilde{\gamma}$; then, $\omega_i \cup \omega_j = \omega_{\phi(i,j)} \subseteqq \tilde{\beta}$. Therefore, since $f(x,y)$ is pseudo doubly productive for $(\tilde{\alpha},\tilde{\beta})$, if we let $i_0$ be an index of $\phi$ as in hypotheses, we have $f(i_0, \phi(i,j)) \epsilon\, \tilde{\alpha} \cap \tilde{\beta} - (\omega_i \cup \omega_j)$; and so, by the hypothesis on $i_0$, $f(i_0, \phi(i,j)) \epsilon\, \tilde{\alpha} \cap \tilde{\beta} \cap \tilde{\gamma} - (\omega_i \cup \omega_j)$. But $\tilde{\alpha} \cap \tilde{\beta} \cap \tilde{\gamma} = (\alpha \cup \beta)^{\sim} \cap (\beta \cup \gamma)^{\sim}$; and thus the function $f(i_0, \phi(x,y))$ is pseudo **D.P.**$^+$ for the pair $((\alpha \cup \beta)^{\sim}, (\beta \cup \gamma)^{\sim})$.

Corollary. *Let $(\alpha,\beta)$ be pseudo doubly creative. Then $(\alpha \cup \beta,\beta)$ is pseudo* **D.C.**$^+$.

2. From the foregoing lemma we obtain the pseudo **D.C.**$^+$ character of certain pairs $(\alpha',\beta')$ for which each of $\alpha' - \beta'$ and $\beta' - \alpha'$ (and hence also

*Received January 19, 1963*

$\alpha' \triangle \beta'$) is r.e. but not recursive. To accomplish this, we proceed as follows. Let $\phi(x,y)$ be as in the proof of the lemma; and let $\psi(x,y)$, similarly, be a recursive function such that $\omega_{\psi(i,j)} = \omega_i \cap \omega_j$, for all $i$ and $j$. Let $\alpha$ and $\beta$ be *disjoint* pseudo doubly creative sets (hence doubly creative in the sense of the revised notion of double productivity indicated in Note 3), with, say, $f(x,y)$ pseudo doubly productive for $(\alpha,\beta)$. It is obvious that $\alpha \cup \beta$ is not simple; so let $\delta$ be an infinite recursive subset of $(\alpha \cup \beta)^{\sim}$, and let $\zeta$ be any r.e. but not recursive subset of $\delta$. Let $d_0$ be an index of $\delta$; $d_1$ an index of $\tilde{\delta}$; then, it is easy to check that $g(x,y) = f(\phi(x,d_0), \psi(y,d_1))$ is again pseudo doubly creative for $(\alpha,\beta)$. Furthermore, if $i_0$ is an index of $\phi$, then $\{g(i_0, y) \mid \omega_y \subseteqq \tilde{\beta}\}$ is disjoint from $\delta$ and so from $\zeta$. Therefore, applying the lemma, the pair $(\alpha' = \alpha \cup \beta,\ \beta' = \beta \cup \zeta)$ is pseudo **D.C.**$^+$; and both $(\alpha \cup \beta) - (\beta \cup \zeta) = \alpha$ and $(\beta \cup \zeta) - (\alpha \cup \beta) = \zeta$ are r.e. nonrecursive.

(It might be asked whether $\alpha'$, $\beta'$ can be taken disjoint. We don't know, at present, the answer to this question; in particular, the arguments used in the neighborhood of [1, p. 115] do not seem quite adaptable to a proof of the negative. If it were the case that $(\tilde{\alpha},\tilde{\beta})$ pseudo **D.P.**$^+ \Rightarrow (\tilde{\alpha},\tilde{\beta})$ *weakly* pseudo **D.P.**$^+$, using the definitions 2 and 2' of [1, pp. 120-121] ('$\sim$' is missing from the first '$\alpha$' and '$\beta$' in the statement of 2'), the negative reply would follow easily. But, this implication is not (despite line 5 from the bottom on p. 121 of [1]) obviously true.)

Remark. *Let $(\gamma,\gamma)$ be pseudo* **D.C.**$^+$, *say under the recursive function $f(x,y)$, and let $\alpha$, $\beta$ be recursive subsets of $\gamma$. Then the pair $(\gamma - \alpha, \gamma - \beta)$ is pseudo* **D.C.**$^+$.

Proof. Let $\psi(x,y)$ be as above; and let $k_0$ be an index of $\tilde{\alpha}$, $j_0$ an index of $\tilde{\beta}$. Then, for all $i,j$, $\omega_i \cap \tilde{\alpha} = \omega_{\psi(i,k_0)}$ and $\omega_j \cap \tilde{\beta} = \omega_{\psi(j,j_0)}$, and it is a routine matter to verify that the pair $(\gamma - \alpha, \gamma - \beta)$ is pseudo **D.C.**$^+$ under the function $f(\psi(x,k_0), \psi(y,j_0))$.

