A THEOREM ON HARTOGS' ALEPHS

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If $\mathfrak{m}$ is a cardinal number which is not finite, we denote by $\aleph(\mathfrak{m})$ the least aleph such that it is not $\leq \mathfrak{m}$. Such alephs are called Hartogs' alephs. It is well-known, that in the set theory the following five theorems concerning the properties of $\aleph(\mathfrak{m})$:

$\mathfrak{A}$. *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists* $\aleph(\mathfrak{m})$.

$\mathfrak{B}$. *If* $\mathfrak{m} = \aleph_\alpha$, i.e., $\mathfrak{m}$ is an aleph, *then* $\aleph(\mathfrak{m}) = \aleph_{\alpha+1}$.

$\mathfrak{C}$. *If* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are cardinal numbers such that* $\mathfrak{n} < \aleph(\mathfrak{m})$, *then* $\mathfrak{n} \leq \mathfrak{m}$.

$\mathfrak{D}$. *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then* $\mathfrak{m} < \mathfrak{m} + \aleph(\mathfrak{m})$.

$\mathfrak{E}$. *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there is no cardinal* $\mathfrak{p}$ *such that* $\mathfrak{m} < \mathfrak{p} < \aleph(\mathfrak{m})$.

are provable without the aid of the axiom of choice.[1]

The aim of this note is to show a fact, which as far as I know has not been noticed, that a formula in some respect analogous to $\mathfrak{E}$, viz.:

A. *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there is no cardinal* $\mathfrak{p}$ *such that* $\aleph(\mathfrak{m}) < \mathfrak{p} < \mathfrak{m} + \aleph(\mathfrak{m})$.

is equivalent to the axiom of choice.[2]

It is obvious that the axiom of choice implies **A**, since it follows from the said axiom that an arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph, say $\aleph_\alpha$. Hence, by $\mathfrak{B}$, $\aleph(\mathfrak{m}) = \aleph_{\alpha+1}$ and, therefore, the consequence of **A** is true.

The proof that **A** implies the axiom of choice requires that the following lemma:

L. *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m} < \mathfrak{n}$ *and* $\mathfrak{n} = \mathfrak{n}^2$, *then either* $\mathfrak{m}$ *is an aleph or* $\mathfrak{m}$ *is finite.*

is a consequence of **A**.

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Proof: Assume the conditions of **L**, viz. that $\mathfrak{m}$ and $\mathfrak{n}$ are arbitrary cardinal numbers and

$$(1)\ \mathfrak{m} < \mathfrak{n} \quad \text{and} \quad (2)\ \mathfrak{n} = \mathfrak{n}^2$$

If $\mathfrak{n} = 1$, then $\mathfrak{m}$ must be 0 and **L** is true. If $\mathfrak{n} \neq 1$, then (2) implies that $\mathfrak{n}$ is transfinite cardinal[3] and, therefore, $\mathfrak{n}$ is also not finite. Hence, by $\mathfrak{A}$, there is $\aleph(\mathfrak{n})$. Therefore, (1) and the elementary laws of set theory give:

$$(3)\ \aleph(\mathfrak{n}) \leq \mathfrak{m} + \aleph(\mathfrak{n}) \leq \mathfrak{n} + \aleph(\mathfrak{n})$$

The first case of (3), viz. $\aleph(\mathfrak{n}) = \mathfrak{m} + \aleph(\mathfrak{n})$, implies that $\aleph(\mathfrak{n}) \geq \mathfrak{m}$, i.e. that *either* $\mathfrak{m}$ *is an aleph or* $\mathfrak{m}$ *is finite*. Thus, for this case **L** is proved.

