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A SIMPLE PROOF OF FUNCTIONAL COMPLETENESS IN MANY-VALUED LOGICS BASED ON ŁUKASIEWICZ'S *C* AND *N*

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Past investigations, [1], [2] and [3], have used the integers $1, 2, \ldots, n$ as truth-values for an n-valued logic. In such a logic, the truth-functions associated with C and N have the following definitions

$$C(p, q) = \max(1, q - p + 1); \qquad N(p) = n - p + 1.$$

Here we shall use $n + 1$ - valued logics with truth-values $0, 1, \ldots, n$. As a result, the above definitions simplify to

$$C(p, q) = \max(0, q - p); \qquad N(p) = n - p.$$

Not only does this simplify the computations involved, but also makes a simple line of proof apparent. No logical tools are used, and the only non-trivial number-theoretic result used is "If $(a, b) = d$,[1] then there are integers x and y for which $ax + by = d$."

Theorem 1. Any function[2] which takes the value 0 once and n otherwise is generated by C and N.

1. $C(p, p) = 0$.
2. $N(0) = n$.
3. $\alpha_m(p_1, \ldots, p_m) = \min(n, p_1 + p_2 + \ldots + p_m)$ is generated for $m \geq 1$.

   Proof is by induction.

   $C(0, p_1) = p_1 = \min(n, p_1) = \alpha_1(p_1)$.

   Suppose that α_k is generated for $k \geq 1$.

   $N(\alpha_k(p_1, \ldots, p_k)) = \max(0, n - (p_1 + \ldots + p_k))$.

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[1] $(a, b) = d$ means that d is the greatest common divisor of a and b.

[2] All functions used in this paper will have $0, 1, \ldots, n$ as the domain for each argument and will take values in this set.

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$C(p_{k+1}, N(\alpha_k(p_1, \ldots, p_k))) = \max(0, n - (p_1 + \ldots + p_k) - p_{k+1}) = \max(0, n - (p_1 + \ldots + p_{k+1}))$.

$N(C(p_{k+1}, N(\alpha_k(p_1, \ldots, p_k)))) = \min(n, p_1 + \ldots + p_{k+1}) = \alpha_{k+1}(p_1, \ldots, p_{k+1})$.

4. $\beta_m(p) = \min(n, mp)$ is generated for $m \geq 0$.

$\beta_0(p) = \min(n, 0) = C(p, p)$

$\beta_m(p) = \alpha_m(p, \ldots, p)$ for $m \geq 1$.

5. $\gamma_m(p) = \begin{cases} n, & p = m \\ 0, & p \neq m \end{cases}$, is generated for $0 < m \leq n$.

Proof is by induction.

$\beta_n(p) = \begin{cases} 0, & p = 0 \\ n, & p \neq 0 \end{cases}$; $N(\beta_n(p)) = \gamma_0(p)$.

Suppose that $\gamma_i(p)$ has been generated for $0 \leq i < k \leq n$.

$$\alpha(p) = \alpha_k(\gamma_0(p), \ldots, \gamma_{k-1}(p)) = \begin{cases} n, & 0 \leq p < k \\ 0, & k \leq p \leq n \end{cases} \qquad \begin{pmatrix} 0, \ldots, k-1, & k, \ldots, n \\ n \text{---------} n, & 0 \text{-------} 0 \end{pmatrix}$$

There is an $s \geq 0$, for which $0 \leq sk < n \leq (s + 1)k$.

