ON RECURSIVE TRANSCENDENCE

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1. Let $P_n\ (x)$ be the n^{th} polynomial in an enumeration of all one-variable polynomials with integral coefficients; let $||z|| = ||x + iy|| = |x| + |y|$ be called the *norm* of a rational complex number $z = x + iy$ and let $\{s_n\}$ be a sequence of rational real or complex numbers. Then $\lim s_n$ is transcendental if

$$(r)\ (\exists\ k)\ (\exists\ N)\ (n)\ \{n \geqslant N \rightarrow ||P_r\ (s_n)|| > 2^{-k}\} \tag{1.1}$$

The convergence of $\{s_n\}$ is expressed by the condition:

$$(k)\ (\exists\ \nu)\ (n)\ \{n \geqslant \nu \rightarrow ||s_n - s_\nu|| < 2^{-(k+2)}\}\ . \tag{1.2}$$

Let $\nu\ (k)$ be the least value of ν for which (1.2) holds, so that $n \geqslant \nu\ (k) \rightarrow ||s_n - s_{\nu(k)}|| < 2^{-(k+2)}$, and let k_r and N_r be the least values of k and N for which (1.1) holds, so that

$$n \geqslant N_r \rightarrow ||P_r\ (s_n)|| > 2^{-k_r}\ . \tag{1.3}$$

Now if $M = \max_{0 \leqslant r \leqslant \nu(1)} \{||s_r|| + 1\}$, and if $P_r^*(x)$ is the sum of the absolute values of the terms of $P_r'\ (x)$, the first derivative of $P_r\ (x)$, then

$$||P_r\ (s_m) - P_r\ (s_n)|| < ||s_m - s_n||\ P_r^*(M)\ ,$$

and, calling the exponent of the least power of 2 which exceeds $P_r^*\ (M)$, c_r, we have

$$m,\ n \geqslant \nu\ (k + c_r) \rightarrow ||P_r\ (s_m) - P_r\ (s_n)\ || < 2^{-k-1}\ . \tag{1.4}$$

If s_n is general recursive and general recursively convergent, so that the function $\nu\ (k)$ is general recursive, and if further, the functions N_r and k_r in (1.3) are both general recursive, then the general recursive real (complex) number $\{s_n\}$ is said to be *general recursively transcendental.*

If s_n, $\nu\ (k)$, N_r and k_r are all primitive recursive (p.r.), then the p.r. real (complex) number $\{s_n\}$ is said to be *primitive recursively* (p.r.) *transcendental.*

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In particular, taking $P_r(x)$ to be a linear function of x, we obtain the corresponding definitions of irrationality.

From (1.3) and (1.4), taking k_r for k and $N_r + \nu(k_r + c_r + 1)$ for n, we find, writing $\nu_r(k)$ for $\nu(k + c_r + 1)$, that

$$||P_r(s_{\nu_r(k_r)})|| > 2^{-k_r-1}, \quad \text{whence}$$

$$(r)(\exists\, k)\{||P_r(s_{\nu_r(k)})|| > 2^{-k-1}\}. \tag{1.5}$$

If $\{s_n\}$ is general recursive and general recursively convergent, and if $\lim s_n$ is transcendental, then $\{s_n\}$ is general recursively transcendental. For, by hypothesis, s_n and $\nu_r(k)$ are general recursive and so if λ_r is the least value of k satisfying (1.5) then λ_r is general recursive, and

$$||P_r(s_{\nu_r(\lambda_r)})|| > 2^{-\lambda_r-1}.$$

Using (1.4) again with $k = \lambda_r + 1$ we have

$$n \geqslant \nu_r(\lambda_r) \to ||P_r(s_n)|| > 2^{-\lambda_r-2}$$

which proves that $\{s_n\}$ is general recursively transcendental. Of course it is not the case that a p.r. number which is transcendental is necessarily p.r. transcendental. However, we shall prove that e and π are p.r. transcendental in the sense that any p.r. real number whose classical limit is e or π is p.r. transcendental.

