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COMMUTATIVE RECURSIVE WORD ARITHMETIC IN THE ALPHABET OF PRIME NUMBERS*

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In memory of Thoralf A. Skolem

Like the recursive arithmetic $\Sigma(1)$ of Skolem[7,5,6] over the familiar Dedekind-Peano word system $\Delta(1)$ in the one-sign alphabet $\{1\}$ (an interpretation of the formal Dedekind-Peano word system $\Delta(a)$ in the alphabet $\{a\}$, the commutative recursive word arithmetic $\Pi(\mathbf{P})$ over the commutative word system $\nabla(\mathbf{P})$ in the alphabet of prime numbers $\mathbf{P}$ (an interpretation of the formal commutative word system $\nabla(\{a_1,a_2,\ldots\})$ of Vučković)[9] is a quantifier-free word arithmetic. As such, $\Pi(\mathbf{P})$ constitutes a version of the Skolem method $\Sigma(1)$ of treating the foundations of elementary number theory. In this paper, we outline a development of $\Pi(\mathbf{P})$ along the lines established by Vučković.[9] However, we take propositional calculus as primitive in lieu of using the Vučković word version of the so-called logic-free equation calculus of Goodstein.[1] In general, $\Pi(\mathbf{P})$ has at its disposal aside from the methods of propositional calculus, substitution for free variables, elementary properties of equality, the usual initial word functions, method of proof by stage induction, and two methods of introducing new word functions, namely, by means of the composition and primitive recursive word schemes. Our formalism follows that of Skolem.[7] In particular, we employ the following notation: ^{non} (negation); $\vee$ (disjunction); $\wedge$(conjunction); $\Longleftrightarrow$ (equivalence); μ(minimalization). Finally, on occasion we employ the following Skolem relation on finite sequences.[7] Let the class of positive integers $1, 2, 3, \ldots$ be denoted by $\mathbf{N}$. The prerequisite relation $x_{i_k}^{(\nu)}$ $(\nu, i_k, k \in \mathbf{N})$ is defined as follows:

$$x_{i_k}^{(\nu)} = x_{i_k} \quad \text{for} \quad k < \nu,$$
$$= x_{i_{k+1}} \quad \text{for} \quad k \geqq \nu.$$

The Skolem relation $\mathsf{I}(x_i, x_j; m, n)$ $(m, n \in \mathbf{N})$ is defined by the following scheme:

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$$\mathsf{I}(x_i, x_j; 1, 1) \Leftrightarrow x_{i_1} = x_{j_1},$$

$\mathsf{I}(x_i, x_j; 1, n)$ false for $n > 1$,

$\mathsf{I}(x_i, x_j; m, 1)$ false for $m > 1$,

$$\mathsf{I}(x_i, x_j; m+1, n+1) \Leftrightarrow \bigvee \nu \leqq n \{x_{i_{m+1}} = x_{j_\nu}\} \wedge \mathsf{I}(x_i, x_j^{(\nu)}; m, n).$$

§1. **Word systems** $\nabla(\mathbf{A})$, $\nabla(\mathbf{P})$. Firstly, we construct the formal commutative word system $\nabla(\mathbf{A})$ introduced by Vučković.[9] We note that we have parted with the Vučković notation $\Omega(\mathbf{A})$ since in the literature this notation is at the moment generally used to denote the formal non-commutative word system in the alphabet $\mathbf{A}$.

Let $\mathbf{A} = \{a_1, a_2, \ldots\}$ denote a denumerable alphabet of signs and Λ the empty word. Let there be given a denumerable class of generating functions

(1.1) $\mathcal{F} = \{F_1(X), F_2(X), \ldots\}$

satisfying the following axioms ($\mu, \nu \in \mathbf{N}$):

(1.2) $F_\mu(X) = a_\mu X$,

(1.3) $a_\mu a_\nu X \neq a_\nu a_\mu Y \vee X = Y$,

(1.4) $a_\mu \neq a_\nu \vee \mu = \nu$.

