# STUDIES IN THE AXIOMATIC FOUNDATIONS OF BOOLEAN ALGEBRA

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## Section III

In the preceding sections we have shown that a deductive system, to be referred to as System 𝔄, can be constructed with *A1* as the only axiom and with R1 – R5 as the rules of inference. This system is strong enough to yield ordinary systems of Boolean Algebra. In contradistinction to such systems we can describe System 𝔄 as a system of Boolean Algebra with definitions since it is the rules of definition, R4, and R5 in particular, that are the distinguishing characteristic of the system.

In the present Section two other systems of Boolean Algebra with definitions are outlined and shown to be inferentially equivalent to System 𝔄. The one, to be known as System 𝔅, is based on the functor of strong inclusion as the only undefined term. Its only axiom takes the form of the following thesis:

*B1.* $[a\,b] :: a \sqsubset b \,.\equiv .\therefore [\exists\, c] \,.\, c \sqsubset a \therefore [c\, d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset .$
$[\exists\, e] \,.\, e \sqsubset d \,.\, e \sqsubset b$

As its rules of inference we have R1 – R4, and instead of R5 we have BR5, which allows us to add to the system new theses of the form

XII $[a \ldots] :: a \sqsubset x \,.\equiv .\therefore a \sqsubset a \therefore [b] : b \sqsubset a \,.\supset .\, [\exists\, c] \,.\, c \sqsubset b$
$\phi\,(c)$

provided they satisfy certain conditions analogous to those postulated by R5.

The other system, which we shall call System ℭ, makes use of the functor of partial inclusion as the only undefined term.

Thesis

*C1.* $[a\,b] :: a \,\Delta\, b \,.\equiv .\therefore [\exists\, c] \therefore c \,\Delta\, a \therefore [d] : c \,\Delta\, d \,.\supset .\, a \,\Delta\, d \,.\, b \,\Delta\, d$

serves as the only axiom of the system, whose rules of inference are the same as those of System 𝔄 except that instead of R5 we have CR5.

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In virtue of CR5 we can add to the system new theses of the form

XIII $[a \ldots] :: a \Delta x . \equiv :\cdot: [\exists b] :\cdot: a \Delta b :\cdot: [c] :: b \Delta c . \supset \therefore [\exists d] \therefore d \Delta d \therefore [e] : d \Delta e . \supset . c \Delta e \therefore \phi(d)$

provided they satisfy certain conditions analogous to those set out in R5.

In order to establish inferential equivalence between System $\mathfrak{A}$ and System $\mathfrak{B}$ we have to show that *B1* and, say,

*BD1.* $[a\, b] \therefore a \subset b . \equiv : [c] : c \sqsubset a . \supset . c \sqsubset b$

which could be used as a definition of weak inclusion in terms of strong inclusion, can be deduced within the framework of System $\mathfrak{A}$. We also have to prove that any thesis that could be added to System $\mathfrak{B}$ in virtue of BR5, is derivable in System $\mathfrak{A}$. Then starting with *B1* and *BD1* we have to deduce *A1* and *D4*. In addition we have to satisfy ourselves that any thesis that could be added to System $\mathfrak{A}$ in virtue of R5, could also be obtained within the framework of System $\mathfrak{B}$.

We assume that all the theses to be found in Sections I and II have been derived within System $\mathfrak{A}$. We continue the outline of our deductions as follows:

*T54.* $[a\, b\, c] : a \subset b . c \sqsubset a . \supset . c \sqsubset b$

*Proof:*

$[a\, b\, c] \therefore$
(1) $a \subset b .$
(2) $c \sqsubset a . \supset :$
(3) $[\exists d] . \sim(c \subset d) :$ } [*D4*, 2]
(4) $c \subset a .$ } [*D4*, 2]
(5) $c \subset b :$ [*S2*, 4, 1]
$c \sqsubset b$ [*D4*, 3, 5]

*T55.* $[a\, b\, c\, d\, e] :: [f] : f \sqsubset a . \supset . f \sqsubset b \therefore \sim(c \subset d) . c \subset e . c \subset a \therefore \supset . [\exists f\, g] . \sim(f \subset g) . f \subset e . f \subset b$

*Proof:*

$[a\, b\, c\, d\, e] ::$
(1) $[f] : f \sqsubset a . \supset . f \sqsubset b \therefore$
(2) $\sim(c \subset d) .$
(3) $c \subset e .$
(4) $c \subset a \therefore \supset :$
(5) $c \sqsubset a .$ [*D4*, 2, 4]
(6) $c \sqsubset b .$ [1, 5]
(7) $c \subset b :$ [*D4*, 6]
$[\exists f\, g] . \sim(f \subset g) . f \subset e . f \subset b$ [2, 3, 7]

*T56.* $[a\, b] \therefore [c] : c \sqsubset a . \supset . c \sqsubset b : \supset . a \subset b$

*Proof:*

$[a\, b] :$

(1) $[c] : c \sqsubset a . \supset . c \sqsubset b : \supset \therefore$

(2) $[c\ d\ e] : \sim (c \subset d) . c \subset e . c \subset a . \supset . [\exists\ f\ g] . \sim (f \subset g) .$

$f \subset e . f \subset b \therefore$ [T55, 1]

$a \subset b$ [T16, 2]

T57 = BD1. $[a\ b] \therefore a \subset b . \equiv : [c] : c \sqsubset a . \supset . c \sqsubset b$ [T54, T56]

T58. $[a\ b] : a \sqsubset b . \supset . a \sqsubset a$

*Proof:*

$[a\ b] \therefore$

(1) $a \sqsubset b . \supset :$

(2) $[\exists\ c] . \sim (a \subset c) :$ [D4, 1]

$a \sqsubset a$ [D4, 2, S1]

T59. $[a\ b\ c\ d] : a \sqsubset b . c \sqsubset d . c \sqsubset a . \supset . [\exists\ e] . e \sqsubset d . e \sqsubset b$

*Proof:*

$[a\ b\ c\ d] \therefore$

(1) $a \sqsubset b .$

(2) $c \sqsubset d .$

(3) $c \sqsubset a . \supset :$

(4) $a \subset b .$ [D4, 1]

(5) $c \sqsubset b :$ [T54, 4, 3]

$[\exists\ e] . e \sqsubset d . e \sqsubset b$ [2, 5]

T60. $[a\ b\ f\ g\ h] :: [c\ d] : c \sqsubset d . c \sqsubset a . \supset . [\exists\ e] . e \sqsubset d . e \sqsubset b \therefore$
$\sim (f \subset g) . f \subset h . f \subset a \therefore \supset . [\exists\ i\ j] . \sim (i \subset j) . i \subset h . i \subset b$

*Proof:*

$[a\ b\ f\ g\ h] ::$

(1) $[c\ d] : c \sqsubset d . c \sqsubset a . \supset . [\exists\ e] . e \sqsubset d . e \sqsubset b \therefore$