2. We start by showing that every algebraic number is a p.r. algebraic number, i.e. that to each root of a polynomial, $f(x) = \sum_{r=o}^{m} a_r x^r$, there corresponds a p.r. real (complex) number, Θ_n, such that $f(\Theta_n) \to 0$ primitive recursively.

Firstly, considering real roots, we note that if $a_m \geqq 1$, and if $|x| > A = \sum_{r=o}^{m} |a_r|$, then $|x| > 1$ and $|f(x)| > 0$; i.e. all the roots of $f(x)$ lie in the circle $|x| < A$.

Let $F(x) = \sum_{r=o}^{l} b_r x^r$ be the quotient on dividing $f(x)$ by the highest common factor of $f(x)$ and $f'(x)$; then the b_r are rational functions of the a_r. Let α_i $(1 \leqslant i \leqslant \mu \leqslant l)$ denote the real roots of $f(x)$ (hence of $F(x)$) and, if $\mu < l$ let α_i $(\mu < i \leqslant m)$ denote the complex roots. Supposing $\mu > 1$, if $h < k \leqq \mu$, then

$$\prod_{i<j\leqslant l} (\alpha_i - \alpha_j)^2 = \Delta^2 < (\alpha_h - \alpha_k)^2\{(2A)^{\frac{1}{2}l(l-1)-1}\}^2$$

and so $$|\alpha_h - \alpha_k| > |\Delta|/\{(2A)^{\frac{1}{2}l(l-1)-1}\} = \delta ,$$

say, where δ is rational since $|\Delta|$ is rational. Divide $(-A, A)$ into subintervals of length at most δ by points δ_0 $(= -A)$, δ_1 $(= -A + \delta)$, . . . , δ_{K-1} $(= -A + (\kappa - 1)\delta)$, δ_K $(= A)$ and evaluate $F(\delta_j)$ for each j $(0 \leqslant j \leqslant \kappa)$.

If (i) $F(\delta_j) = 0$ for some j, then we define $\Theta_n = \delta_j$, and no other real root lies in either (δ_{j-1}, δ_j) or (δ_j, δ_{j+1});

(ii) $F(\delta') < 0$, $F(\delta'') > 0$ where δ', δ'' are the end points of some subinterval, then there is just one root of $f(x)$ in this interval. Let $\rho_0 = (\delta' + \delta'')/2$; if $F(\rho_0) = 0$ then define $\Theta_n = \rho_0$; if $F(\rho_0) > 0$ define $\Theta_0 = \delta'$ and if $F(\rho_0) < 0$ define $\Theta_0 = \delta''$. To complete the recursive definition of $\{\Theta_n\}$, let $\rho_{n+1} = (\rho_n + \Theta_n)/2$ and then

$$\begin{aligned}
\Theta_{n+p+1} &= \rho_{n+1}, \ (p \geqslant 0) \text{ if } F(\rho_{n+1}) = 0,\\
\Theta_{n+1} &= \Theta_n \text{ if } F(\rho_{n+1}) \text{ has the same sign as } F(\rho_n),\\
\Theta_{n+1} &= \rho_n \text{ if } F(\rho_{n+1}) \text{ has the opposite sign to } F(\rho_n).
\end{aligned}$$

$\{\Theta_n\}$ satisfies $n \geqslant \nu \rightarrow |\Theta_n - \Theta_\nu| < \delta 2^{-\nu}$, and so it is p.r. convergent. Further

$$|F(\Theta_n)| < |F(\Theta_n) - F(\rho_n)| < |\Theta_n - \rho_n| \sum_{j=o}^{l} |b_j| j A^{j-1}$$

$$= |\Theta_n - \rho_n| A^*, \text{ say.}$$

Thus $n \geqslant \nu \rightarrow |F(\Theta_n)| < A^*\delta\, 2^{-\nu}$, showing that $F(\Theta_n)$ — and hence also $f(\Theta_n)$ — tends p.r. to zero. A subinterval with end points δ', δ'' contains no root of $f(x)$ if $F(\delta')$ and $F(\delta'')$ have the same sign.