Next define

$$(1.5)\quad \begin{cases} H_0 = \{\Lambda\}. \\ H_{n+1} = \{F_\mu(X) \mid X \in H_n \wedge \mu \in \mathbf{N}\}. \end{cases}$$

In turn, the formal commutative word system $\nabla(\mathbf{A})$ in the alphabet $\mathbf{A}$ is defined as follows:

$$(1.6)\quad \nabla(\mathbf{A}) = \bigcup_{k=1}^{\infty} H_k.$$

The following theorems are easy consequences of the construction:

(1.7) (Stage Induction Theorem)

$$\frac{\Lambda \in \mathcal{S} \qquad X \notin \mathcal{S} \vee a_\mu X \in \mathcal{S}}{\mathcal{S} = \nabla(\mathbf{A})}$$

(1.8) (Regularity Theorem)

$$a_{i_r} a_{i_{r-1}} \ldots a_{i_1} = a_{j_s} a_{j_{s-1}} \ldots a_{j_1} \Leftrightarrow \mathsf{I}(i, j; r, s).$$

(1.9) (Unicity Theorem)

$$a_{i_r} a_{i_{r-1}} \ldots a_{i_1} \neq a_{j_s} a_{j_{s-1}} \ldots a_{j_1} \vee \mathsf{I}(a_i, a_j; r, s).$$

Next, assuming the usual initial word functions, the primitive recursive word scheme of $\nabla(\mathbf{A})$ is defined as follows:

$$(1.10)\quad \begin{cases} F(X, \Lambda) = A(X), \\ F(X, a_\mu Y) = B_\mu(X, Y, F(X, Y)) \quad (\mu \in \mathbf{N}), \end{cases}$$

where $A(X)$ and $B_\mu(X, Y, Z)$ $(\mu \in \mathbf{N})$ are previously defined word functions. Clearly, every new word function defined by scheme (1.10) must satisfy one of the following conditions $(\mu, \nu \in \mathbf{N})$:

(1.11) $F(X, a_\mu a_\nu Y) = F(X, a_\nu a_\mu Y)$,

(1.12) $B_\mu(X, a_\nu Y, B_\nu(X, Y, F(X, Y))) = B_\nu(X, a_\mu Y, B_\mu(X, Y, F(X, Y)))$.

Following the author's earlier paper,[3] we now construct an arithmetical interpretation of the word system $\nabla(\mathbf{A})$, which is exactly the class of all possible Gödel numbers of $\nabla(\mathbf{A})$.

Let $p_1, p_2, p_3, \ldots$ denote the class of consecutive prime numbers with $p_1 = 2$. Make a one-to-one correspondence between the alphabets $\mathbf{A} = \{a_1, a_2, \ldots\}$ and $\mathbf{P} = \{p_1, p_2, \ldots\}$, that is, let a_μ correspond to p_μ $(\mu \in \mathbf{N})$, and let Λ correspond to 1. Next, define

(1.13) $F_\mu(X) = p_\mu X = p_\mu \cdot X$ $(\mu \in \mathbf{N})$,

with $F_\mu(1) = p_\mu$. Clearly, axioms (1.3) and (1.4) are satisfied. Finally, we define the word system $\nabla(\mathbf{P})$ in the alphabet $\mathbf{P}$ as follows:

(1.14) $\nabla(\mathbf{P}) = \bigcup_{k=1}^{\infty} H_k$.

In turn, we have the following theorems:

(1.15) (Stage Induction Theorem)

$$\frac{\begin{array}{c} 1 \in \mathfrak{T} \\ X \notin \mathfrak{T} \vee p_\mu X \in \mathfrak{T} \end{array}}{\mathfrak{T} = \nabla(\mathbf{P})}$$

(1.16) (Unicity Theorem)

$$p_{i_r} p_{i_{r-1}} \cdots p_{i_1} \neq p_{j_s} p_{j_{s-1}} \cdots p_{j_1} \vee \mathrm{I}(p_i, p_j; r, s).$$

Note, by means of theorems (1.16), we have reproduced in $\nabla(\mathbf{P})$ the prime-number unique factorization theorem of $\Sigma(1)$.

Clearly, the primitive recursive word scheme (1.10) carries over into $\nabla(\mathbf{P})$.

Lastly, restricting ourselves to the alphabet $\{p_1\}$, it is not difficult to see that we can also construct a word system $\nabla(p_1)$ in the one-sign alphabet $\{p_1\}$ with the empty word 1 along the lines given in this section. Clearly, the class of all words of the word system $\nabla(p_1)$ is a subclass of the class of the class of words of $\nabla(\mathbf{P})$.

§2. **Recursive word arithmetic** $\Pi(\mathbf{P})$. In this section, we commence our development of the commutative recursive word arithmetic $\Pi\nabla(\mathbf{P})$, that is, the commutative recursive word arithmetic over the word system $\nabla(\mathbf{P})$, which we briefly denote as $\Pi(\mathbf{P})$. On occasion in this and the following section the notation $\Pi\nabla(\mathbf{P})$ will be used to refer to both $\Pi\nabla(\mathbf{P})$ and $\nabla(\mathbf{P})$.

