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RECURSIVE LINEAR ORDERINGS AND HYPERARITHMETICAL FUNCTIONS

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The main purpose of this note is to give an alternative proof to a theorem by Spector [1] which answers a question raised by Kleene [3, p. 25]. There are two by-products. The first (Theorem 1) specifies a sufficient condition for a set linearly ordered by a recursive ordering to have a well-ordered segment of a certain order type.[1] The second (Theorem 2) is a generalization, in some sense, of a theorem of Kleene [4, XXVL]. This enables us to apply Kleene's [3, Theorem 2] directly in our proof of Spector's theorem (Theorem 3 in this note). So it seems that the proof becomes much shorter.[2]

We first introduce some notations. $f \in \mathbf{L} \equiv \{f$ is a Gödel number of some recursive linear ordering $\overset{f}{\prec}$ which orders some set $M_f\}$ [2]. $f \in \mathbf{W} \equiv \{f \in \mathbf{L}$ & M_f is well-ordered by $\overset{f}{\prec}\}$ [2]. $\mathbf{S}(f, n)$ is a primitive recursive function such that $f \in \mathbf{L}$ implies (i) $\mathbf{S}(f, n) \in \mathbf{L}$ for all n, (ii) if $n \notin M_f$, $M_{\mathbf{S}(f, n)}$ is empty, (iii) if $n \in M_f$, $M_{\mathbf{S}(f, n)}$ is a segment $\hat{x}(x \overset{f}{\prec} n)$ of M_f and $x \overset{\mathbf{S}(f, n)}{\prec} y \equiv x \overset{f}{\prec} y$ for all $x, y \in M_{\mathbf{S}(f, n)}$ [2, p. 156]. $\|f\|$ is the order type of $\overset{f}{\prec}$ if $f \in \mathbf{L}$, $|b|$ is the order type named by b, if $b \in 0$ [2]. y^* stands for 2^y, $H_y(u)$ is defined as in [2].

Theorem 1. If $f \in \mathbf{L}$, $f \notin \mathbf{W}$, $y \in 0$ and for every function $\alpha(i)$ recursive in $H_{y^{**}}$, (i) $(\alpha(i+1) \overset{f}{\prec} (i))$, then for every $b \in 0$ with $|b| < |y|$, there is some $n \in M_f$ such that $|b| = \|\mathbf{S}(f, n)\|$.

Proof (by induction on the ordinal $|b|$). The proof for the case $|b| = 0$ is simple.

Suppose $0 < |b| < |y|$. Let enm(i) be a primitive recursive function which enumerates all the numbers $<_0 b$ [6]. By the induction hypothesis, for every i, there is some $n_i \in M_f$ such that $|\mathrm{enm}(i)| = \|\mathbf{S}(f, n_i)\|$. Let n_i be determined as a total function of i by $n_i = \mu z(z \in M_f \;\&\; |\mathrm{enm}(i)| = \|\mathbf{S}(f, z)\|)$. Note that $|\mathrm{enm}(i)^{**}| \leqq |b^*| \leqq |y|$ we see that n_i is recursive in H_y by [2, Theorem 3 and Theorem 5].

Since $\mathbf{S}(f, n_i) \in \mathbf{W}$ for every i and by the supposition of the theorem,

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$f \notin \mathbf{W}$, there must be some (indeed, infinitely many) $x \in M_f$ such that $(i)\ (n_i \overset{f}{\prec} x)$. Let a total function $\delta(i)$ be defined by

$$\delta(0) = \mu z(i)\ (n_i \overset{f}{\prec} z);$$

$$\delta(i+1) = \delta(i), \text{ if } \overline{(Ez)}\ ((i)\ (n_i \overset{f}{\prec} z)\ \&\ z \overset{f}{\prec} \delta(i));$$

$$\delta(i+1) = \mu z((i)(n_i \overset{f}{\prec} z)\ \&\ z \overset{f}{\prec} \delta(i)), \text{ otherwise.}$$

It can be seen that $\delta(i)$ is recursive in $(Ez)\ ((i)\ (n_i \overset{f}{\prec} z)\ \&\ z \overset{f}{\prec} x)$. Since as has been shown, n_i is recursive in H_y, $(Ez)\ (i)\ (n_i \overset{f}{\prec} z)\ \&\ z \overset{f}{\prec} x)$ is recursive in H_{y**} (by [5, Lemma 1] and the definition $H_{b*}(u) \equiv (Ec)\ T_1^{H_b}(c, c, u))$. So $\delta(i)$ is recursive in H_{y**}.

There must be some number, say, i_0 such that $\delta(i_0 + 1) = \delta(i_0)$. For otherwise, by the definition of $\delta(i)$, $(i)\ (\delta(i+1) \overset{f}{\prec} \delta(i))$. This contradicts the hypothesis of the theorem. Then $\delta(i_0)$ is the least t (in the sense of $\overset{f}{\prec}$) in M_f such that $(i)\ (n_i \overset{f}{\prec} t)$ and therefore $\|\mathbf{S}(f, \delta(i_0))\|$ is the least ordinal ζ such that $(i)\ (\|\mathbf{S}(f, n_i)\| < \zeta)$. Since $|\text{enm}(i)| = \|\mathbf{S}(f, n_i)\|$ and $|b|$ is the least ordinal ζ such that $(i)\ (|\text{enm}(i)| < \zeta)$, then $|b| = \|\mathbf{S}(f, \delta(i_0))\|$. This completes the proof.