If $\mu \leqslant 1$, the same construction can be carried out, though of course, the δ will not have its previous importance.

If $\alpha + i\beta$ is a root of $f(x)$, there are polynomials $P(x, y)$, $Q(x, y)$ such that $P(\alpha,\beta) = Q(\alpha,\beta) = 0$, from which we arrive at $R_1(\alpha) = 0$ on eliminating β and $R_2(\beta) = 0$ on eliminating α, where R_1 and R_2 are polynomials obtained rationally from P, Q. Since α and β are thus p.r. algebraic real numbers, then $\alpha + i\beta$ is a p.r. algebraic complex number.

3. If $\{\alpha_n\} = \alpha$, $\{\beta_n\} = \beta$ are two p.r. real numbers, we write $\alpha = \beta$ (and say α, β are p.r. equal) if there is a p.r. function eq (k) such that $n \geqslant \text{eq}(k) \rightarrow |\alpha_n - \beta_n| < 2^{-k}$; we write $\alpha < \beta$ if there are integers i, j such that

$$n \geqslant j \rightarrow \beta_n - \alpha_n \geqslant 2^{-\iota} ;$$

and $\alpha > \beta$ if $\beta < \alpha$.

Using the results of para. 2 we now construct a decision procedure for deciding of two algebraic real numbers α, β which of $\alpha < \beta$, $\alpha = \beta$, $\alpha > \beta$ holds (the proof also ensures that one of these relations must hold.)

3.1 Given a primitive recursive real number $\alpha = \{\alpha_n\}$, a root of $a_m x^{m+1} + \ldots + a_1 x^2 + x$ (rational a_l), then it is decidable whether $\alpha_n \rightarrow 0$ or not ($\alpha = 0$ or not).

Proof: Choose k so large that $2^{k-1} > \sum_{r=1}^{m} |a_r|$, then if $|x| < 2^{-k}$, $|\sum_{r=1}^{m} a_r x^r| < 2^{-k} \cdot 2^{k-1} = \frac{1}{2}$, whence

$$|\sum_{r=1}^{m} a_r x^r + 1| > \tfrac{1}{2}\,.$$

Choose n_1 such that $n > n_1 \to |\alpha_n - \alpha_{n_1}| < 3^{-1} \cdot 2^{-k}$; then

(i) if $|\alpha_{n_1}| < 2^{-k-1}$ we have $|\alpha_n| < 2^{-k}$ $(n \geqslant n_1)$ so that $|\alpha_n \{\sum_{r=1}^{m} a_r \alpha_n{}^r + 1\}| > |\alpha_n|/2$,

whence $\alpha_n \to 0$ primitive recursively, since the left-hand side does so. I.e. $\alpha = 0$.

(ii) if $|\alpha_{n_1}| \geqslant 2^{-k-1}$, then for all $n \geqslant n_1$, $|\alpha_n| > 6^{-1} \cdot 2^{-k}$

showing that $\alpha_n \not\to 0$, i.e. $\alpha \neq 0$.

In case (ii) it follows of course that

$$|\alpha_n \{\sum_{r=1}^{m} a_r \alpha_n{}^r + 1\}| > 6^{-1} \cdot 2^{-k} \,|\sum_{r=1}^{m} a_r \alpha_n{}^r + 1|$$

i.e. that α is a root of $a_m x^m + \ldots + a_1 x + 1$.