(2) $\sim (f \subset g) .$

(3) $f \subset h .$

(4) $f \subset a \therefore \supset \therefore$

(5) $f \sqsubset h .$ [D4, 2, 3]

(6) $f \sqsubset a \therefore$ [D4, 2, 4]

$[\exists\ e] :$

(7) $e \sqsubset h .$

(8) $e \sqsubset b :$ } [1, 5, 6]

(9) $[\exists\ j] . \sim (e \subset j) :$

(10) $e \subset h .$ } [D4, 7]

(11) $e \subset b \therefore$ [D4, 8]

$[\exists\ i\ j] . \sim (i \subset j) . i \subset h . i \subset b$ [9, 10, 11]

T61. $[a\ b\ h] :: h \sqsubset a \therefore [c\ d] : c \sqsubset d . c \sqsubset a . \supset . [\exists\ e] . e \sqsubset d . e \sqsubset b$
$\therefore \supset . a \sqsubset b$

*Proof:*

$[a\ b\ h] ::$

(1) $b \sqsubset a \therefore$

(2) $[c\ d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset .\, [\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b \therefore \supset \therefore$

(3) $b \subset a :$

$[\exists\ c] \therefore$ } [$D4$, 1]

(4) $\sim (b \subset c) \,.$

(5) $\sim (a \subset c) \therefore$ [$S2$, 3, 4]

(6) $[f\ g\ h] : \sim (f \subset g) \,.\, f \subset h \,.\, f \subset a \,.\supset .\, [\exists\ i\ j] \,.\sim (i \subset j) \,.\, i \subset h \,.\, i \subset b \therefore$ [$T60$, 2]

(7) $a \subset b \therefore$ [$T16$, 6]

$a \sqsubset b$ [$D4$, 5, 7]

$T62 = B1.$ $[a\ b] :: a \sqsubset b \,.\equiv \therefore [\exists\ c] \,.\, c \sqsubset a \therefore [c\ d] : c \sqsubset d \,.\, c \sqsubset a \,. \supset .\, [\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b$ [$T58$, $T59$, $T61$]

$T63.$ $[a\ b\ c\ \phi] :\therefore [d] \therefore d \subset a \,.\equiv : [e\ f] : \sim (e \subset f) \,.\, e \subset d \,.\supset .\, [\exists\ g\ h] \,.\, \sim (g \subset h) \,.\, g \subset e \,.\, \phi (g) :: b \sqsubset a \,.\, c \sqsubset b :: \supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi (d)$

*Proof:*

$[a\ b\ c\ \phi] :\therefore$

(1) $[d] \therefore d \subset a \,.\equiv : [e\ f] : \sim (e \subset f) \,.\, e \subset d \,.\supset .\, [\exists\ g\ h] \,.\sim (g \subset h) \,.\, g \subset e \,.\, \phi (g) ::$

(2) $b \sqsubset a \,.$

(3) $c \sqsubset b :: \supset \therefore$

(4) $b \subset a \therefore$ [$D4$, 2]

$[\exists\ f] :$ } [$D4$, 3]

(5) $\sim (c \subset f) \,.$

(6) $c \subset b :$

$[\exists\ g\ h] \,.$ } [1, 4, 5, 6]

(7) $\sim (g \subset h) \,.$

(8) $g \subset c \,.$

(9) $\phi (g) \,.$

(10) $g \sqsubset c \therefore$ [$D4$, 7, 8]

$[\exists\ d] \,.\, d \sqsubset c \,.\, \phi (d)$ [10, 9]

$T64.$ $[b\ e\ f\ \phi] :: [c] : c \sqsubset b \,.\supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi (d) \therefore \sim (e \subset f) \,.\, e \subset b \therefore \supset .\, [\exists\ g\ h] \,.\sim (g \subset h) \,.\, g \subset e \,.\, \phi (g)$

*Proof:*

$[b\ e\ f\ \phi] ::$

(1) $[c] : c \sqsubset b \,.\supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi (d) \therefore$

(2) $\sim (e \subset f) \,.$

(3) $e \subset b \therefore \supset \therefore$

(4) $e \sqsubset b \therefore$ [$D4$, 2, 3]

$[\exists\ d] :$ } [1, 4]

(5) $d \sqsubset e \,.$

(6) $\phi (d) :$

(7) $[\exists\ h] \,.\sim (d \subset h) :$ } [$D4$, 5]

(8) $d \subset e \therefore$

$[\exists\ g\ h] \,.\sim (g \subset h) \,.\, g \subset e \,.\, \phi (g)$ [7, 8, 6]

*T*65. $[a\ b\ \phi] :\cdot: [d] \therefore d \subset a . \equiv : [e\ f] : \sim (e \subset f) . e \subset d . \supset . [\exists\ g\ h] . \sim (g \subset h) . g \subset e . \phi(g) :: h \sqsubset b \therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :: \supset . b \sqsubset a$

*Proof:*

$[a\ b\ \phi] :\cdot:$

(1) $[d] \therefore d \subset a . \equiv : [e\ f] : \sim (e \subset f) . e \subset d . \supset . [\exists\ g\ h] . \sim (g \subset h) . g \subset e . \phi(g) ::$

(2) $h \sqsubset b \therefore$

(3) $[c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :: \supset \therefore$

(4) $[e\ f] : \sim (e \subset f) . e \subset b . \supset . [\exists\ g\ h] . \sim (g \subset h) . g \subset e . \phi(g) \therefore$ [*T*64, 3]

(5) $b \subset a :$ [1, 4]

(6) $[\exists\ c] . \sim (b \subset c) \therefore$ [*D*4, 2]

$b \sqsubset a$ [*D*4, 6, 5]

*T*66. $[a\ \phi] :\cdot: [d] \therefore d \subset a . \equiv : [e\ f] : \sim (e \subset f) . e \subset d . \supset . [\exists\ g\ h] . \sim (g \subset h) . g \subset e . \phi(g) \therefore \supset :: [b] :: b \sqsubset a . \equiv \therefore b \sqsubset b \therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d)$ [*T*58, *T*63, *T*65]

It is clear from *T*62 and *T*57 that *B*1 and *BD*1 are derivable within the framework of System 𝔄. Moreover, *T*66 shows that any thesis that could be added to System 𝔅 in virtue of *BR*5, could also be derived in System 𝔄.

We now proceed to show that within System 𝔅 we can derive *A*1, *D*4, and any thesis that could be obtained in System 𝔄 by applying R5.