By virtue of **A** the second case of (3), viz. $\aleph(\mathfrak{n}) < \mathfrak{m} + \aleph(\mathfrak{n}) < \mathfrak{n} + \aleph(\mathfrak{n})$, is excluded. Hence, it remains to investigate the third case of (3), viz.

$$(4)\ \mathfrak{m} + \aleph(\mathfrak{n}) = \mathfrak{n} + \aleph(\mathfrak{n})$$

The known theorem, which says that:

T1. *If* $\mathfrak{m}$, $\mathfrak{p}$ *and* $\mathfrak{q}$ *are cardinal numbers such that* $\mathfrak{m} + \mathfrak{p} = \mathfrak{m} + \mathfrak{q}$, *then there exist cardinal numbers* $\mathfrak{n}$, $\mathfrak{p}_1$ *and* $\mathfrak{q}_1$ *such that* $\mathfrak{p} = \mathfrak{n} + \mathfrak{p}_1$; $\mathfrak{q} = \mathfrak{n} + \mathfrak{q}_1$; $\mathfrak{m} + \mathfrak{p}_1 = \mathfrak{m} = \mathfrak{m} + \mathfrak{q}_1$.

and which is provable without the aid of the axiom of choice,[4] allows us to deduce from (4), that *there are cardinal numbers* $\mathfrak{p}$, $\mathfrak{a}$ *and* $\mathfrak{b}$ *such that*

$$(5)\ \mathfrak{n} = \mathfrak{p} + \mathfrak{a}; \qquad (6)\ \mathfrak{m} = \mathfrak{p} + \mathfrak{b}; \qquad (7)\ \aleph(\mathfrak{n}) + \mathfrak{a} = \aleph(\mathfrak{n})$$

Hence, by (5) and (7),

$$(8)\ \mathfrak{a} \leq \mathfrak{n} \quad \text{and} \quad (9)\ \mathfrak{a} \leq \aleph(\mathfrak{n})$$

and, therefore, we can conclude from (8) and (9) that

(10) *either* $\mathfrak{a}$ *is an aleph* $\leq \mathfrak{n}$ *or* $\mathfrak{a}$ *is a finite cardinal* $\leq \mathfrak{n}$.

Since, by (2), $\mathfrak{n}$ is a transfinite cardinal, both cases of (10) show that $\mathfrak{n} = \mathfrak{n} + \mathfrak{a}$, which by virtue of (5) gives

$$(11)\ \mathfrak{n} + \mathfrak{a} = \mathfrak{p} + \mathfrak{a}$$

If $\mathfrak{a}$ is finite, then by virtue of the known theorem, which says that:

T2. *For any cardinal numbers* $\mathfrak{n}$, $\mathfrak{p}$ *and* $\mathfrak{q}$, *if* $\mathfrak{n}$ *is finite and* $\mathfrak{p} + \mathfrak{n} = \mathfrak{q} + \mathfrak{n}$, *then* $\mathfrak{p} = \mathfrak{q}$.

and which is provable without the aid of the axiom of choice,[5] it follows from (11) that $\mathfrak{n} = \mathfrak{p}$. Hence, by (6), $\mathfrak{m} = \mathfrak{n} + \mathfrak{b} \geq \mathfrak{n}$ which contradicts (1). Therefore, the second case of (10), i.e. that $\mathfrak{a}$ is a finite cardinal $\leq \mathfrak{n}$, is impossible and, therefore, we can establish that:

$$(12) \quad \mathfrak{a} \textit{ is an aleph } \leq \mathfrak{n}$$

Since we have (2), (5) and (12), the following deduction yields:

$$\mathfrak{p} + \mathfrak{a} = \mathfrak{n} = \mathfrak{n}^2 = (\mathfrak{p} + \mathfrak{a})^2 = \mathfrak{p}^2 + 2\mathfrak{p}\mathfrak{a} + \mathfrak{a}^2 \geq \mathfrak{p}\mathfrak{a} = \mathfrak{p}(1 + \mathfrak{a}) = \mathfrak{p} + \mathfrak{p}\mathfrak{a} \geq \mathfrak{p} + \mathfrak{a}$$

which gives at once:

$$(13) \quad \mathfrak{p} + \mathfrak{a} = \mathfrak{p}\mathfrak{a}$$

Since, by (12), $\mathfrak{a}$ is an aleph, the formula (13), as is well-known,[6] implies without the aid of the axiom of choice that

$$(14) \quad \textit{either } \mathfrak{p} > \mathfrak{a} \textit{ or } \mathfrak{a} \geq \mathfrak{p}$$