3.2 If $|\alpha_n| > \epsilon > 0$ for all n, choose n_2 such that $n \geqslant n_2 \to |\alpha_n - \alpha_{n_2}| < \epsilon/3$. Then

(i) if $\alpha_{n_2} > \epsilon/2$ we have

$$n \geqslant n_2 \to \alpha_n > \epsilon/6\,,\ \text{i.e. } \alpha > 0\ ;$$

and

(ii) if $\alpha_{n_2} \leqslant \epsilon/2$ we have

$$n \geqslant n_2 \to \alpha_n < 5\epsilon/6, \ \text{ but}$$

$|\alpha_n| > \epsilon$ and so $\alpha_n < -\epsilon$ $(n \geqslant n_2)$, i.e. $\alpha < 0$.

3.3 Given two p.r. algebraic real numbers $\alpha = \{\alpha_n\}$ and $\beta = \{\beta_n\}$, roots of integral polynomials $f(x)$ and $g(x)$ respectively, then $\gamma = \{\gamma_n\} = \{\alpha_n - \beta_n\}$ is also a p.r. real number and we can construct rationally from $f(x)$ and $g(x)$ a polynomial with integral coefficients having γ as a root. For $f(\beta + \gamma) = 0$ and can be expressed in the form

$$f_m(\gamma)\beta^m + f_{m-1}(\gamma)\beta^{m-1} + \ldots + f_0(\gamma).$$

We also have $g(\beta) = 0$, whence, on eliminating β we arrive at the desired polynomial with γ as a root. Using (3.1) and (3.2) on γ we can thus decide $\alpha < \beta$, $\alpha = \beta$ or $\alpha > \beta$.

4. In Goodstein [1] and [2] the p.r. irrationality of p.r. sequences with (classical) limits e^x (rational x) and π was established: here we prove the p.r. transcendence of sequences for e and π.

We use the p.r. real numbers $\mathbf{E}\,(n, x)$ (rational x) defined by

$$\mathbf{E}\,(0, x) = 1,\ \mathbf{E}\,(n+1, x) = \mathbf{E}\,(n, x) + x^{n+1}/(n+1)!$$

The following inequalities are needed:

$$\mathbf{E}\,(n, m) \leqslant \{\mathbf{E}\,(n, 1)\}^m \qquad (\text{integral } m \geqslant 0) \tag{4.1}$$

Proof: by induction on m.

$$\{\mathbf{E}\,(n, 1)\}^m \leqslant \mathbf{E}\,(mn, m) \tag{4.2}$$

Proof: by induction on m using the easily proved

$$\mathbf{E}\,(p,a)\,.\,\mathbf{E}\,(q,b) \leqslant \mathbf{E}\,(p+q,\ a+b)$$

(p, q, a, b integers $\geqslant 0$.)

For rational x and y, and $n > 2\,(|x| + |y|)$

$$|\mathbf{E}\,(n,x)\,.\,\mathbf{E}\,(n,y) - \mathbf{E}\,(n,x+y)| \leqslant \frac{2(|x| + |y|)^{n+1}}{(n+1)!} \tag{4.3}$$

Proof: procedure obvious.

$$\mathbf{E}\,(n,m) < 3^m \tag{4.4}$$

Proof: by (4.1) and the familiar comparison with a geometric series.

To prove that $\mathbf{E}\,(n, 1)$ is p.r. transcendental, define $\phi\,(x)$ to be the polynomial

$$\frac{x^{p-1}}{(p-1)!}\left[\prod_{r=1}^{m}(x-r)\right]^p = \frac{1}{(p-1)!}\sum_{r=p-1}^{\nu} c_r x^r$$

where $\nu = mp + p - 1$ and the c_r ($p - 1 \leqslant r \leqslant \nu$) are integers.

Evidently $\phi\,(0) = \phi^{(k)}\,(0) = 0,\ 1 \leqslant k \leqslant p-2;\ \phi^{(p-1)}\,(0) = \{(-1)^m\, m!\}^p \not\equiv 0 \pmod p$ if p be taken prime and $p > m$; also

$$\phi^{(p+r)}\,(0) = \frac{(p+r)!}{(p-1)!}\, c_{p+r} \equiv 0 \pmod p,\ 0 \leqslant r \leqslant mp - 1.$$

Let $L\,(\phi(x)) = \sum_{k=o}^{\nu} \phi^{(k)}\,(x)$, then $L\,(\phi(0)) \not\equiv 0 \pmod p$ but is an integer and so non-zero.