*T*66**BD*2. $[a] :: a \sqsubset \wedge . \equiv \therefore a \sqsubset a \therefore [b] : b \sqsubset a . \supset . [\exists\ c] . c \sqsubset b . \sim (c \sqsubset c)$ [by applying BR5]

*T*66*1. $[a\ b] : a \sqsubset b . \supset . a \sqsubset a$ [*B*1 = *T*62]

*T*66*2. $[a] : a \sqsubset \wedge . \supset . \sim (a \sqsubset \wedge)$

*Proof:*

$[a] :$

(1) $a \sqsubset \wedge . \supset .$

(2) $a \sqsubset a .$ [*T*66**BD*2, 1]

$\sim (a \sqsubset \wedge)$ [*T*66**BD*2, 2, *T*66*1]

*T*66*3. $[a] . \sim (a \sqsubset \wedge)$ [*T*66*2]

*T*66*4. $[a\ b\ c\ d\ e] : b \sqsubset c . a \sqsubset b . d \sqsubset e . d \sqsubset a . \supset . [\exists\ f] . f \sqsubset e . f \sqsubset c$

*Proof:*

$[a\ b\ c\ d\ e] ::$

(1) $b \sqsubset c .$

(2) $a \sqsubset b .$

(3) $d \sqsubset e .$

(4) $d \sqsubset a . \supset \therefore$

(5) $[f\ g] : f \sqsubset g . f \sqsubset a . \supset . [\exists\ h] . h \sqsubset g . h \sqsubset b \therefore$ [*B*1 = *T*62, 2]

(6) $[f\ g] \mathbin{:}\ f \sqsubset g \,.\, f \sqsubset b \,.\supset.\, [\exists\ h] \,.\, h \sqsubset g \,.\, h \sqsubset c \mathbin{\therefore}$ [$B1 = T62$, 1]
(7) $[\exists\ h] \,.\, h \sqsubset e \,.\, h \sqsubset b \mathbin{\therefore}$ [5, 3, 4]
$[\exists\ f] \,.\, f \sqsubset e \,.\, f \sqsubset c$ [6, 7]

*T66*5.* $[a\ b\ c] : b \sqsubset c \,.\, a \sqsubset b \,.\supset.\, a \sqsubset c$

*Proof:*

$[a\ b\ c] : :$
(1) $b \sqsubset c \,.$
(2) $a \sqsubset b \,.\supset \mathbin{\therefore}$
(3) $[\exists\ d] \,.\, d \sqsubset a \mathbin{\therefore}$ [$B1 = T62$, 2]
(4) $[d\ e] : d \sqsubset e \,.\, d \sqsubset a \,.\supset.\, [\exists\ f] \,.\, f \sqsubset e \,.\, f \sqsubset c \mathbin{\therefore}$ [$T66*4$, 1, 2]
$a \sqsubset c$ [$B1 = T62$, 3, 4]

*T66*6.* $[a\ b] : a \sqsubset b \,.\supset.\, a \subset b$ [$T66*5$, $BD1 = T57$]

*T66*7.* $[a\ b] : a \sqsubset b \,.\supset.\, [\exists\ c] \,.\, \sim(a \subset c)$

*Proof:*

$[a\ b] \mathbin{\therefore}$
(1) $a \sqsubset b \,.\supset :$
$[\exists\ c] \,.$ } [$B1 = T62$, 1]
(2) $c \sqsubset a \,.$
(3) $\sim(c \sqsubset \wedge) :$ [$T66*3$]
(4) $\sim(a \subset \wedge) :$ [$BD1 = T57$, 2, 3]
$[\exists\ c] \,.\, \sim(a \subset c)$ [4]

*T66*8.* $[a\ b\ c] : a \subset b \,.\, \sim(a \subset c) \,.\supset.\, a \sqsubset b$ .

*Proof:*
$[a\ b\ c] : :$
(1) $a \subset b \,.$
(2) $\sim(a \subset c) \,.\supset \mathbin{\therefore}$
(3) $[d] : d \sqsubset a \,.\supset.\, d \sqsubset b \mathbin{\therefore}$ [$BD1 = T57$, 1]
(4) $[\exists\ d] \,.\, d \sqsubset a :$ [$BD1 = T57$, 2]
(5) $a \sqsubset a \mathbin{\therefore}$ [$B1 = T62$, 4]
$a \sqsubset b$ [3, 5]

*T66*9 = D4.* $[a\ b] \mathbin{\therefore} a \sqsubset b \,.\equiv : a \subset b : [\exists\ c] \,.\, \sim(a \subset c)$
[$T66*6$, $T66*7$, $T66*8$]

*T66*10.* $[a\ b\ c] : a \subset b \,.\, c \subset a \,.\supset.\, c \subset b$ [$BD1 = T57$]

*T66*11.* $[a\ b\ c\ d\ e] : a \subset b \,.\, \sim(c \subset d) \,.\, c \subset e \,.\, c \subset a \,.\supset.\, [\exists\ f\ g] \,.\, \sim(f \subset g) \,.$
$f \subset e \,.\, f \subset b$

*Proof:*

$[a\ b\ c\ d\ e] \mathbin{\therefore}$
(1) $a \subset b \,.$
(2) $\sim(c \subset d) \,.$

(3) $c \subset e\, .$
(4) $c \subset a\, .\supset :$
(5) $c \subset b :$ [$T66^{*}10$, 1, 4]
$[\exists\ f\, g]\, .\sim(f \subset g)\, .\, f \subset e\, .\, f \subset b$ [2, 3, 5]

$T66^{*}12$. $[a\, b\, h\, i] :: [c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .$
$f \subset e\, .\, f \subset b\, .\therefore h \sqsubset i\, .\, h \sqsubset a\, .\therefore \supset .\, [\exists\ f]\, .\, f \sqsubset i\, .\, f \sqsubset b$

*Proof:*

$[a\, b\, h\, i] ::$
(1) $[c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .\, f \subset e\, .$
$f \subset b\, .\therefore$
(2) $h \sqsubset i\, .$
(3) $h \sqsubset a\, .\therefore \supset :$
(4) $[\exists\ c]\, .\sim(h \subset c) :$ [$T66^{*}7$, 2]
(5) $h \subset i\, .$ [$T66^{*}6$, 2]
(6) $h \subset a :$ [$T66^{*}6$, 3]
$[\exists\ f\, g]\, .$
(7) $\sim(f \subset g)\, .$
(8) $f \subset i\, .$
(9) $f \subset b\, .$ [1, 4, 5, 6]
(10) $f \sqsubset i\, .$ [$T66^{*}8$, 8, 7]
(11) $f \sqsubset b :$ [$T66^{*}8$, 9, 7]
$[\exists\ f]\, .\, f \sqsubset i\, .\, f \sqsubset b$ [10, 11]

$T66^{*}13$. $[a\, b\, h] :: [c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .$
$f \subset e\, .\, f \subset b\, .\therefore h \sqsubset a\, .\therefore \supset .\, h \sqsubset b$

*Proof:*

$[a\, b\, h] ::$
(1) $[c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .\, f \subset e\, .\, f \subset b\, .\therefore$
(2) $h \sqsubset a\, .\therefore \supset .\therefore$
(3) $[i\, j] : i \sqsubset j\, .\, i \sqsubset a\, .\supset .\, [\exists\ f]\, .\, f \sqsubset j\, .\, f \sqsubset b\, .\therefore$ [$T66^{*}12$, 1]
(4) $a \sqsubset b\, .\therefore$ [$B1 = T62$, 2, 3]
$h \sqsubset b$ [$T66^{*}5$, 4, 2]