$$\beta_s(p) = \begin{cases} sp, & 0 \leq p \leq k \\ n, & k < p \leq n \end{cases}. \qquad \begin{pmatrix} 0, 1, \ldots, k, & k+1, \ldots, n \\ 0, p, \ldots, sk, & n \text{--------} n \end{pmatrix}$$

$$\alpha_2(\alpha(p), \beta_s(p)) = \begin{cases} sk, & p = k \\ n, & p \neq k \end{cases}. \qquad \begin{pmatrix} 0, \ldots, k-1, & k, & k+1, \ldots, n \\ n \text{---------} n, & sk, & n \text{--------} n \end{pmatrix}$$

$$N(\alpha_2(\alpha(p), \beta_s(p))) = \begin{cases} n - sk, & p = k \\ 0, & p \neq k \end{cases} \qquad \begin{pmatrix} 0, \ldots, k-1, & k, & k+1, \ldots, n \\ 0 \text{---------} 0, & n-sk, & 0 \text{---------} 0 \end{pmatrix}$$

Since $sk < n$, $n - sk \geq 1$. Thus there is a t, $0 < t \leq n$, for which $t(n - sk) \geq n$.

$$\gamma_k(p) = \beta_t(N(\alpha_2(\alpha(p), \beta_s(p)))) = \begin{cases} n, & p = k \\ 0, & p \neq k \end{cases}$$

6. $\delta_m(p) = N(\gamma_m(p)) = \begin{cases} 0, & p = m \\ n, & p \neq m \end{cases}$, is generated for $0 \leq m \leq n$.

7. $\epsilon_{k_1, \ldots, k_m}(p_1, \ldots, p_m) = \alpha_m(\delta_{k_1}(p_1), \ldots, \delta_{k_m}(p_m)) = \begin{cases} 0, & k_i = p_i \text{ for } 1 \leq i \leq m \\ n, & \text{otherwise} \end{cases}$

is generated for $m \geq 1$ and $0 \leq k_i \leq n$ for $1 \leq i \leq m$. *Q.E.D.*

Theorem 2. If f is a function of m variables, $m \geq 1$, with the value $c_{k_1, \ldots, k_m}$ in the $k_1, \ldots, k_m$ -th place, then f is generated by C, N, and all the constants $c_{k_1, \ldots, k_m}$.

8. $\pi_{k_1, \dots, k_m}(p_1, \dots, p_m) = C(\epsilon_{k_1, \dots, k_m}(p_1, \dots, p_m), c_{k_1, \dots, k_m}) =$
$$\begin{cases} c_{k_1, \dots, k_m}, & k_i = p_i \text{ for } 1 \leq i \leq m \\ \max(0, c_{k_1, \dots, k_m} - n), & \text{otherwise} \end{cases} = \begin{cases} c_{k_1, \dots, \overline{k_m}}, & k_i = p_i \text{ for } 1 \leq i \leq m \\ 0, & \text{otherwise} \end{cases}$$

9. $\alpha_{(n+1)^m}(\pi_{0, \dots, 0}(p_1, \dots, p_m), \dots, \pi_{k_1, \dots, k_m}(p_1, \dots, p_m), \dots,$
$\pi_{n, \dots, n}(p_1, \dots, p_m)) = c_{k_1, \dots, k_m}$ for $k_i = p_i$, $1 \leq i \leq m$. *Q.E.D.*

Theorem 3. If a and b are constants, $0 \leq a < b \leq n$, then every constant function of the form $ax + by$, where $0 \leq ax + by \leq n$, is generated by C, N, a, b.

Proof. By induction on $|x| + |y|$.

If $|x| + |y| = 0$, $x = 0 = y$. $a0 + b0 = 0 = C(p, p)$.

Suppose the theorem is true if $|x| + |y| < k$. Consider $0 \leq ax + by \leq n$, where $|x| + |y| = k$.

Case 1. $x = 0$. $a0 + by = by = \beta_y(b)$.

Case 2.1. $x > 0$. $a \leq ax + by \leq n$.

Then $0 \leq a(x - 1) + by \leq n - a$. Since $|x - 1| + |y| < |x| + |y| = k$, $a(x - 1) + by$ is generated. Therefore $\alpha_2(a(x - 1) + by, a) = ax + by$.

Case 2.2. $x > 0$. $0 \leq ax + by < a$.