For each k, $1 \leqslant k \leqslant m$, we may write $\phi(x) = \frac{1}{(p-1)!}\sum_{r=p}^{\nu} c_{k,\,r}\,(x-k)^r$ with integral $c_{k,\,r}$, showing that

$$\phi^{(r)}\,(k) = 0,\ 0 \leqslant r \leqslant p-1, \text{ and } \phi^{(p+r)}\,(k) \equiv 0 \pmod p.$$

Therefore $L(\phi(k)) \equiv 0 \pmod{p}$, $1 \leqslant k \leqslant m$.

However $L(\phi(k)) = \sum_{s=o}^{\nu} \sum_{r=s}^{\nu} \phi^{(r)}(0)\, k^{r-s}/(r-s)!$

$$= \sum_{t=o}^{\nu} \phi^{(t)}(0)\, \mathbf{E}(t,k)$$

$$= L(\phi(0))\, \mathbf{E}(n,k) - \sum_{r=o}^{\nu} \phi^{(r)}(0)\, k^r \sum_{s=1}^{n-r} k^s/(r+s)!, \text{ for } n>\nu.$$

Now $\left|\sum_{r=o}^{\nu} \phi^{(r)}(0)\, k^r \sum_{s=1}^{n-r} k^s/(r+s)!\right| < \sum_{r=o}^{\nu} |\phi^{(r)}(0)|\, \frac{k^r}{r!}\, \mathbf{E}(n,k)$

$$= \frac{k^{p-1}}{(p-1)!}\left[\prod_{r=1}^{m}(k+r)\right]^p \mathbf{E}(n,k)$$

$$< \frac{\{\prod_{r=o}^{m}(m+r)\}^p}{(p-1)!\, m}\, \mathbf{E}(n,m).$$

Let a_r $(0 \leqslant r \leqslant m)$ be integers, with $a_m > 0$, then by the above

$$L(\phi(0)) \sum_{k=o}^{m} a_k\, \mathbf{E}(n,k) = \sum_{k=o}^{m} a_k\, L(\phi(k)) + \sum_{k=o}^{m} a_k \left[\sum_{r=o}^{\nu} \phi^{(r)}(0)\, k^r \sum_{s=1}^{n-r} k^s/(r+s)!\right]$$

and $\left|\sum_{k=o}^{m} a_k \left\{\sum_{r=o}^{\nu} \phi^{(r)}(0)\, k^r \sum_{s=1}^{n-r} k^s/(r+s)!\right\}\right|$

$$< \frac{a\{\prod_{r=o}^{m}(m+r)\}^p}{(p-1)!}\, \mathbf{E}(n,m) \quad (\text{where } a = \max_{o\leqslant r\leqslant m} |a_r|)$$

$$< \frac{aM^p}{(p-1)!}\, 3^m \quad (\text{where } M = \prod_{r=o}^{m}(m+r))$$

$$< 1/2 \text{ if } p > 1 + 2M + \frac{M^{2M}}{(2M)!}\, 2aM3^m = U, \text{ say.}$$

Also $\sum_{k=o}^{m} a_k\, L(\phi(k))$ is then a non-zero integer.