$T66^{*}14$. $[a\, b] :: [c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .\, f$
$\subset e\, .\, f \subset b\, .\therefore \supset .\, a \subset b$

*Proof:*

$[a\, b] ::$
(1) $[c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f\, g]\, .\sim(f \subset g)\, .\, f \subset e\, .\, f \subset b$
$.\therefore \supset .\therefore$
(2) $[h] : h \sqsubset a\, .\supset .\, h \sqsubset b\, .\therefore$ [$T66^{*}13$, 1]
$a \subset b$ [$BD1 = T57$, 2]

$T66^{*}15 = A1$ $[a\, b]\, .\therefore a \subset b\, .\equiv : [c\, d\, e] : \sim(c \subset d)\, .\, c \subset e\, .\, c \subset a\, .\supset .\, [\exists\ f$
$g]\, .\sim(f \subset g)\, .\, f \subset e\, .\, f \subset b$ [$T66^{*}11$, $T66^{*}14$]

*T66*16.* $[a\ e\ f\ g\ \phi] \vdots\vdots [b] :: b \sqsubset a . \equiv .\therefore b \sqsubset b .\therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :\cdot: e \subset a . \sim (f \subset g) . f \subset e :\cdot: \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h)$

*Proof:*

$[a\ e\ f\ g\ \phi] \vdots\vdots$

(1) $[b] :: b \sqsubset a . \equiv .\therefore b \sqsubset b .\therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :\cdot:$

(2) $e \subset a .$

(3) $\sim (f \subset g) .$

(4) $f \subset e :\cdot: \supset .\therefore$

(5) $\sim (e \subset g) .$ [*T66*10*, 4, 3]

(6) $e \sqsubset a .$ [*T66*8*, 2, 5]

(7) $f \sqsubset e .\therefore$ [*T66*8*, 4, 3]

$[\exists\ h] :$

(8) $h \sqsubset f .$

(9) $\phi(h) :$ [1, 6, 7]

(10) $[\exists\ i] . \sim (h \subset i) :$ [*T66*7*, 8]

(11) $h \subset f .\therefore$ [*T66*6*, 8]

$[\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h)$ [10, 11, 9]

*T66*17.* $[c\ e\ \phi] :: [f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) .\therefore c \sqsubset e .\therefore \supset . [\exists\ d] . d \sqsubset c . \phi(d)$

*Proof:*

$[c\ e\ \phi] ::$

(1) $[f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) .\therefore$

(2) $c \sqsubset e .\therefore \supset :$

(3) $[\exists\ f] . \sim (c \subset f) :$ [*T66*7*, 2]

(4) $c \subset e :$ [*T66*6*, 2]

$[\exists\ h\ i] .$

(5) $\sim (h \subset i)$

(6) $h \subset c .$ [1, 3, 4]

(7) $\phi(h) .$

(8) $h \sqsubset c$ [*T66*8*, 6, 5]

$[\exists\ d] . d \sqsubset c . \phi(d)$ [8, 7]

*T66*18.* $[a\ e\ j\ \phi] \vdots\vdots [b] :: b \sqsubset a . \equiv .\therefore b \sqsubset b .\therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :\cdot: [f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) .\therefore j \sqsubset e :\cdot: \supset . j \sqsubset a$

*Proof:*

$[a\ e\ j\ \phi] \vdots\vdots$

(1) $[b] :: b \sqsubset a . \equiv .\therefore b \sqsubset b .\therefore [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :\cdot:$

(2) $[f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) .\therefore$

(3) $j \sqsubset e :\cdot: \supset .\therefore$

(4) $[c] : c \sqsubset e . \supset . [\exists\ d] . d \sqsubset c . \phi(d) .\therefore$ [*T66*17*, 2]

(5) $e \sqsubset e$ [*B1* = *T62*, 3]

(6) $e \sqsubset a$ .·. [1, 5, 4]
$j \sqsubset a$ [*T66*5*, 6, 3]

*T66*19*. $[a\ e\ \phi] \vdots \vdots [b] :: b \sqsubset a . \equiv .\!\cdot\!. b \sqsubset b .\!\cdot\!. [c] : c \sqsubset b . \supset . [\exists\ d] .\ d \sqsubset c . \phi(d) :\cdot: [f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) :\cdot: \supset . e \subset a$

*Proof:*

$[a\ e\ \phi] \vdots \vdots$
(1) $[b] :: b \sqsubset a . \equiv .\!\cdot\!. b \sqsubset b .\!\cdot\!. [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :\cdot:$
(2) $[f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h) :\cdot: \supset .\!\cdot\!.$
(3) $[j] : j \sqsubset e . \supset . j \sqsubset a$ .·. [*T66*18*, 1, 2]
$e \subset a$ [*BD1* = *T57*, 3]

*T66*20*. $[a\ \phi] :\cdot: [b] :: b \sqsubset a . \supset .\!\cdot\!. b \sqsubset b .\!\cdot\!. [c] : c \sqsubset b . \supset . [\exists\ d] . d \sqsubset c . \phi(d) :: \supset .\!\cdot\!. [e] .\!\cdot\!. e \subset a . \equiv : [f\ g] : \sim (f \subset g) . f \subset e . \supset . [\exists\ h\ i] . \sim (h \subset i) . h \subset f . \phi(h)$ [*T66*16*, *T66*19*]

From *T66*15*, *T66*9*, and *T66*20* we see that *A1*, *D4*, and any thesis introduced into System $\mathfrak{A}$ by applying R5 are all obtainable within the framework of System $\mathfrak{B}$, which completes the proof that the two systems are inferentially equivalent.

We continue by giving an outline of the proof that System $\mathfrak{C}$ is inferentially equivalent to System $\mathfrak{B}$.

In the first stage of the outline we propose to show that *C1* and, say,

*CD1*. $[a\ b] :: a \sqsubset b . \equiv .\!\cdot\!. a \Delta a .\!\cdot\!. [c] : a \Delta c . \supset . b \Delta c$

which in System $\mathfrak{C}$ could be used as the definition of strong inclusion in terms of partial inclusion, are both derivable within the framework of System $\mathfrak{B}$. In addition we will show that System $\mathfrak{B}$ yields any thesis obtainable in System $\mathfrak{C}$ in virtue of CR5.

