ON THE SINGLE AXIOMS OF PROTOTHETIC

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In this paper I should like to present the results of my unpublished investigations [1] concerning axiom systems of protothetic. Strictly speaking, only a system of protothetic called $\mathfrak{S}_5$, will be considered here. It seems to me that this investigation may interest students of propositional calculus and the related subjects, since the deductions which will be used, sometimes unexpected and rather difficult, not only explain in some degree the structure of protothetic, but can also throw light upon several problems connected with various systems of propositional calculus. Because, generally, protothetic is still a little known theory, at the beginning I have to give several, possibly short, explanations concerning it. Without them the subject of this paper and the proofs presented below would hardly be understandable for the reader. Thus, in the first chapter a short description of protothetic and the necessary information about the rules of procedure of the system $\mathfrak{S}_5$ will be given. There will also be added some history of the researches concerning the single axioms of protothetic and related problems. Especially, I shall discuss here briefly the metatheorems **L** (of Leśniewski) and the stronger **S** (mine). In the second chapter I shall present a combined proof: 1) that my axiom A_n can serve as a single (and probably the shortest) axiom of the system $\mathfrak{S}_5$ of protothetic, and 2) that the above mentioned metatheorem **S** is sufficient to check the completeness of any axiom system of protothetic. In the third and the last chapter it will be shown in the shortest possible way how the classical propositional calculus and the quantification theory for protothetical formulas can be obtained in the field of the system $\mathfrak{S}_5$

Instead of an authentic symbolism of Leśniewski[2] introduced by him mostly in order to formulate the rules of procedure in the most precise way I shall use here a more convenient Peano-Russell symbolism modified in such a manner that it will become adjusted to the requirements of protothetic. Any one learned in logic will understand these modifications without difficulty. Only, in order to avoid possible misunderstandings I would like to note: 1) that the parentheses of the form "[" and "]" will be used here exclusively as the left and right scopes of the general quantifier, and 2) that if in a formula a quantifier is immediately preceded and closed by an even collection of

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dots, then the first collection is always considered as a stronger parenthesis than the second one. Concerning the terminology I would like to remark that the term "thesis" will mean throughout this paper a senseful (i.e. well-formed) formula which is either an axiom or a proved theorem of the system under consideration.

It should be remarked that in all systems of Leśniewski: 1) No thesis has free variables; 2) Vacuous quantifiers do not exist; 3) No formula has a form "$[x_1]\ [x_2]\ \ldots\ [x_n]\ .\ \phi\ (x_1 x_2 \ldots x_n)$", but always in such a case "$[x_1 x_2 \ldots x_n]\ .\ \phi\ (x_1 x_2\ \ldots x_n)$", where a quantifier "$[x_1\ x_2\ \ldots x_n]$ is considered as one unit; 4) If several free variables occur in a formula, they are not ordered in the quantifier. Thus, e.g. the expressions "$[xy]\ .\ \phi\ (xy)$" and "$[yx]\ .\ \phi\ (xy)$" are the equiform formulas. But, obviously, "$[xy]\ .\ \phi\ (xy)$" and $[xy]\ .\ \phi\ (yx)$" are the different ones. A general quantifier preceding a senseful formula will be called the main quantifier of it. For the reasons which will not be discussed here, in protothetic we do not have a particular quantifier [3]. Instead of it, any formula of the form "$[\exists x]\ .\ \phi\ (x)$" has to be expressed by "$\sim([x]\ .\ \sim(\phi\ (x)))$".

In the authentic symbolism of Leśniewski any functor precedes its arguments which must be closed by one or several pairs of reciprocal parentheses. Thus, e.g. using the said symbolism we do not have "$p \equiv q$" but "$\equiv (p\ q)$". In this paper I shall retain this custom only in the specific protothetical situations. Functors whose arguments are closed by several pairs of parentheses are called multi-link functors. [4] In such case each pair of parentheses must differ from the others. E.g., if "ϕ" is such a functor, then ϕ with its arguments can have the form "$\phi \nprec p \nsucc \prec q \succ (r)$". The forms of parentheses which can occur in protothetic are determined by the prescriptions included in the formulation of the rules of procedure of this system. Because this subject is not related strictly to the topic of this paper I shall not further explain this point, remarking only that in any place where the parentheses (except dots) are used, it will be done in conformity with the requirements of the said rules.