Then $\left|L(\phi(0)) \sum_{k=o}^{m} a_k\, \mathbf{E}(n,k)\right| > 1 - 1/2 = 1/2$

for $p > U$ and $n > \nu$, so that

$$|\sum_{k=o}^{m} a_k \mathbf{E}(n,k)| > 1/2\,|L(\phi(0))|, \quad \text{for } n > \nu.$$

Now using inequalities (4.1) and (4.2)

$$|\sum_{k=o}^{m} a_k \{\mathbf{E}(n,1)\}^k - \sum_{k=o}^{m} a_k \mathbf{E}(n,k)|$$

$$\leqslant \sum_{k=o}^{m} |a_k|\{\mathbf{E}(nk,k) - \mathbf{E}(n,k)\}$$

$$< \sum_{k=o}^{m} |a_k| \frac{n(k-1)k^n}{n!} < ma \frac{nmm^n}{n!} \quad (\text{for } n > m)$$

$$< 1/4\,|L(\phi(0))|, \text{ if } n > 1 + v + (m^{\nu}/v!)am^3\, 4|L(\phi(0))|$$

$= V$, say, where $v = 2m$.

Thus for $p > U$ and $n > V$ we have

$$|\sum_{k=o}^{m} a_k\{\mathbf{E}(n,1)\}^k| > (1/2 - 1/4)/|L(\phi(0))| = 1/4|L(\phi(0))|.$$

If $\sum_{k=o}^{m} a_k x^k$ is the ρ^{th} member of some recursive enumeration of the polynomials of one variable with integer coefficients, then the $m, a_o, \ldots, a_m$ are p.r. functions of ρ, and therefore so are the $L(\phi(0))$, U and V, establishing the p.r. transcendence of $\mathbf{E}(n,1)$.

Let y_n be a real root of $\sum_{k=o}^{m} a_k x^k$, then we can find a p.r. function $N(i)$ such that

$$n \geqslant N(i) \to |\sum_{k=o}^{m} a_k y_n^k| < 1/i.$$

Taking $i > 8|L(\phi(0))|$ and $n > \max\{N(i), V\}$ we have

$$|\sum_{k=o}^{m} a_k\{y_n^k - (\mathbf{E}(n,1))^k\}| > 1/8\,|L(\phi(0))|.$$

Now $\quad |\sum_{k=o}^{m} a_k\{y_n^k - (\mathbf{E}(n,1))^k\}|$

$$< \quad |y_n - \mathbf{E}(n, 1)| \sum_{k=o}^{m} |a_k| \, k \, A^{k-1} \quad (\text{where } A = \max\{3, a\})$$

$$= \quad C \, |y_n - \mathbf{E}(n, 1)|, \text{ say.}$$

Thus $|y_n - \mathbf{E}(n, 1)| > 1/8 \, C|L(\phi(0))|$, showing by how much at least $\mathbf{E}(n, 1)$ differs from a given algebraic real number.

5. For the purposes of this section, we need some properties of the norm $||z||$; we take the following for granted:

$$||z + w|| \leqslant ||z|| + ||w||, \tag{5.1}$$

$$||z \pm w|| \geqslant ||z|| - ||w||, \tag{5.2}$$

$$||z \cdot w|| \leqslant ||z|| \cdot ||w||, \tag{5.3}$$

$$|z|^2 \leqslant ||z||^2 \leqslant 2|z|^2, \tag{5.4}$$

$$||z \cdot w|| \geqslant \tfrac{1}{2}||z|| \cdot ||w||. \tag{5.5}$$

An inequality similar to (4.3) but with norms replacing moduli is proved in the same way, using (5.1) – (5.5) above.

Let π_n be the p.r. sequence defined in Goodstein [2] § 2: we shall show that this is p.r. transcendental.

Let α_1 $(= \alpha)_1$ $\alpha_2, \ldots, \alpha_N$ be the roots of $\sum_{r=o}^{N} a_r x^r$ (integral a_r); let $2(|a_o| + \ldots + |a_N|) = A$, and $2NA = B$ (then $||\alpha_r|| < A$). Denote $i\alpha_r$ by β_{2r-1} and $- i\alpha_r$ by β_{2r} $(1 \leqslant r \leqslant N)$, then for $1 \leqslant j \leqslant 2N$, $||\beta_j|| < A$. Next let γ_s $(1 \leqslant s \leqslant 2^{2N} - 1 = M)$ consist of all possible sums of the numbers β_j taken k at a time $(1 \leqslant k \leqslant 2N)$ so that the γ_s will be the roots of a polynomial

$$Q(x) = \sum_{r=o}^{M} b_r x^r \qquad (\text{integral } b_r)$$

and $||\gamma_s|| < B$ $(1 \leqslant s \leqslant M)$.