By applying R4 within System $\mathfrak{B}$ we get

*T66*BD3*. $[a\ b] : a \Delta b . \equiv . [\exists\ c] . c \sqsubset a . c \sqsubset b$

Further deductions proceed as follows:

*T66*21*. $[a\ b] : a \sqsubset b . \supset . a \Delta a$ [*T66*1*, *T66*BD3*]

*T66*22*. $[a\ b\ c] : a \sqsubset b . a \Delta c . \supset . b \Delta c$

*Proof:*

$[a\ b\ c]$ .·.
(1) $a \sqsubset b$ .
(2) $a \Delta c . \supset :$
$[\exists\ d]$ .
(3) $d \sqsubset a$ .
(4) $d \sqsubset c$ . } [*T66*BD3*, 2]
(5) $d \sqsubset b$ : [*T66*5*, 1, 3]
$b \Delta c$ [*T66*BD3*, 5, 4]

$T66^*23$. $[a\ b] \mathbin{:\cdot}\ a \Delta b\ .\supset .\ b \Delta a$ [$T66^*BD3$]

$T66^*24$. $[a\ b\ d\ e] \mathbin{:\cdot} : [c] \mathbin{:\cdot}\ a \Delta c\ .\supset .\ b \Delta c \therefore d \sqsubset e\ .\ d \sqsubset a \therefore \supset .\ [\exists\ f] \mathbin{\cdot\cdot}$
$f \sqsubset e\ .\ f \sqsubset b$

*Proof:*

$[a\ b\ d\ e] : :$
(1) $[c] : a \Delta c\ .\supset .\ b \Delta c \therefore$
(2) $d \sqsubset e\ .$
(3) $d \sqsubset a \therefore \supset :$
(4) $a \Delta e\ .$ [$T66^*BD3$, 3, 2]
(5) $b \Delta e\ .$ [1, 4]
(6) $e \Delta b :$ [$T66^*23$, 5]
$[\exists\ f]\ .\ f \sqsubset e\ .\ f \sqsubset b$ [$T66^*BD3$, 6]

$T66^*25$. $[a\ b] : : a \Delta a \therefore [c] \mathbin{:\cdot}\ a \Delta c\ .\supset .\ b \Delta c \therefore \supset .\ a \sqsubset b$

*Proof:*

$[a\ b] \mathbin{:\cdot} :$
(1) $a \Delta a \therefore$
(2) $[c] : a \Delta c\ .\supset .\ b \Delta c \therefore \supset \therefore$
(3) $[\exists\ c] \mathbin{\cdot\cdot}\ c \sqsubset a \therefore$ [$T66^*BD3$, 1]
(4) $[c\ d] \mathbin{:\cdot}\ c \sqsubset d\ .\ c \sqsubset a\ .\supset .\ [\exists\ e] \mathbin{\cdot\cdot}\ e \sqsubset d\ .\ e \sqsubset b \therefore$ [$T66^*24$, 2]
$a \sqsubset b$ [$B1 = T62$, 3, 4]

$T66^*26 = CD1$. $[a\ b] \mathbin{:\cdot} : a \sqsubset b\ .\equiv \therefore a \Delta a \therefore [c] \mathbin{:\cdot}\ a \Delta c\ .\supset .\ b \Delta c$
[$T66^*21$, $T66^*22$, $T66^*25$]

$T66^*27$. $[a\ b\ c\ d] : c \sqsubset a\ .\ c \sqsubset b\ .\ c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d$ [$T66^*22$]

$T66^*28$. $[a\ b] \mathbin{:\cdot} : a \Delta b\ .\supset \therefore [\exists\ c] \mathbin{\cdot\therefore}\ c \Delta a \therefore [d] : c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d$

*Proof:*

$[a\ b] \mathbin{:\cdot\cdot:}$
(1) $a \Delta b\ .\supset : :$
$[\exists\ c] \therefore$
(2) $c \sqsubset a\ .$ } [$T66^*BD3$, 1]
(3) $c \sqsubset b\ .$
(4) $c \Delta c\ .$ [$T66^*21$, 2]
(5) $a \Delta c\ .$ [$T66^*22$, 2, 4]
(6) $c \Delta a \therefore$ [$T66^*23$, 5]
(7) $[d] \mathbin{:\cdot}\ c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d : :$ [$T66^*27$, 2, 3]
$[\exists\ c] \mathbin{\cdot\therefore}\ c \Delta a \therefore [d] : c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d$ [6, 7]

$T66^*29$. $[a\ b\ c] : : c \Delta a \therefore [d] \mathbin{:\cdot}\ c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d \therefore \supset .\ a \Delta b$

*Proof:*

$[a\ b\ c] : :$
(1) $c \Delta a \therefore$
(2) $[d] \mathbin{:\cdot}\ c \Delta d\ .\supset .\ a \Delta d\ .\ b \Delta d \therefore \supset :$

(3) $[\exists\ d]\,.\,d \sqsubset c :$ [$T66^*BD3$, 1]
(4) $c\ \Delta\ c\ .$ [$T66^*BD3$, 3]
(5) $c \sqsubset a\ .$
(6) $c \sqsubset b :$ } [$T66^*25$, 4, 2]
$a\ \Delta\ b$ [$T66^*BD3$, 5, 6]

$T66^*30 = C1$. $[a\ b] : : a\ \Delta\ b\ . \equiv . \therefore [\exists\ c] \therefore c\ \Delta\ a \therefore [d] : c\ \Delta\ d\ . \supset .\ a\ \Delta\ d\ .\ b\ \Delta\ d$ [$T66^*28$, $T66^*29$]

$T66^*31$. $[a\ f\ g\ \phi] \vdots\vdots [b] : : b \sqsubset a\ . \equiv . \therefore b \sqsubset b \therefore [c] : c \sqsubset b\ . \supset . [\exists\ d]\,.\,d \sqsubset c\ .\ \phi\,(d) :\cdot: f \sqsubset a\ .\ f\ \Delta\ g :\cdot: \supset . \therefore [\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h)$

*Proof:*

$[a\ f\ g\ \phi] \vdots\vdots$
(1) $[b] : : b \sqsubset a\ . \equiv . \therefore b \sqsubset b \therefore [c] : c \sqsubset b\ . \supset . [\exists\ d]\,.\,d \sqsubset c\ .\ \phi\,(d) :\cdot:$
(2) $f \sqsubset a\ .$
(3) $f\ \Delta\ g :\cdot: \supset :\cdot:$
$[\exists\ e] : :$
(4) $e \sqsubset f\ .$
(5) $e \sqsubset g : :$ } [$T66^*BD3$, 3]
$[\exists\ h] \therefore$
(6) $h \sqsubset e\ .$
(7) $\phi\,(h)\ .$ } [1, 2, 4]
(8) $h\ \Delta\ h\ .$ [$T66^*21$, 6]
(9) $h \sqsubset g \therefore$ [$T66^*5$, 5, 6]
(10) $[i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i :\cdot:$ [$T66^*22$, 9]
$[\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h)$ [8, 10, 7]

$T66^*32$. $[a\ e\ \phi] \vdots\vdots [b] : : b \sqsubset a\ . \equiv . \therefore b \sqsubset b \therefore [c] : c \sqsubset b\ . \supset . [\exists\ d]\,.\,d \sqsubset c\ .\ \phi\,(d) :\cdot: e\ \Delta\ a :\cdot: \supset :\cdot: [\exists\ f] :\cdot: e\ \Delta\ f :\cdot: [g] : : f\ \Delta\ g\ . \supset . \therefore [\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h)$