Following van Rootselaar[8] and Vučković,[9] we first define the basic class of word operations of $\Pi(\mathbf{P})$. The Vučković operations $\oplus_\nu X (\nu \in \mathbf{N})$, additive word operations $X \oplus_\nu Y$ $(\nu \in \mathbf{N})$, word addition $X \oplus_1 Y$ $(\nu = 1)$ or briefly $X \odot Y$, multiplicative word operations $X \odot_\nu Y$ $(\nu \in \mathbf{N})$, word multiplication $X \odot_1 Y$ $(\nu = 1)$ or briefly $X \odot Y$, exponential word operations $X \Delta_\nu Y$ $(\nu \in \mathbf{N})$ and word exponentiation $X \Delta_1 Y$ $(\nu = 1)$ or briefly $X \Delta Y$ are defined by the following primitive recursive word schemes $(\mu, \nu \in \mathbf{N})$:

$$\text{(2.1)} \qquad \begin{cases} \oplus_\nu 1 = 1, \\ \oplus_\nu p_\mu X = p_{\mu\cdot\nu}(\oplus_\nu X), \end{cases}$$

$$\text{(2.2)} \qquad \begin{cases} X \oplus_\nu 1 = X, \\ X \oplus_\nu p_\mu Y = p_{\mu\cdot\nu}(X \oplus_\nu Y), \end{cases}$$

$$\text{(2.3)} \qquad \begin{cases} X \oplus 1 = X, \\ X \oplus p_\mu Y = p_\mu(X \oplus Y), \end{cases}$$

$$\text{(2.4)} \qquad \begin{cases} X \odot_\nu 1 = 1, \\ X \odot_\nu p_\mu Y = (X \odot_\nu Y) \oplus (\oplus_{\mu\cdot\nu} X), \end{cases}$$

$$\text{(2.5)} \qquad \begin{cases} X \odot 1 = 1, \\ X \odot p_\mu Y = (X \odot Y) \oplus (\oplus_\mu X), \end{cases}$$

$$\text{(2.6)} \qquad \begin{cases} X \Delta_\nu 1 = p_1, \\ X \Delta_\nu p_\mu Y = (X \Delta_\nu Y) \odot (\oplus_{\mu\cdot\nu} X) \end{cases}$$

$$\text{(2.7)} \qquad \begin{cases} X \Delta 1 = p_1, \\ X \Delta p_\mu Y = (X \Delta Y) \odot (\oplus_\mu X). \end{cases}$$

Obviously, ordinary multiplication of $\Sigma(1)$ is reproduced in $\Pi(\mathbf{P})$ as word addition $X \oplus Y$.

We now list some properties of the above operations, the proofs of which carry over easily from those given by van Rootselaar[8] and Vučković.[9,10] Firstly, we state the properties involving word addition, the Vučković operation and additive word operations $(\mu, \nu \in \mathbf{N})$:

$$\text{(2.8)} \qquad X \oplus Y = XY \text{ (concatenation)},$$

$$\text{(2.9)} \qquad X \oplus 1 = 1 \oplus X = X,$$

$$\text{(2.10)} \qquad X \oplus (Y \oplus Z) = (X \oplus Y) \oplus Z,$$

$$\text{(2.11)} \qquad X \oplus Y = Y \oplus X,$$

$$\text{(2.12)} \qquad (X \oplus Y) \oplus_\nu Z = X \oplus (Y \oplus_\nu Z),$$

(2.13) $\oplus_\nu (\oplus_\mu X) = \oplus_{\nu \circ \mu} X,$

(2.14) $\oplus_\nu (X \oplus Y) = (\oplus_\nu X) \oplus (\oplus_\nu Y),$

(2.15) $\oplus_\nu X = 1 \oplus_\nu X,$

(2.16) $X \oplus_\nu Y = X \oplus (\oplus_\nu Y),$

(2.17) $X \oplus_\nu (\oplus_\mu Y) = X \oplus_{\nu \circ \mu} Y,$

(2.18) $X \oplus_\nu (Y \oplus_\mu Z) = (X \oplus_\nu Y) \oplus_{\nu \cdot \mu} Z.$

The second lot of properties involving the above operations, word multiplicative word operations follow ($\mu,\ \nu \in \mathbf{N}$):