$$\text{Let} \quad \psi(x) = \frac{x^{p-1}}{(p-1)!} b_M^{pM} \{Q(x)\}^p$$

$$= \frac{1}{(p-1)!} \sum_{r=p-1}^{pM+p-1} c_r x^r$$

where p is a prime exceeding both $|b_o|$ and $|b_M|$.

Again write $L(\psi(x)) = \sum_{r=o}^{pM+p-1} \psi^{(r)}(x)$, then $L(\psi(0))$ is an integer not divisible by p. Further, if x is a root of $Q(x)$ then $\psi^{(r)}(x) = 0$ for $r \leqslant p-1$;

and $\sum_{m=1}^{M} \psi^{(r)}(\gamma_m)$ is an integer divisible by p, for $r \geqslant p$.

Let $T_n = \prod_{r=1}^{2N} (1 + \mathbf{E}(n,\beta_r))$ so that

$$T_n = 1 + \sum_{r=1}^{M} \mathbf{E}(n,\gamma_r) + U_n$$

where $||U_n|| < 2\, M \sum_{r=o}^{2n-2} (\mathbf{E}(n,A)^r\, B^{n+1}/(n+1)!$

$$= 2\, M^*\, B^{n+1}/(n+1)!\ , \text{ say.}$$

But $L(\psi(0))\, \mathbf{E}(n,\gamma) = L(\psi(\gamma)) + \sum_{r=o}^{pM+p-1} \psi^{(r)}(0)\, \gamma^r \sum_{s=1}^{n-r} \gamma^s/(r+s)!$

for $n > pM + p - 1 = \mu$, say.

Now, $||\sum_{t=o}^{M} \sum_{r=o}^{\mu} \psi^{(r)}(0)\gamma_t^r \sum_{s=1}^{n-r} \gamma_t^s/(r+s)!|| < M\, \mathbf{E}(n,B) \sum_{r=p-1}^{\mu} |c_r| B^r/(p-1)!$

But $\frac{1}{(p-1)!} \sum_{r=p-1}^{\mu} |c_r| B^r = \frac{B^{p-1}}{(p-1)!} |b_M^{pM}| \{\sum_{k=o}^{M} |b_k| B^k\}^p$

$$\to 0 \text{ as } p \to \infty$$

Therefore $L(\psi(0)) \sum_{t=o}^{M} \mathbf{E}(n,\gamma_t) = L(\sum_{t=o}^{M} \psi(\gamma_t)) + \epsilon_p$

where $\epsilon_p \to 0$ as $p \to \infty$ and $L(\sum_{t=o}^{M} \psi(\gamma_t))$ is an integer divisible by p.

Hence $L(\psi(0))T_n = L(\psi(0)) + L(\sum_{t=o}^{M} \psi(\gamma_t)) + \{\epsilon_p + U_n\}\, L(\psi(0))$.

Choose p so that $||\epsilon_p|| < 1/3$, then for $n \geqslant 6|L(\psi(0))|M^* B^{B+1}/B!$

we have $2|L(\psi(0))|M^*\, B^{n+1}/(n+1)! < 1/3$ and therefore

$$||L(\psi(0))T_n|| > 1 - 1/3 - 1/3.$$

However

$$||T_n|| < ||1 + \mathbf{E}(n,i\alpha)||4^{(2N-1)A} < ||1 + \mathbf{E}(n,i\alpha)||4^B$$

whence $||1 + \mathbf{E}(n,i\alpha)|| > 1/3L(\psi(0))4^B$.