Let $\alpha \in HA$ mean that α is hyperarithmetical, i.e. there is some $y \in O$ such that α is recursive in H_y [4, p. 201].

Corollary. If $f \in \mathbf{L}$, $f \notin \mathbf{W}$ and for all $\alpha \in HA$, $\overline{(i)}\ (\alpha(i+1) \overset{f}{\prec} \alpha(i))$, then for every $b \in O$, there is some $n \in M_f$ such that $|b| = \|\mathbf{S}(f, n)\|$.

Theorem 2. For any recursive $R(\alpha, a, x)$, there is a recursive $R'(s, a)$ such that $(\alpha)\ (Ex)\ R\ (\alpha, a, x) \equiv (\alpha)\ (Ex)\ R'(\bar{\alpha}(x), a)$ and for no $\alpha \in HA$, $(x)\ \overline{R'}(\bar{\alpha}(x), a)$.

Proof. By the technique of [3, Lemma 1], we can find a recursive A such that $(\alpha)\ (Ex)\ A(\alpha, c, x) \equiv (E\alpha)_{\alpha \in HA}\ (x)\ \overline{T}_1^{\alpha}\ (c, c, x)$.

Let $(\alpha)\ (Ex)\ R(\alpha, a, x) \vee (\alpha)\ (Ex)\ A(\alpha, c, x)$ . . . . (a)

$\equiv (\alpha)\ (Ex)\ B(\alpha, a, c, x)$ (with B recursive)

$\equiv (\alpha)\ (Ex)\ T_1^{\alpha}(\sigma(a), c, x)$ (with recursive σ, by [5, Lemma 12]).

By (a), $(\alpha)\ (Ex)\ A(\alpha, \sigma(a), x) \rightarrow (\alpha)\ (Ex)\ T_1^{\alpha}(\sigma(a), \sigma(a), x)$. On the other hand $(\ )\ (Ex)\ A(\alpha, \sigma(a), x) \rightarrow (E\alpha)\ (x)\ \overline{T}_1^{\alpha}\ (\sigma(a), \sigma(a), x)$. Thus we have i) $(\bar{\alpha})\ (Ex)\ A(\alpha, \sigma(a), x)$. By (a) and i), we have ii) $(\alpha)\ (Ex)\ R(\alpha, a, x) \equiv (\alpha)(Ex)\ T_1^{\alpha}\ (\sigma(a), \sigma(a), x)$. By the meaning of A, i) implies iii) for no $\alpha \in HA$, $(x)\ \overline{T}_1^{\alpha}\ (\sigma(a), \sigma(a), x)$. From ii) and iii) we see that $T_1^{1}(s, \sigma(a), \sigma(a), lh(s))$ is a recursive $R'(s, a)$ as required. This completes the proof of Theorem 2.

For the predicate $R'(s, a)$, we find a recursive function $\xi(a)$ such that $\xi(a) \in \mathbf{L}$, $n \in M_{\xi(a)} \equiv \{n$ is a sequence number $\bar{\alpha}(x)\ \&\ (t)_{t<x}\ \overline{R'}\ (\bar{\alpha}(t), a)\}$ and

$\xi(a) \epsilon \mathbf{W} \equiv (\alpha)(Ex) R'(\overline{\alpha}(x), a)$ [2, Theorem 1]. In case $(\overline{\alpha})(Ex) R'(\overline{\alpha}(x), a)$, since for no $\alpha \epsilon HA$, $(x) \overline{R}'(\overline{\alpha}(x), a)$, we have that for no $\alpha \epsilon HA$, $(i)(\alpha(i+1) \overset{\xi(a)}{\prec} \alpha(i))$ by arguments similar to [6, (J)]. Then by the corollary of Theorem 1, we have

Lemma 1. For any recursive $R(\alpha, a, x)$, there is a recursive function $\xi(a)$ such that i) if $(\alpha)(Ex) R(\alpha, a, x)$ then $\xi(a) \epsilon \mathbf{W}$, and ii) if $(\overline{\alpha})(Ex) R(\alpha, a, x)$ then for every $b \epsilon 0$, there is some n such that $|b| = \|\mathbf{S}(\xi(a), n)\|$.

Theorem 3 (by Spector). For any recursive $R(\alpha, a, x)$ there is a recursive $\mathbf{S}(\alpha, a, x)$ such that $(E\alpha)_{\alpha \epsilon HA}(x) \mathbf{S}(\alpha, a, x) \equiv (\alpha)(Ex) R(\alpha, a, x)$.