(2.19) $X \odot 1 = 1 \odot X = 1,$

(2.20) $X \odot p_1 = p_1 \odot X = X,$

(2.21) $X \odot (Y \odot Z) = (X \odot Y) \odot Z,$

(2.22) $X \odot Y = Y \odot X,$

(2.23) $X \odot (Y \oplus Z) = (X \odot Y) \oplus (X \odot Z),$

(2.24) $(X \oplus_\nu Y) \odot Z = (X \odot Z) \oplus_\nu (Y \odot Z),$

(2.25) $X \odot_\nu Y = X \odot (\oplus_\nu Y),$

(2.26) $\oplus_\nu (X \odot Y) = (\oplus_\nu X) \odot Y = X \odot (\oplus_\nu Y),$

(2.27) $X \odot_\nu (Y \oplus_\mu Z) = (X \odot_\nu Y) \oplus_\mu (X \odot_\nu Z),$

(2.28) $X \odot_\nu (Y \odot_\mu Z) = (X \odot_\nu Y) \odot_\mu Z.$

Before we list the last lot of properties, we define the length function $\lambda(X)$ by the following scheme:

(2.29) $$\begin{cases} \lambda(1) = 0, \\ \lambda(p_\mu X) = \lambda(X) + 1 \ (\mu \in \mathbf{N}). \end{cases}$$

In the following, note that $[x,y] = x^y$. We now give the properties involving the exponential word operations and word exponentiation ($\mu \in \mathbf{N}$):

(2.30) $1 \vartriangle X = 1,$

(2.31) $X \vartriangle p_1 = X,$

(2.32) $X \vartriangle_\mu Y = X \vartriangle (\oplus_\mu Y),$

(2.33) $\oplus_{[\mu, \lambda(Y)]} (X \vartriangle Y) = (\oplus_\mu X) \vartriangle Y = X \vartriangle (\oplus_\mu Y),$

(2.34) $\oplus_{[\mu, \lambda(Y)]} (X \vartriangle Y) = X \vartriangle_\mu Y,$

(2.35) $(X \vartriangle Y) \odot (X \vartriangle Z) = X \vartriangle (Y \oplus Z),$

(2.36) $(X \vartriangle Y) \vartriangle (\oplus_{[\mu, 2\lambda(Y)]} Z) = X \vartriangle (Y \odot (\oplus_{[\mu, \lambda(Y)]} Z)).$

Finally, following Vučković,[9] we define in $\Pi(\mathbf{P})$ the notions of word predecessors $p_\nu{}^*X$ ($\nu \in \mathbf{N}$), restricted word subtraction $X [\dot{-}] Y$ and the word inequality relation $X \preccurlyeq Y$ as follows ($\mu \in \mathbf{N}$):

$$(2.37)\qquad \begin{cases} p_\nu{}^* 1 = 1, \\ p_\nu{}^* p_\mu X = X \text{ for } \nu = \mu, \\ \qquad = p_\mu p_\nu{}^* X \text{ for } \nu \neq \mu, \end{cases}$$

$$(2.38)\qquad \begin{cases} X [\dot{-}]\, 1 = X, \\ X [\dot{-}]\, p_\mu Y = p_\mu{}^*(X [\dot{-}] Y), \end{cases}$$

$$(2.39)\qquad X \preccurlyeq Y \Longleftrightarrow X = Y [\dot{-}] (Y [\dot{-}] X).$$

The definition of $X \prec Y$ in $\Pi(\mathbf{P})$ is obvious. Note that restricted word subtraction may be interpreted as a sign deletion operation, for example, $p_{i_r} p_{j_r} p_{i_{r-1}} p_{j_{r-1}} \cdots p_{i_1} p_{j_1} [\dot{-}]\, p_{j_r} p_{j_{r-1}} \cdots p_{j_1} = p_{i_r} p_{i_{r-1}} \cdots p_{i_1}$, so that, clearly the familiar divisibility relation $X | Y$ of $\Sigma(1)$ is reproduced in $\Pi(\mathbf{P})$ as $X \preccurlyeq Y$.

We conclude this section with the following notes with regard to a recursive word arithmetic over the word system $\nabla(p_1)$ in the one-sign alphabet $\{p_1\}$. It is not difficult to see that the definitions in this section given in $\Pi(\mathbf{P})$ carry over in the obvious way to a recursive word arithmetic $\Pi(p_1)$ which has the following primitive recursive word scheme:

$$(2.40)\qquad \begin{cases} f(x, 1) = a(x), \\ f(x, p_1 y) = b(x, y, f(x, y)). \end{cases}$$

On the other hand, we have the following versions of the theorems of Péter[2] and Vučković[10,11] respectively:

(2.41) Every primitive recursive function in $\Pi(p_1)$ can be extended to a primitive recursive function in $\Pi(\mathbf{P})$.

