A DOUBLE-ITERATION PROPERTY OF BOOLEAN FUNCTIONS

CARL LYNGHOLM AND WOLFGANG YOURGRAU

It is the object of this paper to furnish a proof of a theorem

$$f(x) = f(f(f(x))),$$

which is derivable from the fundamental equation for the expansion of a Boolean function of one variable:

$$f(x) = (f(1) \cap x) \cup (f(0) \cap \overline{x}).$$ [1]

From this proposition we may obtain a simple method for rewriting an iterative Boolean function in terms of a non-iterative Boolean function and also for proving the equivalence of two such functions of one variable.

LEMMA 1. $$f(f(1)) = f(0) \cup f(1).$$

PROOF. Using the above fundamental theorem and substituting $f(1)$ for x, we have

$$f(f(1)) = (f(1) \cap f(1)) \cup (f(0) \cap \overline{f(1)})$$

$$= f(1) \cup (f(0) \cap \overline{f(1)}) \qquad [\text{by } x \cap x = x]$$

$$= (f(1) \cup f(0)) \cap (f(1) \cup \overline{f(1)})$$

$$[\text{since } x \cup (y \cap z) = (x \cup y) \cap (x \cup z)]$$

$$= (f(1) \cup f(0)) \cap 1 \qquad [\text{by } x \cup \overline{x} = 1]$$

$$= f(1) \cup f(0). \qquad [\text{since } x \cap 1 = x]$$

LEMMA 2. $$f(f(0)) = f(1) \cap f(0).$$

PROOF. Substituting $f(0)$ for x, we have

$$f(f(0)) = (f(1) \cap f(0)) \cup (f(0) \cap \overline{f(0)}) \qquad [\text{by the fundamental theorem}]$$

$$= (f(1) \cap f(0)) \cup 0 \qquad [\text{by } x \cap \overline{x} = 0]$$

$$= f(1) \cap f(0). \qquad [\text{since } x \cup 0 = x]$$

Again applying the fundamental theorem, we can now arrive at equivalent expressions for $f(f(f(0)))$ and $f(f(f(1)))$.

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LEMMA 3. $f(1) = f(f(f(1)))$.

PROOF. It follows from Lemma *1* that

$$\begin{aligned}
f(f(f(1))) &= f(f(1) \cup f(0)) \\
&= (f(1) \cap (f(1) \cup f(0))) \cup (f(0) \cap \overline{(f(1) \cup f(0))}) && \text{[by the fundamental theorem]} \\
&= f(1) \cup (f(0) \cap \overline{(f(1) \cup f(0))}) && [\text{by } x \cap (x \cup y) = x] \\
&= f(1) \cup (f(0) \cap \overline{\overline{(\overline{f(1)} \cap \overline{f(0)})}}) && [\text{since } x \cup y = \overline{\bar{x} \cap \bar{y}}] \\
&= f(1) \cup (f(0) \cap (\overline{f(1)} \cap \overline{f(0)})) && [\text{by } \bar{\bar{x}} = x] \\
&= f(1) \cup (f(0) \cap (\overline{f(0)} \cap \overline{f(1)})) && [\text{by } x \cap y = y \cap x] \\
&= f(1) \cup ((f(0) \cap \overline{f(0)}) \cap \overline{f(1)}) && [\text{since } x \cap (y \cap z) = (x \cap y) \cap z] \\
&= f(1) \cup (0 \cap \overline{f(1)}) && [\text{by } x \cap \bar{x} = 0] \\
&= f(1) \cup 0 && [\text{by } x \cap 0 = 0] \\
&= f(1)\,. && [\text{since } x \cup 0 = x]
\end{aligned}$$

LEMMA 4. $f(0) = f(f(f(0)))$.

PROOF. According to Lemma *2*,

$$\begin{aligned}
f(f(f(0))) &= f(f(1) \cap f(0)) \\
&= (f(1) \cap (f(1) \cap f(0))) \cup (f(0) \cap \overline{(f(1) \cap f(0))}) && \text{[by the fundamental theorem]} \\
&= ((f(1) \cap f(1)) \cap f(0)) \cup (f(0) \cap \overline{(f(1) \cap f(0))}) && [\text{since } x \cap (y \cap z) = (x \cap y) \cap z] \\
&= (f(1) \cap f(0)) \cup (f(0) \cap \overline{(f(1) \cap f(0))}) && [\text{as } x \cap x = x] \\
&= (f(1) \cap f(0)) \cup (f(0) \cap \overline{\overline{(\overline{f(1)} \cup \overline{f(0)})}}) && [\text{by } x \cap y = \overline{\bar{x} \cup \bar{y}}] \\
&= (f(1) \cap f(0)) \cup (f(0) \cap (\overline{f(1)} \cup \overline{f(0)})) && [\text{since } \bar{\bar{x}} = x] \\
&= (f(1) \cap f(0)) \cup (f(0) \cap (\overline{f(0)} \cup \overline{f(1)})) && [\text{by } x \cup y = y \cup x] \\
&= (f(1) \cap f(0)) \cup ((f(0) \cap \overline{f(0)}) \cup (f(0) \cap \overline{f(1)})) && [\text{since } x \cap (y \cup z) = (x \cap y) \cup (x \cap z)] \\
&= (f(1) \cap f(0)) \cup (0 \cup (f(0) \cap \overline{f(1)})) && [\text{by } x \cap \bar{x} = 0] \\
&= (f(1) \cap f(0)) \cup (f(0) \cap \overline{f(1)}) && [\text{since } x \cup 0 = x] \\
&= (f(0) \cap f(1)) \cup (f(0) \cap \overline{f(1)}) && [\text{by } x \cap y = y \cap x] \\
&= f(0) \cap (f(1) \cup \overline{f(1)}) && [\text{by } x \cap (y \cup z) = (x \cap y) \cup (x \cap z)] \\
&= f(0) \cap 1 && [\text{since } x \cup \bar{x} = 1] \\
&= f(0). && [\text{by } x \cap 1 = x]
\end{aligned}$$

