Notre Dame Journal of Formal Logic
Volume III, Number 3, July 1962

FINITE LIMITATIONS ON DUMMETT'S **LC**

IVO THOMAS

The propositional system **LC** of [1] can be based on axioms for $\supset$ (implication), $\wedge$ (conjunction), a constant **f**, and definitions for $\vee$ (alternation) and $\urcorner$ (negation), as hereunder. In primitive notation, elementary variables and **f** are wffs, and if α, β are wffs so are $(\alpha \supset \beta)$, $(\alpha \wedge \beta)$. To restore primitive notation in the sequel, replace dots by left parentheses with right terminal mates; in a sequence of wffs separated only by implications, restore parentheses by left association; enclose the whole in parentheses. If S is a system, S_c is its implicational fragment, containing only variables and implications. If α is provable (not provable) in S, we write $\vdash_{S} \alpha$ ($\dashv_{S} \alpha$); if α is uniformly valued *0* (is not uniformly valued *0*) by the matrix $\mathfrak{M}$, we write $\vdash_{\mathfrak{M}} \alpha$ ($\dashv_{\mathfrak{M}} \alpha$). As a basis for **LC** we take, with detachment and substitution, the axioms and definitions:

1 $\quad p \supset . q \supset p$
2 $\quad p \supset (q \supset r) \supset . p \supset q \supset . p \supset r$
3 $\quad p \supset q \supset r \supset . q \supset p \supset r \supset r$
4 $\quad \mathbf{f} \supset p$
5 $\quad (p \wedge q) \supset p$
6 $\quad (p \wedge q) \supset q$
7 $\quad p \supset . q \supset (p \wedge q)$

Def. $\vee$ $\quad (\alpha \vee \beta) = (\alpha \supset \beta \supset \beta) \wedge (\beta \supset \alpha \supset \alpha)$
Def. $\urcorner$ $\quad \urcorner\alpha = \alpha \supset \mathbf{f}$

[2] shows that 1-3 suffice for $\mathbf{LC}_c$, and it is well known that 1-2 suffice for $\mathbf{IC}_c$, the positive logic. By [1] the infinite adequate matrix for **LC** is $\mathfrak{M} = \langle M, \{0\}, \wedge, \supset, \mathbf{f} \rangle$ where $M = \{0, 1, 2, \ldots, \omega\}$ and

$$a \wedge b = \max(a, b),$$
$$a \supset b = \begin{cases} 0 \text{ if } a \geqq b, \\ b \text{ if } a < b, \end{cases}$$
$$\mathbf{t} = \omega .$$

Received November 14, 1961

Axioms are now to be given for **LC**n and **LC**n_c with finite adequate matrix $\mathfrak{M}_n = < \{0, \ldots, n\}, \{0\}, \wedge, \supset, \mathsf{f} >$ where n is a natural number, implication and conjunction are valued as by $\mathfrak{M}$, $\mathsf{f} = \max(0, \ldots, n)$. Taking variables '$p_o$', $p_1, \ldots, p_n$ we define:

$$3_n \begin{cases} 3_o = p_o \\ 3_{n+1} = p_n \supset p_{n+1} \supset p_o \supset 3_n \,. \end{cases}$$

Replacing 3 by 3_n we obtain the required axioms. To prove this it will be enough to consider 1-2, 3_n, 4, since conjunction is eliminable by the inferential equivalences:

$$(\alpha \wedge \beta) \supset \gamma \sim \alpha \supset . \beta \supset \gamma;$$
$$\alpha \supset (\beta \wedge \gamma) \sim \alpha \supset \beta, \alpha \supset \gamma;$$
$$\alpha \wedge \beta \sim \alpha, \beta.$$

THEOREM I. **LC**n_c contains **LC**$_c$.

Proof. In 3_n replace p_o by r, p_i by p if i is odd, by q if i is even. Then every antecedent is $p \supset q \supset r$ or $q \supset p \supset r$ except one which is $r \supset p \supset r$, and the consequent is r. Where one or more of these antecedents is missing it may be added by **IC**$_c$, by which also these antecedents can by commuted and reduced so as to obtain:

$$\vdash_{\mathbf{LC}n_c} p \supset q \supset r \supset . \; q \supset p \supset r \supset . \; r \supset p \supset r \supset r \qquad (1).$$

Further

$\vdash_{\mathbf{IC}_c} p \supset q \supset r \supset . \; q \supset p \supset r \supset . \; r \supset p \supset r,$ so that by **IC**$_c$ and (1) we have

$$\vdash_{\mathbf{LC}n_c} 3.$$

THEOREM II. $\mathfrak{M}_n$ verifies **LC**n.

