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A NOTE ON THE REGULAR AND IRREGULAR MODAL SYSTEMS OF LEWIS

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I say that a modal formula α is regular, if after deleting the modal functors L and M,[1] if they occur in α , and after replacing the modal functors for more then one argument, as e.g. $\mathfrak{C}$ and $\mathfrak{E}$, if they occur in α, by the corresponding functors from the classical propositional calculus, throughout α, this formula becomes a thesis of the bi-valued propositional calculus. On the other hand, if after such operations α is transformed into a meaningful propositional formula, but not into a thesis, then α is called an irregular modal formula. Thus, e.g., $\mathfrak{C}Lpp$ is a regular modal formula, but Lewis' C13: MMp[2] is irregular. Correspondingly, the modal systems in which no irregular formula occurs are called regular. And, obviously, the irregular modal systems are such that they contain the irregular theses. Thus, e.g., the systems S1 - S5 and T are regular, but the system S6 of Lewis is irregular.

In this note I shall prove that any Lewis' modal system which contains system T of Feys-von Wright[3] must be regular. On the other hand, it will be shown that there are systems in which the rule:

RI *If α is provable in the system, then also $L\alpha$ is provable in the system.*

holds, and which have irregular, quasi-normal (in the sense of Scroggs) extensions.[4]

1. *System T°*. It is known[5] that an addition of **RI** as a new rule of procedure to S1 of Lewis gives a system inferentially equivalent to system T. In [11] Yonemitzu has proved that an addition to S1 of an arbitrary formula which has the form $LL\alpha$ and is such that $L\alpha$ is a thesis of S1, generates rule **RI** and, therefore, gives a system inferentially equivalent to T.

It can be proved easily[6] that an addition to S1° of an arbitrary formula of the form $LL\alpha$ and such that $L\alpha$ is a thesis of Feys' system S1°[7] as a new axiom constitutes a system, called T°, in which rule **RI** is also provable. Group I of Lewis-Langford[8] shows that formula $LL\alpha$ which satisfies the, above mentioned, condition is independent from the system S1°. On the

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other hand, Group IV verifies the axioms of T°, but falsifies *𝔖pMp*, i.e. the proper axiom of S1 and T. Hence, system T° is a proper extension of S1° and constitutes a proper subsystem of T.

2. *Lemma 1. Let 1 and 0 be the abbreviations of the formulas NKpNp and KpNp respectively. Then, the following formulas:*

H1	*𝔖N01*
H2	*𝔖N10*
H3	*𝔖K111*
H4	*𝔖K010*
H5	*𝔖K100*
H6	*𝔖K000*
H7	*𝔖L11*
H8	*𝔖M00*
H9	*𝔖M11*
H10	*𝔖L00*

are such that H1 - H6 are provable in S1°, *H7 and H8 - in* T°, *but H9 and H10 are provable only in* T.

Proof: It is known[9] that the formulas: *H1 - H6* and:

F1	*LNKpNp*
F2	*𝔖NMNpLp*
F3	*𝔖NLNpMp*
F4	*𝔖𝔖Npq𝔖Nqp*
F5	*𝔖𝔖pNq𝔖qNp*
F6	*CLp𝔖qp*

and the following metarule of procedure:

FI *If the formulas α and $C\alpha\beta$ are the theses of the system, then also β is a thesis of the system.*

are provable in S1°. Since we have *F1* in S1°, formula *LLNKpNp* = *LL1* is provable in T°, and, therefore, *H7* is provable in T° (by *F6, F1* and **FI**). Having *H2, F2, F3* and *F4*, one can deduce *H8* from *H7* at once.[10] Group I shows that *H7* and *H8* are provable neither in S1° nor in S1.

Since we have *F1*, we obtain *𝔖M11* (by *F6* and **FI**) in S1°. Hence, due to it and the proper axiom of T, *𝔖pMp*, we have *H9* in T. And, obviously, *H10* follows from *H9, H1, F3, F4* and *F5*. Thus, since Group IV falsifies *H9* and *H10*, the proof is completed.