We will conclude this section by giving an example of a pseudo **D.C.**$^+$ pair $(\alpha', \beta')$ for which both $\alpha' - \beta'$, $\beta' - \alpha'$ (and hence also $\alpha' \triangle \beta'$) fail to be recursively enumerable (in strong contrast to the examples obtained from the above remark).

Since the implications $(6) \Rightarrow (3) \Rightarrow (1)$ of "Theorem 24" of [1, p. 121] are valid, the construction given by Smullyan ([1, pp. 112-113]) of the **D.U.**$^+$ pair of r.e. sets $(U_1, U_2)$ yields not only a **D.U.**$^+$ but furthermore a pseudo **D.C.**$^+$ pair. Specifically, $U_1 = \{f(x,y,z) \mid z \in \omega_x\}$, $U_2 = \{f(x,y,z) \mid z \in \omega_y\}$, where $f(x,y,z)$ is an arbitrarily prespecified 1-1 recursive function. Here is an informal proof that $U_1 - U_2$ is not r.e.:

Let $\beta$ be any r.e., nonrecursive set; then, $N - \beta$ is not r.e. Let $j_0$ be an index of $N$, $k_0$ an index of $\beta$ (i.e., $N = \omega_{j_0}$, $\beta = \omega_{k_0}$); and suppose $U_1 - U_2$ were r.e. Let $\phi(x)$ be a recursive function such that $U_1 - U_2 = \{\phi(0), \phi(1), \ldots\}$. (It is clear enough that $U_1 - U_2 \neq \phi$.) Since $f(x,y,z)$ is 1-1 recursive, one can effectively determine, for each generated element $\phi(k)$ of $U_1 - U_2$, the unique triple, call it $\ulcorner \langle x,y,z \rangle_{\phi(k)} \urcorner$, such that $\phi(k) = f(x,y,z)$. But the effective sequence $\langle j_0,k_0,0 \rangle$, $\langle j_0,k_0,1 \rangle$, ... can then be compared with the sequence $\langle x,y,z \rangle_{\phi(0)}$, $\langle x,y,z \rangle_{\phi(1)}$, ...; and, thereby, one obtains an effective generation of $N - \beta$: contradiction. $U_2 - U_1$ is dealt with similarly.

3. We might point out, finally, that, as a direct consequence of the lemma of section 1 together with a sufficiency condition for $(\alpha, \beta)$ pseudo doubly productive $\Rightarrow(\tilde{\alpha}, \tilde{\beta})$ doubly-productive-as-in-note 3 (viz., that there be a recursive set $\lambda$ such that $\alpha \cap \beta \subseteqq \lambda \subseteqq \alpha \cup \beta$), we have that if $(\alpha, \beta)$ is pseudo doubly creative then there is no recursive $\lambda$ such that either $\alpha \subseteqq \lambda \subseteqq \alpha \cup \beta$ or $\beta \subseteqq \lambda \subseteqq \alpha \cup \beta$. For, it is easily seen that if $(\alpha, \beta)$ is pseudo doubly creative, then $(\alpha \cup \beta, \beta)$, while pseudo doubly creative by the lemma of section 1, is *not* doubly creative in the revised sense.

## NOTES

1) In private communication.

2) In a letter to Prof. Smullyan. Given therein also was a sufficient condition (stated in section 3 of this note), appreciably weaker than the condition $\alpha \cap \beta = \phi$, for a pseudo doubly productive pair to be doubly productive as in Note 3 below.

3) It has been pointed out by a referee of a previous draft of this paper that, in [2], Smullyan's new notion of double productivity is not quite accurately stated relative to pp. 107-108 of [1]. (We had been working from the definition as given in a private communication from Prof. Smullyan, and did not notice the slip in his abstract.) A correct statement of the revised notion is this:

$(\tilde{\alpha}, \tilde{\beta})$ is doubly productive just in case there is a recursive function $f(x,y)$ such that $\omega_i \subseteqq (\alpha - \beta)^{\sim}$ & $\omega_j \subseteqq (\beta - \alpha)^{\sim}$ & $\omega_i \cap \omega_j = \phi \Rightarrow f(i,j) \epsilon \tilde{\alpha} \cap \tilde{\beta} \cap \tilde{\omega}_i \cap \tilde{\omega}_j$.

4) '$\Delta$' denotes the operation of *symmetric difference*.

## REFERENCES

[1] R. M. Smullyan, *Theory of formal systems*, Princeton, 1961.

[2] R. M. Smullyan, *On double productivity*, Abstract 588-28, Notices Amer. Math. Soc., February, 1962.

*University of Illinois*
*Urbana, Illinois*