(2.42) The recursive word arithmetic $\Pi(p_1)$ is contained in the recursive word arithmetic $\Pi(\mathbf{P})$.

It is not difficult to see that the familiar relation $x \leqslant y$ of $\Sigma(1)$ is reproduced in $\Pi(p_1)$ as $x \preccurlyeq y$, however note that the familiar operations of addition and multiplication of $\Sigma(1)$ are not reproduced in $\Pi(p_1)$ by $x \oplus y$ and $x \odot y$.

§3. **Recursive word arithmetic** $\Pi(\mathbf{P})$ **(continued).** In this section, we show in particular that the familiar inequality relation $\leqslant$ and ordinary addition of $\Sigma(1)$ are recursively word definable in $\Pi(\mathbf{P})$. However, to show this, we need the following development of the recursive word arithmetic $\Pi(p_1)$. First, we give the following pertinent definitions:

$$(3.1)\qquad y \,\|\, x \Longleftrightarrow \bigvee z \leqslant x \,\{x = y \odot z \wedge z \neq 1\}$$

(this is the word-divisibility relation of $\Pi(p_1)$);

$$(3.2)\qquad \mathrm{pw}(x) \Longleftrightarrow x > p_1 \wedge \bigvee z \leqslant x \,\{z \overset{non}{\|} x \vee (z = x \vee z = p_1)\}$$

($\mathrm{pw}(x)$ means that x is a primitive word in $\Pi(p_1)$);

$$(3.3)\qquad \begin{cases} \mathfrak{n}_1 = p_1, \\ \mathfrak{n}_{n+1} = p_1 \oplus \mathfrak{n}_n, \end{cases}$$

($\mathfrak{n}_n$ is the nth word in $\nabla(p_1)$);

$$(3.4)\qquad \begin{cases} \mathfrak{p}_1 = p_1 p_1, \\ \mathfrak{p}_{n+1} = \mu z \leqslant \mathfrak{n}_2 \Delta(\mathfrak{n}_2 \Delta \mathfrak{n}_{n+1}) \{\mathfrak{p}_\mu < z \wedge \mathrm{pw}(z)\} \end{cases}$$

($\mathfrak{p}_n$ is the nth primitive word of $\Pi(p_1)$). For example, $\mathfrak{p}_1 = p_1 p_1$, $\mathfrak{p}_2 = p_1 p_1 p_1$, $\mathfrak{p}_3 = p_1 p_1 p_1 p_1$ and so on. Note that $\lambda(\mathfrak{p}_\mu) = p_\mu$ and $\mathfrak{n}_{p_\mu} = \mathfrak{p}_\mu$ $(\mu \in \mathbf{N})$.

Secondly, we state several theorems of $\Pi(p_1)$:

$$(3.5)\qquad p \ {}^{non}\| (x \odot y) \vee {}^{non}\mathrm{pw}(p) \vee p \| x \vee p \| y.$$

(3.6) (Primitive-word unique resolution theorem of $\Pi(p_1)$)

$$\mathfrak{p}_{i_r} \odot \mathfrak{p}_{i_{r-1}} \odot \ldots \odot \mathfrak{p}_{i_1} \neq \mathfrak{p}_{j_s} \odot \mathfrak{p}_{j_{s-1}} \odot \ldots \odot \mathfrak{p}_{j_1} \vee \mathrm{I}(\mathfrak{p}_i, \mathfrak{p}_j;\ r, s).$$

The proofs of these theorems are the same as for the parallel theorems of $\Sigma(1)$. In turn, note that the words $x \in \nabla(p_1)$ can be also uniquely represented in the following form:

$$(3.7)\qquad x = (\mathfrak{p}_{i_r} \Delta \mathfrak{n}_{j_r}) \odot (\mathfrak{p}_{i_{r-1}} \Delta \mathfrak{n}_{j_{r-1}}) \odot \ldots \odot (\mathfrak{p}_{i_1} \Delta \mathfrak{n}_{j_1}).$$

Finally, we define the following useful primitive recursive function of $\Pi(p_1)$:

$$(3.8)\qquad \exp(\mathfrak{p}_\nu, x) = \mu z \leqslant x \{(\mathfrak{p}_\nu \Delta p_1 z) \ {}^{non}\| x\},$$

that is, the greatest word-exponent function.