*Proof:*

$[a\ e\ \phi] \vdots\cdot\vdots$
(1) $[b] : : b \sqsubset a\ . \equiv . \therefore b \sqsubset b \therefore [c] : c \sqsubset b\ . \supset . [\exists\ d]\,.\,d \sqsubset c\ .\ \phi\,(d) :\cdot:$
(2) $e\ \Delta\ a :\cdot: \supset \vdots\vdots$
$[\exists\ f] :\cdot:$
(3) $f \sqsubset e\ .$
(4) $f \sqsubset a\ .$ } [$T66^*BD3$, 2]
(5) $f\ \Delta\ f\ .$ [$T66^*21$, 3]
(6) $e\ \Delta\ f :\cdot:$ [$T66^*22$, 3, 5]
(7) $[g] : : f\ \Delta\ g\ . \supset . \therefore [\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h) \vdots\vdots$ [$T66^*31$, 1, 4]
$[\exists\ f] :\cdot: e\ \Delta\ f :\cdot: [g] : : f\ \Delta\ g\ . \supset . \therefore [\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h)$ [6, 7]

$T66^*33$. $[d\ e\ f\ \phi] \vdots\vdots [g] : : f\ \Delta\ g\ . \supset . \therefore [\exists\ h] \therefore h\ \Delta\ h \therefore [i] : h\ \Delta\ i\ . \supset .\ g\ \Delta\ i \therefore \phi\,(h) :\cdot: e \sqsubset f\ .\ d \sqsubset e :\cdot: \supset . [\exists\ h]\,.\,h \sqsubset d\ .\ \phi\,(h)$

*Proof:*

$[d\ e\ f\ \phi] \mathrel{::}$

(1) $[g] \mathrel{::} f \Delta g \,.\supset \therefore [\exists\ h] \therefore h \Delta h \therefore [i] : h \Delta i \,.\supset .\, g \Delta i \therefore \phi(h) \mathrel{:\cdot:}$

(2) $e \sqsubset f\,.$

(3) $d \sqsubset e \mathrel{:\cdot:} \supset \mathrel{::}$

(4) $d \sqsubset f\,.$ [T66*5, 2, 3]

(5) $d \sqsubset d\,.$ [T66*1, 4]

(6) $f \Delta d \mathrel{::}$ [T66*BD3, 4, 5]

$[\exists\ h] \mathrel{::}$

(7) $h \Delta h \therefore$

(8) $[i] : h \Delta i \,.\supset .\, d \Delta i \therefore$ [1, 6]

(9) $\phi(h) \mathrel{::}$

(10) $h \sqsubset d \mathrel{::}$ [T66*25, 7, 8]

$[\exists\ h] \,.\, h \sqsubset d \,.\, \phi(h)$ [10, 9]

*T66*34*. $[a\ e\ f\ \phi] \mathrel{::} [b] \mathrel{::} b \sqsubset a \,.\equiv \therefore b \sqsubset b \therefore [c] : c \sqsubset b \,.\supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi(d) \mathrel{:\cdot:} e \Delta f \mathrel{:\cdot:} [g] \mathrel{::} f \Delta g \,.\supset \therefore [\exists\ h] \therefore h \Delta h \therefore [i] : h \Delta i \,.\supset .\, g \Delta i \therefore \phi(h) \mathrel{:\cdot:} \supset .\, e \Delta a$

*Proof:*

$[a\ e\ f\ \phi] \mathrel{::}$

(1) $[b] \mathrel{::} b \sqsubset a \,.\equiv \therefore b \sqsubset b \therefore [c] : c \sqsubset b \,.\supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi(d) \mathrel{:\cdot:}$

(2) $e \Delta f \mathrel{:\cdot:}$

(3) $[g] \mathrel{::} f \Delta g \,.\supset \therefore [\exists\ h] \therefore h \Delta h \therefore [i] : h \Delta i \,.\supset .\, g \Delta i \therefore \phi(h) \mathrel{:\cdot:} \supset \mathrel{::}$

$[\exists\ g] \therefore$

(4) $g \sqsubset e\,.$ [T66*BD3, 2]

(5) $g \sqsubset f\,.$

(6) $g \sqsubset g \therefore$ [T66*1, 4]

(7) $[d] : d \sqsubset g \,.\supset .\, [\exists\ h] \,.\, h \sqsubset d \,.\, \phi(h) \therefore$ [T66*33, 3, 5]

(8) $g \sqsubset a \mathrel{::}$ [1, 6, 7]

$e \Delta a$ [T66*BD3, 4, 8]

*T66*35*. $[a\ \phi] \mathrel{:\cdot:} [b] \mathrel{::} b \sqsubset a \,.\equiv \therefore b \sqsubset b \therefore [c] : c \sqsubset b \,.\supset .\, [\exists\ d] \,.\, d \sqsubset c \,.\, \phi(d) \mathrel{::} \supset \mathrel{::} [e] \mathrel{::} e \Delta a \,.\equiv \mathrel{:\cdot:} [\exists\ f] \mathrel{:\cdot:} e \Delta f \mathrel{:\cdot:} [g] \mathrel{::} f \Delta g \,.\supset \therefore [\exists\ h] \therefore h \Delta h \therefore [i] : h \Delta i \,.\supset .\, g \Delta i \therefore \phi(h)$

[T66*32, T66*34]

It is evident from *T66*30*, *T66*26*, and *T66*35* that any thesis derivable in System $\mathfrak{C}$ is also derivable in System $\mathfrak{B}$. In order to complete the outline of the proof that System $\mathfrak{B}$ and System $\mathfrak{C}$ are inferentially equivalent, we now go on to derive *B1* = *T62* and *T66*BD3* from *C1* = *T66*30* and *CD1* = *T66*26*. We shall also have to satisfy ourselves that any thesis that could be added to System $\mathfrak{B}$ in virtue of BR5, is obtainable in System $\mathfrak{C}$ as well. Our deductions proceed as follows:

*T66*35*1*. $[a\ b] : a \Delta b \,.\supset .\, a \Delta a$ [C1 = T66*30]

*T66*35*2*. $[a\ b] : a \Delta b \,.\supset .\, b \Delta a$ [C1 = T66*30]

$T66{*}35{*}3.\quad [a\ b] : a \Delta b \,.\supset.\, b \Delta b$ $\qquad$ $[T66{*}35{*}2,\ T66{*}35{*}1]$

$T66{*}35{*}4.\quad [a\ b] : a \Delta b \,.\supset.\, [\exists\ c] \,.\, c \sqsubset a \,.\, c \sqsubset b$

*Proof:*

$[a\ b] ::$

(1) $a \Delta b \,.\supset ::$

$[\exists\ c] \therefore$

(2) $c \Delta a \therefore$

(3) $[d] : c \Delta d \,.\supset.\, a \Delta d \therefore$ $\qquad$ $[C1 = T66{*}30,\ 1]$

