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ON AN EXTENSION OF A THEOREM OF FRIEDBERG

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In [1], Friedberg showed that any essentially r.e. set (i.e., any r.e. but nonrecursive set) is effectively decomposable into the disjoint union of a pair of essentially r.e. sets.

A careful examination of Friedberg's proof shows that by a slight modification of it, one can specify a method for effectively decomposing an essentially r.e. set ω_u into the union of $\aleph_0$ pairwise-disjoint, essentially r.e. subsets. What is needed, for the modification, is simply to redefine the notion of an index being *satisfied*, so that *satisfaction of e at step a* is defined relative to *the first e + 1* "*a*-parts", P_0^a, P_1^a, . . . , P_e^a, of the components in the decomposition. Minor readjustments in the remainder of the argument then give the desired result; and, indeed, it can be seen that each component is "effectively covered" by the whole set, in a sense we shall shortly define.

However, the bookkeeping details which would attend a formalization of the argument appear formidable; and, therefore, it seems worthwhile to take note of a less technically rococo proof, available in the special case of *creative* sets, of the existence of an infinitary decomposition, with effectively indexed components[1] and with each component "effectively covered" by the whole set. We emphasize, however, that the theorem does also hold for those essentially r.e. sets which do not happen to be creative, though for such sets the more complicated variation on Friedberg's argument, or something like it, seems to be needed.

DEFINITION. *Let α be an essentially r.e. set. An essentially r.e. set β is said to be an effective cover of* $\alpha \longleftrightarrow \mathrm{df}\ \alpha \subseteq \beta \;\&\; \overline{\overline{\beta - \alpha}} = \aleph_0 \;\&\sim (\exists\gamma)$ (γ is a recursive set & $\alpha \subset \gamma \subset \beta$).

We proceed now to the theorem which is the object of this note.

THEOREM. *Let β be a creative set. Then, there is an effectively enumerable class, $\Gamma = \{\omega_{f(i)} \mid i \in N\}$ (here N is the set of all natural numbers, and f a recursive function), such that the $\omega_{f(i)}$ are pairwise-disjoint, essentially r.e. sets, $\beta = \bigcup\Gamma$, and each $\omega_{f(i)}$ is effectively covered by β.*

PROOF. Since β is creative, $\tilde{\beta}$ has a productive function; and indeed, by results of Dekker ([2, p. 135]) and Myhill ([2, p. 149], [5], and [3, p. 32]),

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$\widetilde{\beta}$ has a 1-1 increasing, *total* recursive productive function, say, $p(x)$. As in [4, p. 69], let $h(x, y)$ be a recursive function such that, for all $i, j \in N$, $h(i, j) > j$ and $\omega_{h(i,j)} = \omega_i$. Define a recursive function $h^*(x)$ as follows:

$$h^*(0) = df\ h(0, 0);$$

$$h^*(s+1) = df \begin{cases} h(k, h^*(s)), \text{ in case } s+1 \text{ is the } k\text{-th non-(positive power of a prime);} \\ h(j, h^*(s)), \text{ if } s+1 \text{ is the } (j+1)\text{-st power of some prime.} \end{cases}$$

Thus, an effective subsequence of range(h^*), enumerating a list of indices for all the r.e. sets, corresponds to (i) the sequence of non-(positive power of a prime) numbers, and (ii) each of the various sequences of powers q^k, $k \geqq 1$, q some fixed prime.

Now, let $h^*_{-1}(x)$ be that recursive function which generates, in order of magnitude, those elements of range(h^*) corresponding to non-(positive power of a prime) numbers; and, for each $i \in N$, let $h^*_i(x)$ be that recursive function which generates, in order of magnitude, those elements of range(h^*) corresponding to positive powers of the i-th prime (in order of magnitude of primes). It is easily verified that all of the functions $p \circ h^*_i$, where $i = -1$ or $i \in N$, are 1-1 increasing total recursive productive functions for $\widetilde{\beta}$, and that they possess pairwise-disjoint ranges. In fact, for $i \in N$, we have:

$$\text{range}(p \circ h^*_i) = \{x \mid (\exists z)\ (z > 0 \ \&\ x = p(h^*((i\text{-th prime})^z)))\}.$$

Hence, by [4, p. 71], the operation $\Phi(\omega_i) = df$ range $(p \circ h^*_i)$, $i \in N$, is effective; i.e., there is a recursive function $g(x)$ such that, for all $i \in N$, range$(p \circ h^*_i) = \omega_{g(i)}$.

We now assert that, for $i \in N$, $\beta \cap$ range$(p \circ h^*_i)$ is essentially r.e. and effectively covered by β.[2] That $\beta \cap$ range$(p \circ h^*_i)$ is r.e. is obvious. Suppose there were a recursive set γ such that $(\beta \cap \text{range}(p \circ h^*_i)) \subseteqq \gamma \subset \beta$. Then, since range$(p \circ h^*_i)$ is recursive ($p \circ h^*_i$ being 1-1 increasing), we have: $\widetilde{\beta} \cap \text{range}(p \circ h^*_i) = \widetilde{\gamma} \cap \text{range}(p \circ h^*_i)$ = a recursive set; and this, since $p \circ h^*_i$ is productive for $\widetilde{\beta}$, leads to a contradiction.