We now give the second array of definitions leading to the recursive word definibility in $\Pi(\mathbf{P})$ of the inequality relation and the operation of addition of $\Sigma(1)$. First, we identify in $\Pi(\mathbf{P})$ the words of $\nabla(p_1)$ as follows:

$$(3.9)\qquad \begin{cases} N(1) = 1, \\ N(p_\mu X) = p_1(N(X)) \ (\mu \in \mathbf{N}), \end{cases}$$

$$(3.10)\qquad \mathrm{num}(X) \Longleftrightarrow X = N(X)$$

($\mathrm{num}(X)$ means that X is a numeral word in $\Pi(\mathbf{P})$). Clearly, the class of all numeral words in $\Pi(\mathbf{P})$ constitutes the class of all words in $\nabla(p_1)$ with the exception of the empty word.

Secondly, it is easy to see that the class of all primitive words in $\Pi(p_1)$ is also recursively definable in $\Pi(\mathbf{P})$. We shall refer to the primitive words of $\Pi(p_1)$ defined in $\Pi(\mathbf{P})$ as the prime-numeral words of $\Pi(\mathbf{P})$. Further, note that we shall employ the same notation used in (3.3) and (3.4) for these words in $\Pi(\mathbf{P})$.

We point out that the familiar factorization $p_{i_r}^{k_r} p_{i_{r-1}}^{k_{r-1}} \ldots p_{i_1}^{k_1}$ in $\Sigma(1)$ is reproduced in $\Pi(\mathbf{P})$ as $(p_{i_r} \odot \mathfrak{n}_{k_r}) \oplus \ldots \oplus (p_{i_1} \odot \mathfrak{n}_{k_1})$.

Lastly, we define the following basic word functions of $\Pi(\mathbf{P})$:

$$(3.11)\qquad \begin{cases} \Gamma(1) = p_1, \\ \Gamma(p_\mu X) = \mathfrak{p}_\mu \odot \Gamma(X) \quad (\mu \in \mathbf{N}), \end{cases}$$

$$(3.12)\qquad \begin{cases} \Gamma^*(X) = 1 \quad \text{if} \quad X \neq \mathrm{num}(X), \\ \Gamma^*(X) = (\exp(\mathfrak{p}_{\delta(X)}, X) \,\Delta\, p_{\delta(X)}) \oplus \ldots \oplus (\exp(\mathfrak{p}_1, X) \,\Delta\, p_1) \\ \qquad\qquad \text{if} \quad X = \mathrm{num}(X), \end{cases}$$

where $\delta(X)$ denotes the index of the greatest prime-numeral word which word-divides X, which is also primitive recursive in $\Pi(\mathbf{P})$. Clearly $\Gamma(X)$ is a primitive recursive function in $\Pi(\mathbf{P})$, and $\Gamma^*(X)$ is also primitive recursive in $\Pi(\mathbf{P})$.

At this point, we can give the recursive word-theoretical definition in $\Pi(\mathbf{P})$ of the relation $\leqslant$ of $\Sigma(1)$ as follows:

$$(3.13)\qquad X \leqslant Y \Longleftrightarrow \Gamma(X) \preccurlyeq \Gamma(Y).$$

The definition in $\Pi(\mathbf{P})$ of the relation $<$ of $\Sigma(1)$ is obvious. In turn, we define the number-theoretic function $+$ of $\Sigma(1)$ as a primitive recursive word function in $\Pi(\mathbf{P})$ as follows:

$$(3.14)\qquad X + Y = \Gamma^*(\Gamma(X) \oplus \Gamma(Y)).$$

We conclude this section with a discussion of the primitive words in $\Pi(\mathbf{P})$ leading to the primitive-word unique resolution theorem in $\Pi(\mathbf{P})$, and finally we state several properties of the even words in $\Pi(\mathbf{P})$.

First, we define the recursively definable class of words called the primitive words of $\Pi(\mathbf{P})$ as follows:

$$(3.15)\qquad Y \| X \Longleftrightarrow \bigvee Z \leqslant X \{X = Y \odot Z \ \wedge\ Z \neq 1\};$$

$$(3.16)\qquad \mathrm{pw}(X) \Longleftrightarrow X > p_1 \wedge \bigwedge Z \leqslant X \{Z \overset{non}{\|} X \vee (Z = X \vee Z = p_1)\}$$

($\mathrm{pw}(X)$ means that X is a primitive word in $\Pi(\mathbf{P})$);

$$(3.17)\qquad \begin{cases} P_1 = p_2, \\ P_{n+1} = \mu Z \leqslant \mathfrak{n}_2 \Delta (\mathfrak{n}_2 \,\Delta\, \mathfrak{n}_{n+1}) \{Z > P_n \wedge \mathrm{pw}(Z)\} \end{cases}$$