CHAPTER I

§ 1. A system of the bi-valued logic of propositions, called protothetic, was constructed by Leśniewski in 1923. In the papers [5], [6], [7] he described this theory, outlined a history (up to 1937) of its development and modifications, formulated in a very strict way the rules of procedure of its final formalization, called the system $\mathfrak{S}_5$, and showed how it is possible to deduce the propositional calculus from the first axiom system (found by himself) of $\mathfrak{S}_5$. And anyone who would like to become more familiar with this theory should study thoroughly the papers mentioned above. Besides Leśniewski's own writings a rather popular characterization of protothetic is given by Słupecki [20]. Unfortunately, in preparing his paper the author preferred to use rather the notes of former students of Leśniewski than the original papers. For this reason, in spite of several valuable remarks and proofs which can be found in Słupecki's paper, many important questions discussed by Leśniewski either are presented inadequately or are even omitted completely (e.g. the whole problem of the, so called, computable protothetical systems). The brief descriptions of protothetic are given in Prior [16] and in Church [1]. In [2] Grzegorczyk discusses some problems connected with this theory; it seems that several of his remarks are too hastily formulated. The important results from the field of protothetic obtained by Tarski are published in [28], [29], [30], [31] and some contributions of Sobociński in [22], [23], [25] [5]

Here, I have no intention to give a full description of protothetic or to discuss problems unrelated to the topic of this paper. The sole purpose of the explanations which will be presented below is to introduce readers unfamiliar with Leśniewski's system and his methods of deduction *in medias res* and to make the subject comprehensible to them.

As far as I know, Russell was the first who in [18] studied the possibility of introducing quantifiers binding the propositional variables into the calculus of propositions. But, he did not develop this idea and made no use of it in [34]. Then the propositional calculus with quantifiers was constructed by Łukasiewicz [9], [13], [32], who not only formulated the adequate rules of procedure for such a system, but remarked also that it is a stronger theory than the classical calculus of propositions. But Łukasiewicz limited his investigations to the system based on implication as a sole primitive term, and up to now the systems with quantifiers are not elaborated so fully as the ordinary propositional calculi. In 1921 Leśniewski, having his two other theories (mereology and ontology [6]) ready, wanted to formalize his system of the foundations of mathematics by basing it on as strong as possible a logic of propositions [7]. For this end he began to investigate the possibility of constructing a system of propositional logic to which would be added not only quantifiers but also variable proposition-forming functors. I.e., in the simplest case he added to the propositional calculus such variables as assume the values of four constant functors: negation (N), verum (Vr), falsum (Fl) and assertium (As). As is well known, in the bi-valued logic we have these only and only these constant functors for one propositional argument. This is shown by the following matrix (where *0* represents a false and *1* a true proposition):

M I

p	Np	Vrp	Flp	Asp
0	1	1	0	0
1	0	1	0	1

These considerations and some results obtained by Tarski [31] enabled Leśniewski to construct a system which is actually a subsystem of the proper protothetic and may be called the restricted system of it. This theory for which Leśniewski formulated suitable rules of procedure can be based, e.g., on the following axiomatic assumptions:

a) An axiom system of the complete implicational calculus of propositions and

b) An additional axiom $Z1$:

$$Z1.\ [pf] \therefore f([u].u).\supset: f([u].u.\equiv.[u].u).\supset.f(p)$$

The thesis $Z1$ called the principle of bivalency for propositions [9], has, obviously, the following meaning: Assume that an expression "$f(q)$" is a senseful, propositional function, in which "q" is a propositional variable; moreover, that "f" is not an abbreviation of a formula in which "q" occurs, but a real variable for which any senseful formula having q as a free variable can be substituted. Then, $Z1$ says that if "f" is satisfied by "$[u]\ .\ u$" (i.e. by 0) and by "$[u]\ .\ u\ .\equiv.\ [u]\ .\ u$" (i.e. by 1), then it is satisfied by any proposition.