(4) $[d] : c \Delta d \,.\supset.\, b \Delta d \therefore$

(5) $c \Delta c ::$ $\qquad$ $[T66{*}35{*}1,\ 2]$

$[\exists\ c] \,.\, c \sqsubset a \,.\, c \sqsubset b$ $\qquad$ $[CD1 = T66{*}26,\ 5,\ 3,\ 4]$

$T66{*}35{*}5.\quad [a\ b\ c\ d] : c \sqsubset a \,.\, c \sqsubset b \,.\, c \Delta d \,.\supset.\, a \Delta d \,.\, b \Delta d$ $\qquad$ $[CD1 = T66{*}26]$

$T66{*}35{*}6.\quad [a\ b] : c \sqsubset a \,.\, c \sqsubset b \,.\supset.\, a \Delta b$

*Proof:*

$[a\ b] ::$

(1) $c \sqsubset a \,.$

(2) $c \sqsubset b \,.\supset \therefore$

(3) $a \Delta c \,.$ $\qquad$ $[CD1 = T66{*}26,\ 1]$

(4) $c \Delta a \therefore$ $\qquad$ $[T66{*}35{*}2,\ 3]$

(5) $[d] : c \Delta d \,.\supset.\, a \Delta d \,.\, b \Delta d \therefore$ $\qquad$ $[T66{*}35{*}5,\ 1,\ 2]$

$a \Delta b$ $\qquad$ $[C1 = T66{*}30,\ 4,\ 5]$

$T66{*}35{*}7 = T66{*}BD3.\quad [a\ b] : a \Delta b \,.\equiv.\, [\exists\ c] \,.\, c \sqsubset a \,.\, c \sqsubset b$ $\qquad$ $[T66{*}35{*}4,\ T66{*}35{*}6]$

$T66{*}35{*}8.\quad [a\ b] : a \sqsubset b \,.\supset.\, a \sqsubset a$ $\qquad$ $[CD1 = T66{*}26]$

$T66{*}35{*}9.\quad [a\ b\ c] : b \sqsubset c \,.\, a \sqsubset b \,.\supset.\, a \sqsubset c$ $\qquad$ $[CD1 = T66{*}26]$

$T66{*}35{*}10.\quad [a\ b\ c\ d] : a \sqsubset b \,.\, c \sqsubset d \,.\, c \sqsubset a \,.\supset.\, [\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b$

*Proof:*

$[a\ b\ c\ d] \therefore$

(1) $a \sqsubset b \,.$

(2) $c \sqsubset d \,.$

(3) $c \sqsubset a \,.\supset :$

(4) $c \sqsubset b :$ $\qquad$ $[T66{*}35{*}9,\ 1,\ 3]$

$[\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b$ $\qquad$ $[2,\ 4]$

$T66{*}35{*}11.\quad [a\ b\ f] :: [c\ d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset.\, [\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b \therefore a \Delta f \therefore\supset.\, b \Delta f$

*Proof:*

$[a\ b\ f] ::$

(1) $[c\ d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset.\, [\exists\ e] \,.\, e \sqsubset d \,.\, e \sqsubset b \therefore$

(2) $a \,\Delta\, f \therefore \supset \therefore$
$[\exists\, g] :$
(3) $g \sqsubset a \,.$ [T66*35*4, 2]
(4) $g \sqsubset f :$
$[\exists\, e] \,.$
(5) $e \sqsubset f \,.$ [1, 4, 3]
(6) $e \sqsubset b \therefore$
$b \,\Delta\, f$ [T66*35*6, 6, 5]

*T66*35*12.* $[a\, b\, f] :: f \sqsubset a \therefore [c\, d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset.\, [\exists\, e] \,.\, e \sqsubset d \,.\, e$
$b \therefore \supset .\, a \sqsubset b$

*Proof:*

$[a\, b\, f] ::$
(1) $f \sqsubset a \therefore$
(2) $[c\, d] : c \sqsubset d \,.\, c \sqsubset a \,.\supset.\, [\exists\, e] \,.\, e \sqsubset d \,.\, e \sqsubset b \therefore \supset \therefore$
(3) $a \,\Delta\, a \therefore$ [T66*35*5, 1]
(4) $[c] : a \,\Delta\, c \,.\supset.\, b \,\Delta\, c \therefore$ [T66*35*11, 2]
$a \sqsubset b$ [CD1 = T66*26, 3, 4]

*T66*35*13 = B1.* $[a\, b] :: a \sqsubset b \,.\equiv\therefore [\exists\, c] \,.\, c \sqsubset a \therefore [c\, d] : c \sqsubset d \,.\, c$
$a \,.\supset.\, [\exists\, e] \,.\, e \sqsubset d \,.\, e \sqsubset b$
[T66*35*8, T66*35*10, T66*35*12]

*T66*35*14.* $[a\, b\, c\, \phi] :\cdot: [e] :: e \,\Delta\, a \,.\equiv :\cdot: [\exists\, f] :\cdot: e \,\Delta\, f :\cdot: [g] :: f \,\Delta\, g \,.$
$\supset \therefore [\exists\, h] \therefore h \,\Delta\, h \therefore [i] : h \,\Delta\, i \,.\supset.\, g \,\Delta\, i \therefore \phi(h) :\cdot: b \sqsubset a \,.$
$c \sqsubset b :\cdot: \supset .\, [\exists\, d] \,.\, d \sqsubset c \,.\, \phi(d)$

*Proof:*

$[a\, b\, c\, \phi] :\cdot:$
(1) $[e] :: e \,\Delta\, a \,.\equiv :\cdot: [\exists\, f] :\cdot: e \,\Delta\, f :\cdot: [g] :: f \,\Delta\, g \,.\supset \therefore [\exists\, h] \therefore h \,\Delta\, h$
$\therefore [i] : h \,\Delta\, i \,.\supset.\, g \,\Delta\, i \therefore \phi(h) :\cdot:$
(2) $b \sqsubset a \,.$
(3) $c \sqsubset b :\cdot: \supset ::$
(4) $c \sqsubset a \,.$ [T66*35*9, 2, 3]
(5) $c \sqsubset c \,.$ [T66*35*8, 3]
(6) $c \,\Delta\, a ::$ [T66*35*6, 5, 4]
$[\exists\, f] :\cdot:$
(7) $c \,\Delta\, f :\cdot:$ [1, 6]
(8) $[g] :: f \,\Delta\, g \,.\supset \therefore [\exists\, h] \therefore h \,\Delta\, h \therefore [i] : h \,\Delta\, i \,.\supset.\, g \,\Delta\, i \therefore$
$\phi(h) :\cdot:$ [1, 6]
(9) $f \,\Delta\, c ::$ [T66*35*2, 7]
$[\exists\, h] \therefore$
(10) $h \,\Delta\, h \therefore$
(11) $[i] : h \,\Delta\, i \,.\supset.\, c \,\Delta\, i \therefore$ [8, 9]
(12) $\phi(h) \,.$
(13) $h \sqsubset c ::$ [CD1 = T66*26, 10, 11]
$[\exists\, d] \,.\, d \sqsubset c \,.\, \phi(d)$ [13, 12]

*T66*35*15.* $[b\ f\ g\ \phi] : : [c] : c \sqsubset b\ .\supset .\ [\exists\ d]\ .\ d \sqsubset c\ .\ \phi(d)\ .\therefore f \sqsubset b\ .\ f\ \Delta\ g\ .\therefore \supset .\therefore [\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$

*Proof:*

$[b\ f\ g\ \phi]$ :·:

(1) $[c] : c \sqsubset b\ .\supset .\ [\exists\ d]\ .\ d \sqsubset c\ .\ \phi(d)\ .\therefore$

(2) $f \sqsubset b$ .