Since 3_n alone involves an addition to **LC**, we need only consider this. Let $\overline{p_i}$ be the value of p_i. Then for all n, 3_n—containing $n + 1$ variables—fails to obtain the value 0 if and only if $0 < \overline{p_o} < \overline{p_1} < \ldots < \overline{p_n}$, i.e. if and only if it is valued by some $\mathfrak{M}_m$ with $m > n$.

THEOREM III. $\vdash_{\mathbf{LC}_c} 3_n \supset \subset 3'_n$, with $3'_n$ defined:

$$3'_n \begin{cases} 3_o = p_o \\ 3_{n+1} = p_{n+1} \supset p_n \supset . \; p_n \supset p_{n+1} \supset p_{n+1} \supset 3'_n \,. \end{cases}$$

Proof. By induction on n. From right to left there is required the **LC**$_c$-thesis:

$\vdash_{\mathbf{LC}_c} \quad p \supset q \supset q \supset r \supset . p \supset q \supset r \supset r .$

If a wff is of the form $\alpha \supset \beta \supset . \beta \supset \alpha \supset \alpha \supset \gamma$ we shall write $\alpha \to \beta \supset . \gamma$; and where we have $\alpha_n \to \alpha_{n-1} \supset . . . . \supset . \alpha_1 \to \alpha_0 \supset . \beta$ $(n > 0)$ we shall say that there is an n-length arrow chain to α_0 among the antecedents. Using this terminology, for $n > 0$, $3'_n$ has an n-length arrow chain to p_o, and consequent p_o.

We now modify the normal forms of [2] for $\mathbf{LC}_c$-wffs by adding the productions:

(A) $\pi \supset \rho \supset . \rho \supset \pi \supset \alpha$ yields $\alpha\, \pi/\rho$
(B) antecedents $\alpha \supset \beta$, $\beta \to \gamma$ add antecedents $\alpha \to \gamma$
(C) . . . . $\alpha \to \beta$, $\beta \supset \gamma$ . . . . $\alpha \to \gamma$

without loss of inferential equivalence. For the reader's information we note that any normal form not provable in $\mathbf{LC}_c$ has all its antecedents $\tau \supset \nu \supset \nu$ or $\rho \supset \sigma$, and consequent ϕ, with ρ, σ, τ, ν, ϕ elementary variables. Not both $\tau \supset \nu \supset \nu$, $\tau \supset \nu$ are present, and if $\rho \supset \sigma$, $\sigma \supset \tau$ are both present, so is $\rho \supset \tau$. We can now state:

THEOREM IV. If α is an $\mathfrak{M}$-rejected normal form in $\mathbf{LC}_c$ with consequent π_o, and the longest arrow chain to π_o in α is of length $n \geqq 1$, rejection can be effected in the range of values $0, . . . , n + 1$ and α is inferentially equivalent by $\mathbf{LC}_c$ to $3'_n$. If there is no arrow-chain to π_o, α is rejected in the values 0, 1 and is inferentially equivalent to $3'_o$.

Proof. (Case 1) α has a tail $\pi_n \to \pi_{n-1} \supset . . . . \supset . \pi_1 \to \pi_o \supset . \pi_o$. Associated with the antecedents by (B), (C) will be $\pi_i \to \pi_j$ for all i, j such that $n \geqq i > j \geqq 0$. By elementary combinatory considerations and the conditions on normal forms, all possible further antecedents are covered by the following six types:

$\rho_1 \to \rho_2$, $\rho_2 \to \rho_3$, ... , $\rho_k \to \pi_i$; $i < n$, $k \leqq n - i$.

$\pi_i \to \sigma_l$, $\sigma_l \to \sigma_{l-1}$, ... , $\sigma_2 \to \sigma_1$, $\sigma_1 \to \pi_j$; $l \leqq i - j$, $n \geqq i > j \geqq 1$.

$\pi_i \to \tau_1$, $\tau_1 \to \tau_2$, ... , $\tau_{m-1} \to \tau_m$; $i \leqq n$, and not $\tau_a \supset \pi_j$ for any $a \leqq m$, $j \leqq n$.