As has been shown by Tarski [10] the axiom $Z1$ in this axiom system can be replaced by a thesis:

$$Z2\quad [pqf]: p\equiv q.\supset.f(p)\supset f(q)$$

which, evidently, is nothing else than the law of extensionality for propositions.

It is obvious that the restricted protothetic is not the strongest possible system of the bi-valued logic of propositions, since it can be strengthened by the addition of other variable functors, e.g. for two or more propositional variables or the variable proposition-forming functors at least one argument of which belongs to a higher type. In other words the system can be evidently enriched in an analogous way to that which is known in the field of the functional calculi. Therefore, protothetic is a system of the logic of propositions in which besides the propositional variables we have variable proposition-forming functors of any type, if it has a sense according to the theory of semantical categories for protothetic.[11] Moreover, the quantification theory and the laws of extensionality (in the systems $\mathfrak{S}_2$-$\mathfrak{S}_5$) for variables of any category is assumed in this theory.

I shall entirely omit here the history of the transformation of restricted protothetic into the full system or the gradual modification of the rules of procedure which Leśniewski effected for this end [12]. Also no explanations will be given concerning the reasons which induced him to adopt this or that position in the final formalization of protothetic.

In 1922 Tarski showed [13] that in the propositional logic having quantifiers and variable functors it is possible to define conjunction by equivalence, e.g. in the following way:

$$Z3\quad [pq] \therefore p.q.\equiv:[f]:f(p)\equiv f(q).\equiv p$$

And this fact gives a guarantee that any constant functor belonging to propositional calculus can be defined by equivalence alone, since, banally, negation can be obtained as follows:

$$Z4\quad [p] \therefore \sim p.\equiv: p\equiv.[q].q$$

For several reasons Leśniewski preferred to profit by this result and adopt equivalence as a sole primitive term of protothetic, although for this purpose we can take some other functors, as implication, Scheffer's functions etc. The successive formalization of protothetic based mostly on this primitive term were named the system $\mathfrak{S}_1$ to $\mathfrak{S}_5$, and the last of them was accepted as final. Besides this Leśniewski constructed the systems of so called, computable protothetic. These systems, whose foundations differ sharply from those used in the construction of ordinary protothetic will not be discussed here. But it has to be noted that the construction of the computable systems of protothetic allowed Leśniewski to prove that $\mathfrak{S}_5$ is a consistent and complete system [14] The completeness of protothetic is such that for any senseful formula in which all variables are bound either this formula or its negation is a thesis of $\mathfrak{S}_5$.

§ 2. In order to understand a formalization of a theory belonging to Leśniewski's system we have to note that this system differs from others in that it gives no specification as regards the semantical categories (logical types) to which the expressions of the system may belong. Instead of such specification the rules of procedure establish how a new semantical category can be introduced into the system. Every expression occurring in a senseful formula of a theory, except the brackets and quantifiers, belongs to a defined semantical category. The semantical categories to which the expressions occurring in an axiom-system, belong are called the primitive categories of the system. According to the requirements of Leśniewski in a well-constructed axiom system of any theory the number of different primitive semantical categories should be as small as possible.[15] This demand was fulfilled in the axiom system of $\mathfrak{S}_1$ constructed by Leśniewski in 1922 which contains the three following theses:

A1. $[pqr] \,.\!\because.\, p \equiv r \,.\!\equiv.\, q \equiv p :\equiv.\, r \equiv q$

A2. $[pqr] \,.\!\because.\, p \equiv.\, q \equiv r :\equiv: p \equiv q \,.\!\equiv r$

A3. $[gp] :\!\because: [f] :: g(pp) \,.\!\equiv\!.\!\because. [r] : f(rr) \,.\!\equiv.\, g(pp) :\equiv$
$: [r] : f(rr) \,.\!\equiv.\, g(p \equiv.\, [q] \,.\, q, p) ::\equiv.\, [q] \,.\, g(qp)$ [16]

