A FORMAL SYSTEM

G. Y. RAINICH

The present paper arose in connection with my teaching of projective geometry. In trying to get away from too much reliance on drawings I used the notation (ABC) to indicate that the three points A, B, C are collinear; it is possible to express the hypotheses and the conclusions of all the theorems in terms of collinearity; it then appeared possible to dispense not only with drawings but also with other aspects of intuition, and I became interested in how far one can go in that direction. I have not found in literature a formal system that fitted my requirements, so I attempted to devise my own. This attempt is described in what follows. Its origin in connection with projective geometry influences the exposition but it is hoped that the point of view is applicable to more general situations.

In discussing formal systems it is possible roughly to distinguish four stages in the development: in the first stage, we deal with objects; in the second stage, having given names to the objects, we lead a discussion using these names; in the third stage we still handle these names but we have achieved a certain degree of abstraction, the names, or symbols apply to generalizations; they still stand for objects but there may be different sets of objects to which the same names apply and what we say applies to all these sets. In the fourth stage we do not consider that the signs or symbols stand for something - we just operate on them according to certain rules (in a way, we deal with symbols as objects so that the fourth stage resembles the first).

There is importance that can be attached to activity in any of these stages, and it also in important to investigate the transitions from one stage to another but in this essay we are interested principally in the fourth stage.

In order to emphasize the formal character of the system we shall speak of *letters* rather than symbols the word symbol having the connotation that it stands for something, that it symbolizes some object. Further, we will not speak of sentences or propositions; instead of that we'll speak of *rows* (of letters), and instead of theory we'll speak of a *page*. Corresponding to hypotheses and conclusions of a proposition we'll speak of the left side and the right side of a row; we'll separate them by the letter $\supset$. We are not

Received March 24, 1960

going to speak of the truth of propositions (as in stage 2) or assertion (as in stage 3). Instead we'll have rules that permit to add new lines to those already on the page. We'll not discuss questions like what the possible consequences of given propositions are; we'll just consider that a page must be *checked* as to whether the rules have been observed that govern the addition of new rows. In order to make possible the formulations of the rules it is convenient to distinguish three types of letters; Roman letters, Greek letters, and Gothic letters (Gothic letters will not be letters in the ordinary sense; by Gothic letters we will mean signs that are not either Roman or Greek letters. Such a sign is, for instance $\supset$ introduced above whose purpose is to separate the left side of a row from the right.). Gothic letters will be irreplaceable; a Greek letter will be irreplaceable within a page; and the rules for replacing Roman letters are as follows: it is permitted to obtain from a given row a new row by replacing Roman letters by other Roman letters provided that if a letter appears in the old row more than once it must be replaced by the same letter in all places of the new row; that is, if the letter A appears in the old row and we want to replace it by a letter S that letter S must replace A in every position in which A appears. On the other hand, it is permitted to replace different letters by the same letter unless this is explicitly prohibited (see below under negation).

In addition to obtaining new rows by replacement of letters we'll have the following fundamental rule for obtaining new rows from old rows. If the right side of one row is the same as the left side of another row it is permitted to write a new row whose left side is the left side of the first and whose right side is the right side of the second (this corresponds, of course, to a kind of syllogism we use in the second or the third stage).

It is permitted in rewriting a line to omit part of the right side or to add anything to the left side.

The following gives additional information about permissible structure of a page together with motivation. There are several situations that occur in earlier stages that can not be taken over into the fourth stage without reinterpretation because they deal with the meaning of symbols.

Existence. The existence of objects satisfying some conditions can not be taken over since we are not supposed to have objects in mind. The formal aspect of the situation is, however, clear; it is that a new letter (or new letters) appears in the right side of a row that does not appear in the left side of that row. For instance, the assertion that there exists a difference of two numbers, that is, that given two numbers a and b there exists a number x such that $a = b + x$ simply means that it is permitted to write $a = b + x$ in the right side of a row if a and b appear, and x does not appear in the left side.

Uniqueness also loses its original meaning, of course: it has no sense to say that there is only one object satisfying some condition if the existence has lost its meaning; neither can we say that if we introduce two letters satisfying certain conditions they stand for the same thing; but we *can* say that if two letters appear in the same surroundings on the left they appear in the same surroundings (one as the other) on the right. Leibniz, I believe, said that the statement that two things are identical means that everything we can say about one we can also say about the other. Here we have what might be called relative identity: if it is permitted to make a statement involving x it must be permitted to make the same statement but involving y instead of x. For instance, uniqueness of subtraction would be translated into a row that would contain $a = b + x$ and $a = b + y$ and also, in addition to that, $p + q = x$ on the left and $p + q = y$ on the right. In other words, instead of saying that if $a = b + x$, $a = b + y$ then any relation that involves x implies the same relation with y substituted for x we must actually write out these relations. This would mean that x is unique with respect to, or relative to these relations; there is no such a thing as absolute uniqueness but we may state uniqueness relative to all relations that occur on our page.

Negation. It is clear that we cannot deny a statement in the sense of stating that it is not true; corresponding to the assertion of a proposition we only permit something *to be written down.* Negation, therefore, would seem to be the prohibition of writing down something. We may consider situations in which we do not use such prohibitions. But if we do wish to include them every new row will have to be checked from both points of view; some affirmative rules will be invoked that permit to write a new row but also we would have to check to see that no prohibition has been broken. As an example we'll use the proposition that if A, B, C are collinear points and A, B, D are collinear then A, C, D are collinear. In this form the proposition is not true in Geometry because if A and B stand for the same point, for instance, a vertex of a triangle, C and D may be the two other vertices and so not be collinear with A. To obtain a true proposition (in the second, or third stage) we must say that A and B are (or stand for) distinct points. This means that they cannot be replaced by the same letter. The prohibition, corresponding to the assertion that B is not A, that is to the negation of the identity of A and B is then in the fourth stage a limitation on the right to replace two distinct letters by the same letter, a prohibition of such a replacement. To make it possible to express simply the corresponding rule we may introduce a new symbol (a "Gothic" letter, as we shall say, although we will write it, to make it easier for the reader to follow, as figure $\neq$). The proposition about collinearity will look like this:

$$A \neq B,\ ABC\ \gamma,\ ABD\ \gamma \supset ACD\ \gamma$$

(Here $ABC\gamma$ is to be read A, B, C, are collinear).

Remark 1. It seems that in Geometry we cannot go far without introducing negation (at least in this form); in Arithmetic, on the other hand, as long as we do not introduce division, or in Group Theory, we can go quite a distance without introducing negation; but sooner or later we have to introduce the prohibition of dividing by zero; and this can also be done by using the symbol $\neq$.

Remark 2. There are other situations in Geometry in which negation is used. As an example we consider the so-called Fano axiom. It states that if in the plane of a triangle ABC we draw a line through each of the three vertices and if all these lines meet in a fourth point O then the meets of these three lines with the three sides of the triangle are *not collinear*. However, this is true only in an actual triangle, that is, if A, B, C are not collinear and an equivalent statement is that if the points P, Q, R such that P is common to AO and BC, Q is common to BO and CQ and R is common to CO and AB are collinear then A, B, and C are collinear. In this form the proposition contains no negation and can be transferred to the fourth stage without changing its form.

It may be that still other situations in which negation is important occur but I have not come across them.

Remark. In earlier stages, for instance in the third stage, one sometimes distinguishes between *elements* and *relations* on these elements. It seems that in the system proposed here this distinction corresponds to the distinction between Roman letter and Greek letters which results in a different rule for replacement.

University of Notre Dame
Notre Dame, Indiana