Proof. By [7, Theorem 1], we can find a recursive function k such that $f \epsilon \mathbf{W} \rightarrow k(f) \epsilon 0$ & $\|f\| \leqq |k(f)|$ and $\|f\| < \|g\| \rightarrow |k(f)| < |k(g)|$. Let $\xi(a)$ be the recursive function of Lemma 1, and $k(\mathbf{S}(\xi(a), n))$ be abbreviated to $y(a, n)$. Let f_0, f_1 be the recursive functions of [3, Theorem 2] so that for any $y \epsilon 0$, $(E\alpha)(x) \overline{T}_1^{\alpha}(f_0(y), t, x)$ or $(E\alpha)(x) \overline{T}_1^{\alpha}(f_1(y), t, x)$ according as $H_y(t)$ or not. Now let us consider the following predicate of γ and β.

$$\text{(A)}\quad (n)(t)[(\gamma(2^n . 3^t) = 0 \ \& \ (x) \overline{T}_1^{\lambda s \beta(2^n . 3^t . 5^s)}(f_0(y(a, n)), t, x)) \vee$$

$$(\gamma(2^n . 3^t) = 1 \ \& \ (x) \overline{T}_1^{\lambda s \beta(2^n . 3^t . 5^s)}(f_1(y(a, n)), t, x))].$$

Case 1. $(\overline{\alpha})(Ex) R(\alpha, a, x)$. Suppose (A) is true, we can show $\gamma \not\epsilon HA$. By the meanings of f_0, f_1, (A) implies that for any fixed $y(a, n) \epsilon 0$, (1) $\lambda t \gamma (2^n . 3^t)$ is the representing function of $\lambda t H_{y(a, n)}(t)$ and therefore (2) $\lambda t H_{y(a, n)}(t)$ is recursive in γ. By Lemma 1, we have (3) that for suitable n, $y(a, n) \epsilon 0$ and $|y(a, n)| > |z|$, any pre-assigned constructive ordinal. Since given any $\gamma' \epsilon HA$ (γ' recursive in, say, H_z), all H_y with $|y| > |z|$ are not recursive in γ', then from (2) and (3) it follows that $\gamma \not\epsilon HA$.

Case 2. $(\alpha)(Ex) R(\alpha, a, x)$. We can find $\gamma, \beta \epsilon HA$ such that γ and β satisfy (A). By Lemma 1, $\xi(a) \epsilon \mathbf{W}$. Then $k(\xi(a)) \epsilon 0$, $y(a, n) \epsilon 0$ and $|y(a, n)| < |k(\xi(a))|$ for every n. By [2, Theorem 5], $\lambda n t H_{y(a, n)}(t)$ is recursive in $H_{k(\xi(a))}$. A function $\gamma \epsilon HA$ *is* defined by $\gamma(x) = 0$ if $x \neq 2^n . 3^t$, and $\gamma(2^n . 3^t) = 0$ or 1 according as $H_{y(a, n)}(t)$ or not. A β is defined by $\beta(x) = 0$ if $x \neq 2^n . 3^t . 5^s$, and $\beta(2^n . 3^t . 5^s) = \{d_j(y(a, n), t)\}(H_{w_j(y(a, n), t)}, s)$ where j is 0 or 1 according as $H_{y(a, n)}(t)$ or not, and d_j, w_j are as defined in [3, Theorem 2]. Then it can be seen that γ and β satisfy (A). $\beta \epsilon HA$ because β is defined in terms of some H_b with $|b| < |k(\xi(a))|$ and then is recursive in $H_{k(\xi(a))}$.

From (A) we get $(x) \mathbf{S}'(\gamma, \beta, a, x)$ by contracting the quantifiers. $\mathbf{S}'(\lambda t(\alpha(t))_0, \lambda t(\alpha(t))_1, a, x)$ is a $\mathbf{S}(\alpha, a, x)$ for Theorem 3.

NOTES

1. We see that the concept $e \not\epsilon \mathbf{W}$ is as complicated as $e \epsilon \mathbf{W}$. We can classify all the numbers $e \epsilon \mathbf{L}$ into as many hierarchies as the constructive ordinals. For any $e, e' \epsilon \mathbf{L}$, we say that e belongs to a hierarchy higher

than that of e' if M_e contains a well-ordered segment larger than that contained by $M_{e'}$. Elsewhere the author classified all $e \in \mathbf{L}$ into countable hierarchies based upon a notion $\mathbf{L}_n(e, z)$ of [7]. We say e belongs to the n-th hierarchy if there is an infinite decreasing sequence .... $\overset{e}{\prec}$ $\alpha(i+1) \overset{e}{\prec} \alpha(i) \overset{e}{\prec} \ldots \overset{e}{\prec} \alpha(0)$ such that $(i)\ (\mathbf{L}_n(e, \alpha(i)))$. This second type of classification can help us to solve the problem raised in [7, p. 25] partly.

2. After reading the first version of this manuscript Dr. Spector showed me a manuscript of Gandy's which contained also a proof of Theorem 3. Theorem 1 of this note is essentially the same as Gandy's except the former contains some contents more specific. Other parts of both proofs were carried out through different routs.

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