(P_n is the nth primitive word in $\Pi(\mathbf{P})$). For example,

$$P_2 = p_1 p_1,\ P_3 = p_3,\ P_4 = p_1 p_2,\ P_5 = p_1 p_1 p_1,\ P_6 = p_1 p_3$$

and so on. Note that not all elements of the alphabet $\mathbf{P}$ are primitive words of $\Pi(\mathbf{P})$. Clearly, the class of all prime-numeral words is a subclass of the class of primitive words in $\Pi(\mathbf{P})$. We now state two theorems of $\Pi(\mathbf{P})$, the proofs of which are similar to the proofs already given in detail earlier by the author:[4]

$$(3.18)\quad P_\nu \overset{non}{\|} (X \odot Y) \vee (\lambda(P_\nu) \neq \lambda(X) \wedge \lambda(P_\nu) \neq \lambda(Y)) \vee (P_\nu \| X \vee P_\nu \| Y).$$

(3.19) (Primitive-word unique resolution theorem of $\Pi(\mathbf{P})$)

$$P_{i_r}\odot\ldots\odot P_{i_1}\neq P_{j_s}\odot\ldots\odot P_{j_1}\vee\bigwedge\nu\leqslant s\{\lambda(P_{i_r})\neq\lambda(P_{j_\nu})\}\vee \mathsf{I}(P_i, P_j; r, s).$$

Note, it follows from Theorem (3.19) that the elements of the alphabet P are also uniquely representable as word products of "primitive prime numbers", i.e., words of the form $p_\mu\in\mathsf{P}$ where $\mu\in\mathsf{P}$.

Lastly, we define and state several properties of the even words of $\Pi(\mathsf{P})$ as follows:

(3.20) $\quad \mathrm{even}(X)\Longleftrightarrow X\neq 1\wedge p_1\leqslant X;$

(3.21) $\quad \mathrm{even}(X\odot Y)\Longleftrightarrow \mathrm{even}(X)\wedge\mathrm{even}(Y);$

(3.22) $\quad \mathrm{even}(X\vartriangle Y)\Longleftrightarrow \mathrm{even}(X)\wedge\mathrm{num}(Y).$

We note that on the strength of theorems (3.21) and (3.19) the even words in $\Pi(\mathsf{P})$ are uniquely represented as word products of even primitive words of $\Pi(\mathsf{P})$. Clearly, the notion of even word in $\Pi(\mathsf{P})$ is a reproduction in $\Pi(\mathsf{P})$ of the familiar notion of even number in $\Sigma(\bar{1})$.

Evidently, aside from the commutative recursive word arithmetic $\Pi(\mathsf{P})$ there is a class of noncommutative recursive word arithmetics over the noncommutative word systems $\Omega(\mathsf{P}^1)$, $\Omega(\mathsf{P}^2)$, ... (see author)[6] which can be constructed along the general lines indicated in this paper, the author's[4] earlier paper, and the paper by Vučković[10] in particular.

§4. **Some problems in $\Pi(\mathsf{P})$.** We conclude this paper with two problems in $\Pi(\mathsf{P})$. Before stating our problems, however, note the following theorem:

(Theorem of Vučković)

Every word function $F(X,Y)$ in $\Pi(\mathsf{P})$ defined by the scheme

$$(4.1)\qquad \begin{cases} F(X, 1) = A(X),\\ F(1, p_\mu Y) = B_\mu(Y),\\ F(p_\nu X, p_\mu Y) = C_{\mu,\nu}(X, Y, F(X, Y))\quad (\mu,\nu\in\mathsf{N})\end{cases}$$

is a primitive recursive word function in $\Pi(\mathsf{P})$.

We sketch a proof of this theorem. Define the following word function:

$$(4.2)\qquad \begin{cases} D_\mu(1, X, Y, Z) = 1,\\ D_\mu(p_\nu U, X, Y, Z) = C_{\mu,\nu}(X, Y, Z)\ (\mu,\nu\in\mathsf{N}).\end{cases}$$

Now, let

$$(4.3)\qquad \begin{cases} Q(1) = 1,\\ Q(p_\nu X) = p_1 (\nu\in\mathsf{N}).\end{cases}$$

Finally, define

$$(4.4)\qquad \begin{cases} G(X, 1) = A(X),\\ G(X, p_\mu Y) = [(p_1[\dot{-}]\ Q(X))\odot B_\mu(Y)]\oplus\\ \qquad\qquad [Q(X)\odot D_\mu(X, X, Y, Z)]\ (\mu\in\mathsf{N}).\end{cases}$$

Clearly, $G(X,Y)$ is a primitive recursive word function in $\Pi(\mathbf{P})$. On the other hand, it is obvious that $F(X,Y) = G(X,Y)$. This completes the proof.