$\nu \to \phi_1$, $\phi_1 \to \phi_2$, ... , $\phi_{q-1} \to \phi_q$; and not $\pi_i \supset \nu$ or $\phi_q \supset \pi_i$ for $i \leqq n$.

$\pi_i \supset \psi \supset \psi$; and no syllogistic chain from ψ to π_i.

$\chi \supset \pi_a$, $\chi \supset \pi_b$, ... ; $a, b, . . . \leqq n$, and not $\pi_i \supset \chi \supset \chi$ for $i \leqq n$.

Therein for all τ, ϕ, ψ we can substitute $\pi \supset \pi$ to obtain antecedents valued 0, while substitution of π_{j+s} for ρ_s and σ_s, of π_r $(r = \max(a, b, ...))$ for χ, produces antecedents already present in or associated with the n-length arrow chain to π_o. We thus obtain an expression $\mathbf{LC}_c$-equivalent to $3'_n$, and which, when π_i is valued $i + 1$, reduces by $\mathfrak{M}$ to the value 1, having used only the values $0, . . . , n + 1$.

(Case 2) Where there is no arrow-chain to π_o the first two types of antecedent are not present. Remaining types can be verified as in Case 1 and we are left with an expression $\mathbf{LC}_c$-equivalent to $3'_o$, reducing by $\mathfrak{M}$ to the value 1, having used only the values 0, 1.

THEOREM V. For all natural n, $\mathbf{LC}n_c$ is complete for $\mathfrak{M}_n$.

Proof. $\mathbf{LC}0_c$ is obviously complete for $\mathfrak{M}_o$. If $\vdash_M \alpha$, then by [2] $\vdash_{\mathbf{LC}} \alpha$ and so (Theorem I) $\vdash_{\mathbf{LC}_{n+1}} \alpha$; while if $\dashv_M \alpha$ and $\vdash_{M_{n+1}} \alpha$, then by [2] and Theorem IV all normal forms of α are either $\mathbf{LC}_c$-provable or have arrow chains to the consequent of length at least $n + 1$. But all such are $\mathbf{LC}_c$-implied by $3'_{n+1}$ and so (Theorem III) by 3_{n+1}.

Taking now f into account, we add to the reduction process of 2:

(D)	$\alpha \supset \mathsf{f}$	$\sim$	$\alpha \supset \pi$ (π not in α),
(E)	$\mathsf{f} \supset \alpha$	$\sim$	$\mathsf{f} \supset . \alpha \supset \alpha$,
(F)	$\mathsf{f} \supset \underline{\alpha} \supset \beta \supset \gamma$	$\sim$	$\beta \supset \gamma$,
(G)	$\mathsf{f} \supset \alpha \supset \beta$	$\sim$	β,
(H)	$\alpha \supset \mathsf{f} \supset \mathsf{f} \supset \beta$	$\sim$	$\mathsf{f} \supset \alpha \supset . \alpha \supset \mathsf{f} \supset \beta$,

(G) not to be used where $\alpha \supset \mathsf{f} \supset \mathsf{f}$ is present. Then in $\mathfrak{M}$-rejected normal forms f can only occur in the positions $\pi \supset \mathsf{f}$ and $\mathsf{f} \rightarrow \rho$ and in any arrow chain only as its opening member.

THEOREM VI. If α is as in Theorem IV with f occurring only as just stated, then etc. as in Theorem IV. Proof is exactly similar, giving f the value $n + 1$.

THEOREM VII. $\mathbf{LC}n$ is complete for $\mathfrak{M}_n$. This follows from Theorem VI, as Theorem V from Theorem IV.

THEOREM VIII. If $3''_n$ is defined by means of $3''_n$ as below, then 3_n may replace 3_n in the axioms of $\mathbf{LC}n$.

$$3''_n \begin{cases} 3''_o = p_o \\ 3''_{n+1} = p_n \supset \mathsf{f} \supset p_o \supset 3_n . \end{cases}$$

MODAL CONSEQUENCES. Using the McKinsey-Tarski translation T of [3] to obtain $T(3_n)$ we have axioms for a denumerably infinite series of modal systems, S4 with $T(3_n)$, between S5 (i.e. S4 with $T(3_1)$) and S4·3 (i.e. S4 with $T(3)$), to use the numeration of [4]. It seems appropriate to call these systems S4·3_n.

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Blackfriars
Oxford, England