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A SET-THEORETICAL FORMULA EQUIVALENT TO THE AXIOM OF CHOICE

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It is obvious that the following set-theoretical formula:[1]

S1 *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, *then there is no cardinal* $\mathfrak{p}$ *such that* $\mathfrak{m} < \mathfrak{p} < \mathfrak{n}$.

is a simple consequence of the theorem:

$\mathfrak{A}$. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, *then* $\mathfrak{m} = \mathfrak{n}$.

which, as it is proved in [3], p. 230, is inferentially equivalent to the axiom of choice. Although at first glance it appears that formula **S1** is weaker than $\mathfrak{A}$, in fact, as I shall show in this note, the former formula implies the axiom of choice, and, therefore, it is inferentially equivalent to $\mathfrak{A}$. For, a proof is given here that the following theorem:

A. *For any cardinal number* $\mathfrak{m}$ *which is not finite, if* $\aleph(\mathfrak{m})$ *is the least Hartogs' aleph with respect to* $\mathfrak{m}$, *then there is no cardinal* $\mathfrak{p}$ *such that* $\aleph(\mathfrak{m}) < \mathfrak{p} < \mathfrak{m} + \aleph(\mathfrak{m})$.

which is inferentially equivalent to the axiom of choice, as it is proved in [2], follows from **S1** without the aid of the said axiom.

Proof: Let us assume **S1** and consider that

(i) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite,*

and that

(ii) $\aleph(\mathfrak{m})$ *is the least Hartogs' aleph with respect to* $\mathfrak{m}$.

Then, obviously, we have

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(iii) $\aleph(\mathfrak{m}) \leqslant \mathfrak{m} + \aleph(\mathfrak{m})$

and, hence (iii) together with the theorem **T1** which is mentioned in [3], p. 229,[2] and which is provable without the aid of the axiom of choice, implies at once

(iv) $\aleph(\aleph(\mathfrak{m})) \leqslant \aleph(\mathfrak{m} + \aleph(\mathfrak{m}))$

Since the following theorem of Tarski[3]

T2 *If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are two cardinals different from 0 and not both finite, then* $\aleph(\mathfrak{m} + \mathfrak{n}) = \aleph(\mathfrak{m}) + \aleph(\mathfrak{n})$

is provable without the aid of the axiom of choice, the case

1. $\aleph(\aleph(\mathfrak{m})) < \aleph(\mathfrak{m} + \aleph(\mathfrak{m}))$

of (iv) is impossible, because it together with (i), (ii), **T2** and the elementary properties of Hartogs' alephs gives at once

2. $\aleph(\aleph(\mathfrak{m})) < \aleph(\mathfrak{m} + \aleph(\mathfrak{m})) = \aleph(\mathfrak{m}) + \aleph(\aleph(\mathfrak{m})) = \aleph(\aleph(\mathfrak{m}))$

Hence, the second case of (iv) holds, viz.

(v) $\aleph(\aleph(\mathfrak{m})) = \aleph(\mathfrak{m} + \aleph(\mathfrak{m}))$

which together with the assumed formula **S1** implies

(vi) *there is no cardinal* $\mathfrak{p}$ *such that* $\aleph(\mathfrak{m}) < \mathfrak{p} < \mathfrak{m} + \aleph(\mathfrak{m})$

Thus, theorem **A** follows from **S1** without the aid of the axiom of choice, and, therefore, our proof is completed.

It should be noted that a slight modification of this proof shows that the following formula:

S2 *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\aleph(\mathfrak{m})$ *and* $\aleph(\mathfrak{n})$ *are the least Hartogs' alephs with respect to* $\mathfrak{m}$ *and* $\mathfrak{n}$ *respectively, and such that* $\aleph(\mathfrak{m}) = \aleph(\mathfrak{n})$, *then it is not true that* $\mathfrak{m} < \mathfrak{n}$.

is also inferentially equivalent to the axiom of choice.

Notes

1. Concerning a definition of the so-called Hartogs' alephs *cf.*, e.g., [3], p. 234, note 1. In the same place there is given a description of the general set theory in the field of which the proofs presented in this paper are carried on.

2. This theorem is due to Tarski, *cf.* [1], p. 311, theorem **77**.

3. *Cf.* [1], p. 311, theorem **76**, and [4], p. 30.

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