But, the first case of (14) is impossible, since if $\mathfrak{a}$ is an aleph and $\mathfrak{p} > \mathfrak{a}$, then $\mathfrak{p} = \mathfrak{p} + \mathfrak{a}$.[7] Hence, by (5), $\mathfrak{n} = \mathfrak{p}$ and, therefore, by (6), $\mathfrak{m} = \mathfrak{n} + \mathfrak{b} \geq \mathfrak{n}$ which contradicts (1). Therefore, the second case of (14) yields, viz. that $\mathfrak{a} \geq \mathfrak{p}$, which, by (12), shows that

$$(15) \quad \mathfrak{a} = \mathfrak{p} + \mathfrak{a}$$

Hence, due to (15), (5) and (1) we obtain $\mathfrak{a} = \mathfrak{p} + \mathfrak{a} = \mathfrak{n} > \mathfrak{m}$ which, by (12), shows that

either $\mathfrak{m}$ *is an aleph or* $\mathfrak{m}$ *is finite.*

Thus, lemma **L** follows from **A**.

It is easy to prove that **L** in turn implies the axiom of choice. For this end assume that $\mathfrak{m}$ is an arbitrary cardinal number which is not finite, and put $\mathfrak{n} = \aleph_0\mathfrak{m}$. Hence

(16) $\mathfrak{n}$ *is not finite*; (17) $\mathfrak{n} = \aleph_0\mathfrak{m} = 2\aleph_0\mathfrak{m} = 2\mathfrak{n}$; (18) $2^{\mathfrak{n}} = 2^{2\mathfrak{n}} = (2^{\mathfrak{n}})^2$

Since the formula $\mathfrak{n} < 2^{\mathfrak{n}}$ is generally true, an application of it, (18) and (16) to **L** gives at once: $\mathfrak{n}$ *is an aleph*. Therefore, since $\mathfrak{n} = \aleph_0\mathfrak{m} \geq \mathfrak{m}$, our arbitrary cardinal number $\mathfrak{m}$ which is not finite must be an aleph too. Hence, formula **L** implies the axiom of choice.

Thus, it was proved that each of the formulas **A** and **L** is equivalent to the axiom of choice. Concerning theorem **A** it is worth while to note that Sierpiński has proved without the use of the axiom of choice the following theorem:[8]

$\mathfrak{F}$. *For any cardinal number* $\mathfrak{m}$ *which is not finite, the difference* $[\mathfrak{m} + \aleph(\mathfrak{m})] - \aleph(\mathfrak{m})$ *does not exist.*

In the first glance it appears that formula $\mathfrak{F}$ is stronger than **A**. In fact, it is just the opposite.

NOTES

1. In [1], Hartogs proved $\mathfrak{A}$, $\mathfrak{C}$ and $\mathfrak{D}$. Theorems $\mathfrak{B}$ and $\mathfrak{E}$ are due to Tarski. *Cf.* [2], p. 311, [7], pp. 28-30, and [3], pp. 407-409 and 413-414.

2. The proofs which are given below are established within the general set theory, i.e. the set theory from which the axiom of choice and all its consequences otherwise unprovable have been removed. It is well-known that if we base a so defined general set theory on an axiomatic system in which the notions of cardinal and ordinal numbers cannot be defined, we have to introduce these concepts into the system by means of special axioms.
3. I.e. that $\mathfrak{n} \geq \aleph_0$.
4. This theorem was announced without proof by Tarski in [2], p. 301, theorem **6**. Sierpiński gave a proof in [4], p. 116. *Cf.* also [3], p. 161.
5. This theorem is due to Sierpiński, *cf.* [3], p. 168, corollary.
6. *Cf.*, e.g., [6], p. 148, lemme 1.
7. *Cf.*, e.g., [3], p. 413.
8. *Cf.*, [5], p. 8. Concerning the definition of the difference of two cardinal numbers, *cf.* [2], p. 306, position **47**, and [3], p. 159.

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