Again, we assert that $\beta - \bigcup_{i \in N} \text{range}(p \circ h^*_i)$ is essentially r.e. and effectively covered by β. It is r.e., since (noting the 1-1 increasing character of $p(x)$ and of the recursive function which picks up, in order of magnitude, the elements of range(h^*) - range(h^*_{-1})) the set $\bigcup_{i \in N}$ range$(p \circ h^*_i)$ is recursive. Suppose there were a recursive set γ such that $(\beta - \bigcup_{i \in N} \text{range}(p \circ h^*_i)) \subseteqq \gamma \subset \beta$; then, $\widetilde{\beta} \cap \text{range}(p \circ h^*_{-1}) = \widetilde{\gamma} \cap \text{range}(p \circ h^*_{-1})$ = a recursive set, and we arrive at the same sort of contradiction as before, since $p \circ h^*_{-1}$ is productive for $\widetilde{\beta}$.

Lastly, we define a recursive function $r(x)$ by:

$$r(0) = df\ j_0, \text{ where } j_0 \text{ is some fixed index of } \beta - \bigcup_{i \in N} \text{range}(p \circ h^*_i);$$
$$r(s+1) = df\ g(s).$$

Then, we have that $\beta = \bigcup \{\omega_{r(i)} \mid i \in N\}$ is an effectively indexed decomposition of β into $\aleph_0$ essentially r.e. components each effectively covered by β.

Q.E.D.

NOTES

1. The question whether an *effectively indexed* infinitary decomposition might, at least in the creative case, be obtained, was put to the writer by R. Montague.

2. The writer's original handling of this portion of the proof was somewhat clumsier. The present form of the argument showing that the sets range $(p \circ h_i^*) \cap \beta$, $i \in N$, are essentially r.e. and effectively covered by β, is the result of a suggestion by William Lambert.

3. (Added October 9, 1962.) (a) As pointed out to us by R. Montague, in the theorem of this note we can take the range of the indexing function f for Γ to be not only r.e., but *recursive*. For, f can be required to be 1-1 increasing. This remark applies both to the exhibited discussion of the creative case, and to the general result for arbitrary essentially r.e. sets.

   (b) The creative case of the theorem can also be proven, as a third alternative, by means of the powerful Myhill isomorphism theorem. (See [5].) Without any reference to that theorem, however, we are able to prove all parts of the following decomposition theorem, from which the creative case of the theorem of this note, with some added embellishments, follows. (For the notation 'Dom( )', see [2].)

   Theorem. Let $2 \leqq k \leqq \aleph_0$, *and let* f *be a partial recursive function productive for a set* α. *Then, there is an infinite recursive set* β *such that:*

   (i) $j \in \beta \to f_j$ *is a recursive function, with* $range(f_j)$ *recursive, which is both productive and contraproductive for* α;

   (ii) $\tilde{\alpha} = \bigcup_{j \in N} (\tilde{\alpha} \cap range(f_j))$;

   (iii) $[j, k \in \beta \ \& \ j \neq k] \to range(f_j) \cap range(f_k) = \phi$;

   (iv) $j \in \beta \to [range(f_j|_{Dom(\alpha)}) \subsetneqq range(f|_{Dom(\alpha)}) \ \& \ f_j$ *is 1-1* $\& \ f_j$ *is increasing on* $Dom(\alpha)]$; *and,*

   (v) $\tilde{\alpha}$ *r.e.* $\to$ *each component,* $\tilde{\alpha} \cap range(f_j)$, *is creative.*

   (c) From the (general version of the) theorem of the present note, taken as applying to *simple* sets, follows a more explicit form of a result due originally to Mučnik. Namely, we can show: if $\mathfrak{M}$ is a *maximal* r.e. set, in the sense of [1], then there is an infinite recursive set Γ such that (i) $[i, j \in \Gamma \ \& \ i \neq j] \to \omega_i \cap \omega_j = \phi$; (ii) $\mathfrak{M} = \bigcup_{i \in \Gamma} \omega_i$; and (iii) $[i, j \in \Gamma \ \& \ i \neq j] \to \omega_i$ and ω_j are recursively, but not effectively, inseparable.

REFERENCES

[1] R. Friedberg, Three Theorems on Recursive Enumeration, *The Journal of Symbolic Logic,* Vol. 23, No. 3, 1958.

[2] J. C. E. Dekker, Productive Sets, *Transactions AMS*, Vol. 78, 1955.

[3] J. C. E. Dekker and J. Myhill, Some Theorems on Classes of Recursively Enumerable Sets, *Transactions AMS*, Vol. 89, 1958.

[4] R. M. Smullyan, *Theory of Formal Systems*, Princeton, 1961.

[5] J. Myhill, Creative Sets, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, Vol. 1, 1955.

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