3. *Theorem 1. Any consistent modal system of Lewis which contains* T *must be regular.*

Proof: Let us assume that α is an arbitrary irregular modal formula. Then, according to the definition of the irregular formulas, there is a meaningful propositional formula, say α', associated with α and such that there is at·

least one substitution of *1* and *0* (i.e. of *NKpNp* and *KpNp* respectively) for its variables which shows that α' is not a thesis of the classical propositional calculus.

Now, suppose that S is an arbitrary consistent Lewis' system which contains T, and that we add formula α as a new axiom to this system. Evidently, we can substitute *1* and *0* (i.e. *NKpNp* and *KpNp* respectively) for the variables occurring in α in the same exactly way, as we made previously in α' in order to show that α' is not a thesis of the bi-valued propositional calculus. Since *H1* - *H10* are provable in S (due to T), their application and the use of the first rule of substitution of Lewis reduces, obviously, α transformed by the, mentioned above, substitution to the formula $KpNp = 0$ which is inconsistent with S, since the latter system contains S1°. Thus, theorem 1 is proved.

4. *Theorem 2. There are the quasi-normal extensions of* T° *which are irregular.*

Proof: We can obtain such extension of T°, say system T^x, by adding the following formula

P1 *MLp*

as a new axiom to T°. Group IV satisfies the axioms of T° and formula *P1*. Hence, system T^x is consistent. Group II shows that *P1* is independent from T°. Therefore, T^x is a proper extension of T°. On the other hand, although the rule of substitution and the rule of detachment for material implication (i.e. metarule **FI**) are preserved in T^x, rule **RI** provable in T° does not hold in T^x. Group IV verifies the axiom *P1* of T^x, but falsifies a formula *LMLp*. Thus, system T^x constitutes a quasi-normal extension in the sense of Scroggs of T°.

I do not know whether it is possible to construct a consistent irregular modal system which would be a normal extension of T°. Also, the question remains open whether it is a necessary condition for a regular modal system to contain T as a subsystem. Since we have *H1* - *H8* provable in T, a proof of McKinsey that there is only one complete extension of S4[11] holds also for system T. It seems to me that this fact indicates that the, mentioned above, condition is rather necessary.

It is worthwhile to note that an addition of *P1* as a new axiom to the systems S2°, S3° and S4°[12] respectively generates three other irregular modal systems. Group IV of Lewis-Langford shows that these systems are consistent.

NOTES

1. In this paper instead of the original symbols of Lewis I use a modification of Łukasiewicz's symbolism which is described in [7] and [6], p. 52. Throughout this paper the term "thesis" means: a formula which is true in a system under consideration.

2. *Cf.* [6], p. 407. Also, see [2], p. 213, where there are given the definitions of the irregular modal systems S6, S7 and S8.
3. Concerning system T *cf.*, e.g., [1], p. 500, note 13, [10], Appendix II, pp. 85-90 and [9].
4. A definition of a normal extension is given in [5], p. 7, definition 3.2. Concerning the quasi-normal extensions of modal systems see [7], p. 112.
5. *Cf.* [9], p. 173.
6. *Cf.* [8], p. 159, where a proof akin to this is given.
7. Concerning Feys' system S1° *cf.* [1], pp. 483-489 and [8].
8. Groups I, II and IV of Lewis-Langford are given in [6], pp. 493-494.
9. *Cf.* [1], pp. 483-489.
10. In [3], p. 126, Theorem 7, McKinsey has shown that *H8*, i.e. $\mathfrak{C}MKpNpKpNp$, is a thesis of S4. In [9], p. 176, I have proved *H8* in system T. Here, a proof is given that *H8* is provable in T°.
11. *Cf.* [4], pp. 42-43, where also a definition of a complete extension of a system is given.
12. Concerning the systems S2°, S3° and S4° *cf.* [8] and [1].

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