Next, note that we defined ordinary addition $X + Y$ as a primitive recursive word function in $\Pi(\mathbf{P})$ by means of the composition scheme of $\Pi(\mathbf{P})$ as follows:

$$(4.5) \qquad X + Y = \Gamma^*(\Gamma(X) \oplus \Gamma(Y)).$$

Problem I is to show whether it is possible to define $X + Y$ either by the primitive recursive word scheme (1.10) or scheme (4.1) or some scheme reducible to scheme (1.10).

For example, ordinary multiplication $X \cdot Y$ can be defined as a primitive recursive word function in $\Pi(\mathbf{P})$ by means of the composition scheme of $\Pi(\mathbf{P})$ as

$$(4.6) \qquad X \cdot Y = \Gamma^*(\Gamma(X) \odot \Gamma(Y)),$$

nonetheless, it is also possible, as we have already shown, to define $X \cdot Y$ by means of the primitive recursive scheme of $\Pi(\mathbf{P})$ as follows:

$$(4.7) \qquad \begin{cases} X \oplus 1 = X, \\ X \oplus p_\mu Y = p_\mu(X \oplus Y) \quad (\mu \in \mathbf{N}). \end{cases}$$

In the same sense, it may be possible also to define $X + Y$ in $\Pi(\mathbf{P})$ by either the primitive recursive word scheme (1.10) or scheme (4.1). If so, then clearly we would have one of the following cases:

$$(4.8) \qquad p_\nu + p_\mu = B_\mu'(p_\nu, 1, p_\nu + 1) \quad (\mu, \nu \in \mathbf{N}),$$

$$(4.9) \qquad p_\nu + p_\mu = C_{\mu,\nu}(1, 1, p_1) \quad (\mu, \nu \in \mathbf{N}).$$

In turn, we can enumerate the even words in $\Pi(\mathbf{P})$, with the exception of p_1 and $p_1 p_1$, as follows:

$$(4.10) \qquad \begin{cases} E_1 = p_1 p_2, \\ E_{n+1} = \mu\, Z \in \nabla(\mathbf{P}) \, \{Z > E_n \wedge \text{even}(Z)\}. \end{cases}$$

Lastly, it is not difficult to see that given a definition of $X + Y$ by means of either scheme (1.10) or (4.1), it would be possible to construct a number-theoretic function $\phi(\mu,\nu)$ $(\mu, \nu \in \mathbf{N} \setminus 1)$ directly determined by B_μ' or $C_{\mu,\nu}'$ such that

$$(4.11) \qquad E_{\phi(\mu,\nu)} = B_\mu'(p_\nu, 1, p_\nu + 1) \quad (\mu, \nu \in \mathbf{N} \setminus 1),$$

or

$$(4.12) \qquad E_{\phi(\mu,\nu)} = C_{\mu,\nu}'(1, 1, p_1) \quad (\mu, \nu \in \mathbf{N} \setminus 1).$$

Obviously, $\phi(\mu,\nu)$ in either case is an effectively calculable function and consequently on the strength of Church's Thesis it is a recursive number-theoretic function.

Assuming that $X + Y$ is definable by scheme (1.10) or scheme (4.1), taking into account the numeration (4.10), *Problem II* is to determine whether

$\phi(\mu, \nu)$ $(\mu, \nu \in \mathbf{N} \setminus 1)$ is a recursive number-theoretic function of the type for which it is possible to show that it is either into or onto $\mathbf{N}$.

If the above solution is possible, then clearly $\phi(\mu,\nu)$ is into if and only if Goldbach's conjecture is false and on the other hand $\phi(\mu,\nu)$ is onto if and only if Goldbach's conjecture is true.

We point out that the word-version of the Goldbach conjecture in $\Pi(p_1)$ is as follows: Every even-length word in $\Pi(p_1)$ greater than $\mathfrak{n}_4$ is expressible as a word-sum of two prime numerals greater than $\mathfrak{p}_1$. It is not difficult to see that this version of the Goldbach conjecture in $\Pi(p_1)$ is equivalent to the Goldbach conjecture in $\Sigma(1)$.

Finally, we note that the Goldbach conjecture for primitive prime numbers is false, since the even number 12 cannot be expressed as a sum of two primitive primes $p_{p_\mu} + p_{p_\nu}(\mu, \nu \in \mathbf{N})$. From this it follows that the restricted function $\phi(p_\mu, p_\nu)$ $(\mu, \nu \in \mathbf{N})$ is into $\mathbf{N}$.

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