(3) $f\ \Delta\ g\ .\therefore \supset : :$

$[\exists\ h]$ :

(4) $h \sqsubset f$ . } [T66*35*4, 3]

(5) $h \sqsubset g$ .

(6) $h \sqsubset b$ : [T66*35*9, 2, 4]

$[\ \ d]$ .

(7) $d \sqsubset h$ . } [1, 6]

(8) $\phi(d)$ .

(9) $d \sqsubset g : :$ [T66*35*9, 5, 7]

$[\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ [CD1 = T66*26, 9, 8]

*T66*35*16.* $[a\ b\ j\ \phi]$ ⁝·⁝ $[e]$ ⁝ ⁝ $e\ \Delta\ a\ .\equiv$ :·: $[\exists\ f]$ :·: $e\ \Delta\ f$ :·: $[g] : : f\ \Delta\ g\ .\supset .\therefore [\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ ⁝·⁝ $[c] : c \sqsubset b\ .\supset .\ [\exists\ d]\ .\ d \sqsubset c\ .\ \phi(d)\ .\therefore b\ \Delta\ j$ ⁝·⁝ $\supset .\ a\ \Delta\ j$

*Proof:*

$[a\ b\ j\ \phi]$ ⁝·⁝

(1) $[e]$ ⁝ ⁝ $e\ \Delta\ a\ .\equiv$ :·: $[\exists\ f]$ :·: $e\ \Delta\ f$ :·: $[g] : : f\ \Delta\ g\ .\supset .\therefore [\exists\ h] : .\ h\ \Delta h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ ⁝·⁝

(2) $[c] : c \sqsubset b\ .\supset .\ [\exists\ d]\ .\ d \sqsubset c\ .\ \phi(d)\ .\therefore$

(3) $b\ \Delta\ j$ ⁝·⁝ $\supset$ ⁝ ⁝

$[\exists\ k] : :$

(4) $k \sqsubset b$ . } [T66*35*4, 3]

(5) $k \sqsubset j$ .

(6) $k \sqsubset k$ . [T66*35*8, 5]

(7) $j\ \Delta\ k$ :·: [T66*35*6, 5, 6]

(8) $[g] : : k\ \Delta\ g\ .\supset . .\ [\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ :·: [T66*35*15, 2, 4]

(9) $j\ \Delta\ a$ ⁝ ⁝ [1, 7, 8]

$a\ \Delta\ j$ [T66*35*2, 9]

*T66*35*17.* $[a\ b\ \phi]$ ⁝·⁝ $[e]$ ⁝ ⁝ $e\ \Delta\ a\ .\equiv$ :·: $[\exists\ f]$ :·: $e\ \Delta\ f$ :·: $[g] : : f\ \Delta\ g\ .\supset .\therefore [\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ ⁝·⁝ $b \sqsubset b\ .\therefore [c] : c \sqsubset b\ .\supset .\ [\exists\ d]\ .\ d \sqsubset c\ .\ \phi(d)$ ⁝·⁝ $\supset .\ b \sqsubset a$

*Proof:*

$[a\ b\ \phi]$ ⁝·⁝

(1) $[e]$ ⁝ ⁝ $e\ \Delta\ a\ .\equiv$ :·: $[\exists\ f]$ :·: $e\ \Delta\ f$ :·: $[g] : : f\ \Delta\ g\ .\supset .\therefore [\exists\ h] .\therefore h\ \Delta\ h\ .\therefore [i] : h\ \Delta\ i\ .\supset .\ g\ \Delta\ i\ .\therefore \phi(h)$ ⁝·⁝

(2) $b \sqsubset b\ .\therefore$

(3) $[c] : c \sqsubset b \,.\supset .\, [\exists\, d] \,.\, d \sqsubset c \,.\, \phi(d) \,\vdots\!\cdot\!\vdots\, \supset \therefore$

(4) $b \,\Delta\, b \therefore$ [$T66^*35^*6$, 2]

(5) $[c] : b \,\Delta\, c \,.\supset .\, a \,\Delta\, c \therefore$ [$T66^*35^*16$, 1, 3]

$b \sqsubset a$ [$CD1 = T66^*26$, 4, 5]

*T66*35*18.* $[a\,\phi] \,\vdots\!\cdot\!\vdots\, [e] \,\vdots\vdots\, e \,\Delta\, a \,.\equiv \,:\cdot:\, [\exists\, f] \,:\cdot:\, e \,\Delta\, f \,:\cdot:\, [g] \,\vdots\, f \,\Delta\, g \,.\supset \therefore [\exists\, h] \therefore h \,\Delta\, h : . [i] : h \,\Delta\, i \,.\supset .\, g \,\Delta\, i \therefore \phi(h) \,\vdots\!\cdot\!\vdots\, \supset :: [b] :: b \sqsubset a \,.\equiv \therefore b \sqsubset b \therefore [c] : c \sqsubset b \,.\supset .\, [\exists\, d] \,.\, d \sqsubset c \,.\, \phi(d)$

[$T66^*35^*8$, $T66^*35^*14$, $T66^*35^*17$]

By deducing *T66*35*13*, which is equiform with *B1*, *T66*35*7*, which is equiform with *T66*BD3*, i.e. with a thesis which defines partial inclusion in terms of strong inclusion, and *T66*35*18* we have shown that System 𝔅 is contained within System ℭ. This completes our outline of the proof that Systems 𝔅 and ℭ are inferentially equivalent.

It is interesting to note that out of our three systems of Boolean Algebra with definitions System 𝔅 appears to be comparatively the neatest. Its axiom contains six elementary propositional functions and the *definitional frame* for writing nominal definitions contains five such functions. In System 𝔄 the axiom and the definitional frame contain seven and six elementary propositional functions respectively. The axiom of System ℭ is shorter than any of the other two axioms as it involves only five elementary propositional functions, but the definitional frame constructed with the aid of the functor of partial inclusion is rather cumbersome. It has the form of an expression which involves seven elementary propositional functions.

What was said just now applies to the present state of affairs. The possibility of finding a way to simplify the foundations of our systems is by no means excluded.

*To be continued.*

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