Since $ax \geq a$, $y < 0$. $b - a \leq a(x - 1) + b(y + 1) < b$; $|x - 1| < |x|$ and $|y + 1| < |y|$, so $|x - 1| + |y + 1| < k$. Therefore $a(x - 1) + b(y + 1)$ is generated. $C(b, a) = a - b$, $\alpha_2(a(x - 1) + b(y + 1), a - b) = ax + by$.

Case 3. $x < 0$. Since $ax < 0$, $y > 0$.

Case 3.1. $b \leq ax + by \leq n$.

Then $0 \leq ax + b(y - 1) \leq n - b$. Since $|y - 1| < |y|$, $|x| + |y - 1| < k$. Therefore $ax + b(y - 1)$ is generated. $\alpha_2(ax + b(y - 1), b) = ax + by$.

Case 3.2. $0 \leq ax + by < b$.

Then $0 \leq b - (ax + by) = a(-x) + b(1 - y)$. $|-x| = |x|$ and $|1 - y| < |y|$, so $|-x| + |1 - y| < k$. Therefore $b - (ax + by)$ is generated. $C(b - (ax + by), b) = ax + by$. Therefore in every case, $ax + by$ is generated. *Q.E.D.*

Theorem 4. If $(i_0, i_1, \dots, i_m) = d$, where $i_0 = n$, then d is generated by C, N, $i_1, \dots, i_m$.

Proof. By induction on m.

If $m = 0$, $d = n = N(C(p, p))$.

Suppose that $(n, i_1, \dots, i_k) = d_k$ has been generated. Then $d_{k+1} = (n, i_1, \dots, i_{k+1}) = ((n, i_1, \dots, i_k), i_{k+1}) = (d_k, i_{k+1})$. Therefore there are integers x and y so that $d_k x + i_{k+1} y = d_{k+1}$. Consequently, by Theorem 3, d_{k+1} is generated.

Theorem 5. If $(n, i_1, \ldots, i_m) = d$, then the set of constant functions generated by C, N, $i_1, \ldots, i_m$ is

$$\mathcal{C}(d) = \left\{kd \mid 0 \leq k \leq \frac{n}{d}\right\}$$

Proof. $\beta_k(d) = kd$ for $0 \leq k \leq \frac{n}{d}$, so $\mathcal{C}(d)$ is generated. Further, if kd and jd are in $\mathcal{C}(d)$, then $C(kd, jd) = \max(0, (j-k)d)$ and $N(kd) = n - kd = \left(\frac{n}{d} - k\right) d$ are in $\mathcal{C}(d)$. Therefore $\mathcal{C}(d)$ is closed under the application of C and N. Also, since d divides i_k for each k, $i_k = \frac{i_k}{d} d$ is in $\mathcal{C}(d)$. Consider any constant function, $f(p_1, \ldots, p_s; i_1, \ldots, i_m)$, generated by C, N, $i_1, \ldots, i_m$. Substitute values from $\mathcal{C}(d)$ for the variables. By the two facts given immediately above, f must take on a value from $\mathcal{C}(d)$ for this substitution. Therefore these are the only constants generated.

Theorem 6. An $n + 1$ - valued logic using the values $0, 1, \ldots, n$ and based on C, N, and the constant functions $i_1, \ldots, i_m$ is functionally complete if and only if $(i_0, i_1, \ldots, i_m) = 1$. $n = i_0$.

Proof. If $(i_0, \ldots, i_m) = 1$, then by Theorem 5, $\mathcal{C}(1)$, which is all the constant functions, is generated. Now applying Theorem 2 we see that every function is generated. If $(i_0, \ldots, i_m) = d > 1$, then by Theorem 5, no constant function outside of $\mathcal{C}(d)$ is generated. In particular, 1 is not generated.

Corollary. An $n + 1$ - valued logic using the values $0, 1, \ldots, n$ and based on C and N is functionally complete if and only if $n = 1$.

Proof. Set $m = 0$ in Theorem 6. The greatest divisor of n is 1 if and only if $n = 1$.

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