We are now able to prove the double-iteration theorem by means of the fundamental theorem and Lemmas *3* and *4*.

THEOREM. *Let f denote a Boolean function of one variable. Then for all x*

(1) $$f(x) = f(f(f(x))).$$

PROOF. We have already stated Boole's fundamental theorem:

$$f(x) = (f(1) \cap x) \cup (f(0) \cap \bar{x}).$$

We can thus infer that for another function, g,

$$g(x) = (g(1) \cap x) \cup (g(0) \cap \bar{x}).$$

Hence, if $f(1) = g(1)$, and $f(0) = g(0)$, then $f(x) = (g(1) \cap x) \cup (g(0) \cap x) = g(x)$. Let

$$f(f(f(x))) = g(x).$$

Now, we have demonstrated above that

$$f(1) = f(f(f(1)))$$

and that $$f(0) = f(f(f(0))).$$

We may therefore conclude that

$$f(x) = (f(f(f(1))) \cap x) \cup (f(f(f(0) \cap \bar{x}) \quad \text{[by F. T. and Lemmas 3 and 4]}$$
$$= f(f(f(x))).^2 \quad \text{[by F. T.]}$$

This result can be generalized to k variables by using a simple construction. We shall, for the purpose of lucidity, adopt the following definitions:

(2) $$f(f(f(x))) =_{df} f^3(x).$$

(3) $$f_i^3(x_1, x_2 \ldots x_k) =_{df} f(x_1, x_2 \ldots x_{i-1}, f(x_1, x_2 \ldots x_{i-1}, f(x_1, x_2 \ldots x_k), x_{i+1} \ldots x_k), x_{i+1} \ldots x_k). \quad [i = 1 \ldots k]$$

Invoking (1) and (3) we have, on iteration with respect to x_1,

(4) $$f(x_1, x_2 \ldots x_k) = f_1^3(x_1, x_2 \ldots x_k).$$

Further, iterating with respect to x_2, we get

(5) $$f_1^3(x_1, x_2 \ldots x_k) = (f_1^3)_2^3(x_1, x_2 \ldots x_k).$$

Repeating this process k times, we obtain

(6) $$(\ldots((f_1^3)_2^3)\ldots)_{k-1}^3 = (\ldots((f_1^3)_2^3)\ldots)_k^3 .$$

Hence, by the law of transitivity our corollary of the theorem reads:

(7) $$f(x_1, x_2 \ldots x_k) = (\ldots((f_1^3)_2^3)\ldots)_k^3(x_1, x_2 \ldots x_k).^3$$

Although the method here applied can hardly claim elegance, the formal deducibility of the double-iteration property immediately suggests an analogous theorem for certain n-valued algebras.

NOTES

[1] G. BOOLE, *An Investigation of the Laws of Thought . . .*, Dover (New York), p. 72. The theorem in the original version reads

$$f(x) = f(1)\,x + f(0)\,(1 - x).$$

[2] A referee has kindly drawn our attention to the following proof:

$$f(x) = ax + bx.\ f(f(x)) = a(ax + b\bar{x}) + b(\bar{a}x + \bar{b}\bar{x}) = (a + b)\,x + ab\bar{x}.$$

$$f(f(f(x))) = a((a + b)\,x + ab\bar{x}) + b(\bar{a}\bar{b}x + (\bar{a} + \bar{b})\,\bar{x}) = ax + ab\bar{x} +$$

$$\bar{a}b\bar{x} = ax + b\bar{x} = f(x),$$

where "ab" = "$a \cap b$" and "$a + b$" = "$a \cup b$".

[3] We very much appreciate a personal communication by W. V. Quine in which he suggests the following simplification of our proof:

$$f^3_k =_{df} f(w_1, \ldots, w_k, f(w_1, \ldots, w_k, f(w_1, \ldots, w_k, x))) .$$

$$=_{df} F(F(F(x))) .$$

Taking x variously and applying Boole's law of development several times, he arrives at $F(F(F(x))) = F(x)$. However, his result, which appears in our notation as an intermediary step, viz. $f^3_i(x_1, x_2 \ldots x_k) = f_i(x_1, x_2 \ldots x_k)$ for all i, is not the same as (7) above.

Carleton College,
Northfield, Minnesota

and

Smith College,
Northampton, Massachusettes