Now for $n \geqslant 14$ (see Goodstein [2]),

$$||1 + \mathbf{E}(2n+1, i\pi_n)|| < 1/10^{n-1} < 1/12|L(\psi(0))|4^B$$

if $n \geqslant |L(\psi(0))|4^{B+1}$. Therefore

$$||\mathbf{E}(2n+1, i\alpha) - \mathbf{E}(2n+1, i\pi_n)|| > 1/|L(\psi(0))|4^{B+1}$$

for $n \geqslant c = \max\{|L(\psi(0))|4^{B+1},\ 3|L(\psi(0))|M\, B^{B+1}/B!\}$.

Since $||i\pi_n|| < 4$ and $||i\alpha|| < A$, taking C to be $\max\{4, A\}$, we see that

$$||\mathbf{E}(2n+1), i\alpha) - \mathbf{E}(2n+1, i\pi_n)|| \leqslant ||\alpha - \pi_n||\mathbf{E}(2n, C)$$
$$< ||\alpha - \pi_n||3^C$$

and therefore $||\alpha - \pi_n|| > 1/|L(\psi(0))|4^B\, 3^C$ for $n \geqslant c$.

It then follows that

$$||\sum_{r=0}^{N} a_r \pi_n^r|| \geqslant \{||a_N|| \cdot \prod_{r=1}^{N} ||\alpha_r - \pi_n||\}/2^N$$
$$> ||a_N||/|L(\psi(0))|^N\, 2^{2(B+1)N}\, 3^{NC} \text{ for } n \geqslant c.$$

6. Having proved the p.r. transcendence of $\mathbf{E}(n, 1)$ and π_n, we must show that any other p.r. real numbers with classical limit e or π are also p.r. transcendental. This follows from:

6.1 Any two p.r. numbers which are classically equal are also p.r. equal; and

6.2 if $\{\alpha_n\}$ and $\{\beta_n\}$ are p.r. equal and $\{\alpha_n\}$ is p.r. transcendental, then $\{\beta_n\}$ is p.r. transcendental.

Proof of (6.1) Classical equality of $\{\alpha_n\}$ and $\{\beta_n\}$ is expressed by $(k)(\exists N)(n)\{n \geqslant N \to ||\alpha_n - \beta_n|| < 2^{-k-3}\}$. There is a p.r. $\nu(k)$ such that

$$n \geqslant \nu(k) \to ||\alpha_n - \alpha_{\nu(k)}|| < 2^{-k-3} \ \&$$
$$||\beta_n - \beta_{\nu(k)}|| < 2^{-k-3}$$

Then it follows that $||\alpha_{\nu(k)} - \beta_{\nu(k)}|| < 3.2^{-k-3}$,

whence $n \geqslant \nu(k) \to ||\alpha_n - \beta_n|| < 2^{-k}$.

Proof of (6.2) For some p.r. N_r, k_r, in the notation of para. 1

$$n \geqslant N_r \to ||P_r(\alpha_n)|| > 2^{-k_r} \ldots \qquad \text{(i)}$$

There is a p.r. eq(k) such that

$$n \geqslant \mathrm{eq}(k) \rightarrow ||\alpha_n - \beta_n|| < 2^{-k}.$$

$$||P_r(\alpha_n) - P_r(\beta_n)|| \leqslant ||\alpha_n - \beta_n||\ P_r^*\ (\mathbf{S})$$

where $\mathbf{S}$ is an upper bound for $||\alpha_n||$ and $||\beta_n||$, and if 2^{c_r} is the least power of 2 to exceed $P_r^*\ (\mathbf{S})$, then

$$n \geqslant \mathrm{eq}(k_r + c_r + 1) \rightarrow ||P_r(\alpha_n) - P_r(\beta_n)|| < 2^{-k_r-1}.$$

From this and (i) follows

$$n > \max\{N_r,\ \mathrm{eq}(k_r + c_r + 1)\} \rightarrow ||P_r(\beta_n)|| > 2^{-k_r-1}.$$

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