It can be easily verified that in *A1* – *A3* we have a sole primitive term, viz. equivalence. And, that in these theses there occur only the expressions belonging to two semantical categories, viz. to the category of propositions (e.g. "p", "q" etc.) and to the category of proposition-forming functors for two propositional arguments, namely the variables "g" and "f" and a constant " $\equiv$ ". Hence, these two categories are primitive in the system $\mathfrak{S}_1$.[17]

As Leśniewski proved [18], the axioms *A1* and *A2* together with the customary rules of substitution and detachment (adjusted only to the system without free variables) constitutes a complete axiom system of the bi-valued equivalential calculus of propositions. He called this theory the system $\mathfrak{S}$. The axiom *A3* comprises:

a) The principle of bivalency expressed by equivalence and variable functors for two arguments.

b) Some forms of the law of extensionality for propositions which together with $A1$ and $A2$ enable one to obtain a complete propositional calculus.[19]

It is rather easy to show that on the base of the rules of $\mathfrak{S}_1$ this axiom system is inferentially equivalent with the assumption of the restricted protothetic, given in § 1. In order to obtain a complete system of protothetic Leśniewski did not have to change this axiom system, but he had only to reformulate and to reinforce the rules of procedure of $\mathfrak{S}_1$. It will become clear, if we remark that the restricted protothetic may be obtained by the addition of the principle of bivalency for propositions (i.e. the thesis $Z1$) to the complete propositional calculus. In addition to $\mathfrak{S}_1$ of the analogous theses guaranteeing bivalency for the expressions belonging to other semantical categories are carrying the similar results giving more and more strong system of protothetic. Since, theoretically, in the complete system any possible semantical category should have a place, it can be assured only by a suitable rule of procedure which under some conditions allows one to add theses expressing the principle of bivalence for other semantical categories than the category of propositions. The system $\mathfrak{S}_1$ does not have such a rule, but the various formulations of it appear in the next systems. In the complete system of protothetic $\mathfrak{S}_5$ the rules of procedure obtained their final formulation.[20] Below, I shall describe briefly only such points concerning the rules of $\mathfrak{S}_5$ as are indispensible for the purposes of this paper. No explanation will be given why this or that formulation was adopted by Leśniewski. But, I should like to remark here that Leśniewski who tried to have as strong as possible a system of logic (except for existential assumptions) formulated his rules of procedure in an exceptionally constructive manner. It permits one to check their operation by simple inductive reasoning at any stage of the development of a theory.

Strictly speaking, system $\mathfrak{S}_5$ has only one rule of procedure divided into five points. This rule is formulated in such a way that it becomes automatically adjusted to the last thesis belonging to the system. This means that any expression occurring in the axiom system or in the already proved theses is senseful in respect to this rule, when we are using it in order to prove a new thesis. The rule says that a formula A which is senseful in respect to the last thesis of the system can be added to it as a new proved thesis when and only when A results from one and only one of the following five conditions:

α) A is a well formed definition.

β) There is in the system such a thesis B that A can be obtained from B by a distribution of its main quantifier.

γ) There are in the system theses B and C such that A can be obtained by the detachment of C from B.

δ) There is in the system such a thesis B that A can be obtained from B by substitution.

ϵ) A is a thesis of extensionality in respect to the expressions belonging to a semantical category which is not a category of propositions, but has already a sense in the system.[21]

In protothetic as in all Leśniewski's systems a definition is not considered as a pure abbreviation, but as a real thesis of the theory. Therefore, there are no prescriptions here concerning the use of definitions, but there are given only the conditions of their construction. A well formed definition added to the system becomes automatically a proved thesis. Speaking superficially a definition has always the following form:

$$[a_1\ a_2 \ldots a_n]\ .\ \phi \equiv \psi$$

where the equivalence is the main functor, ϕ a definiens and ψ a definiendum. The variables a_1, $a_2 \ldots a_n$ from the main quantifier of the definition must belong to semantical categories already having a sense in the system, and they must occur in both ϕ and ψ. But there can be cases when a definition does not possess a main quantifier, e.g. "$[u]\ .\ u\ . \equiv 0$".[22] A definiens must be a senseful formula in which can occur also the variables bound by the quantifiers belonging to it. The constants occurring in ϕ have to be either the primitive functor or previously defined terms. Any variable occurring in a definiendum has to belong to the main quantifier of a definition and can occur only once in ψ. No quantifier occurs in a definiendum, which is either a new constant without arguments or a new constant followed by its variable arguments. The first case takes place, when a definition does not have its main quantifier. In the second case the variables are closed by one or several pairs of suitable parentheses.[23] The first form does not require further explanation. The second alternative gives the multi-link functors. In order to make these notions more familiar, I shall present here a concrete example. Let us assume that in our system a customary conjunction is defined already and "$\wedge$" is its symbol.[24] Then, obviously, we can define a conjunction for more propositional arguments, e.g.:

$$Z5.\ \ [pqrs] : \wedge(p \wedge (q \wedge (rs)))\ . \equiv .\ \wedge_1 (pqrs)$$

where a new constant "$\wedge_1$" belongs to a new semantical category, viz. proposition-forming functors for four propositional arguments. But, besides the definition $Z5$ the rule allows one to construct also the following definitions:

$$Z6\ \ [pqrs] : \wedge(p \wedge (q \wedge (rs)))\ . \equiv ..\ \wedge_2 \langle pqr \rangle (s)$$

$$Z7\ \ [pqrs] : \wedge(p \wedge (q \wedge (rs)))\ . \equiv .\ \wedge_3 \langle py \rangle (ps)$$

$$Z8\ \ [pqrs] : \wedge(p \wedge (q \wedge (rs)))\ . \equiv .\ \wedge_4 \langle\!\langle p \rangle\!\rangle \langle qs \rangle (r)$$

$$Z9\ \ [pqrs] : \wedge(p \wedge (q \wedge (rs)))\ . \equiv .\ \wedge_5 \langle\!\langle\!\langle p \rangle\!\rangle\!\rangle \langle\!\langle s \rangle\!\rangle \langle r \rangle (p)$$

and any other possible combinations.

It should be remarked that the constants $\wedge_2 - \wedge_5$ defined above belong to different semantical categories. For instance, the category of "$\wedge_2$" is a functor-forming functor with three propositional arguments for one propositional argument. But the whole expression "$\wedge_2 \langle pqr \rangle$" belongs to the category of proposition-forming functors for one propositional argument. The multi-link functors were introduced by Leśniewski[25] in order to have as full as possible a theory of logical types, i.e. his theory of semantical categories. Moreover, this kind of functor plays an important rôle in the formulation of substitution in his system, as we shall see below. Ending these very superficial remarks about the definitions in protothetic, I should note that if a newly defined constant belongs to a semantical category which was not yet represented in the system, then by this very fact this precise category obtains a sense in the theory, and not only the defined constant, but also variables belonging to this category can be used in substitutions.

The point concerning distribution of quantifiers says that having in the system a thesis B of the form:

$$B \quad [a_1 a_2 \ldots a_n] : \phi \,.\!\equiv\!.\, \psi$$

i.e. a thesis whose main functor is an equivalence and the variables bound by its main quantifier occur either in ϕ or in ψ or in both, we can distribute all or some of these variables between the arguments of the formula. Hence, if we have in the main quantifier of B n distinct variables (which, evidently, can belong to various semantical categories) we have $2^n - 1$ possible combinations obtainable from B:

$$A_1 \; [a_2 \ldots a_n] : [a_1] \,.\, \phi \,.\!\equiv\!.\, [a_1] \,.\, \psi$$

$$A_2 \; [a_1 a_3 \ldots a_n] : [a_2] \,.\, \phi \,.\!\equiv\!.\, [a_2] \,.\, \psi$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_3 \; [a_3 \ldots a_n] : [a_1 \, a_2] \,.\, \phi \,.\!\equiv\!.\, [a_1 a_2] \,.\, \psi$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{2^n - 1} \; [a_1 a_2 \ldots a_n] \,.\, \phi \,.\!\equiv\!.\, [a_1 \, a_2 \ldots a_n] \,.\, \psi$$

and every such formula can be added to the system as a new thesis. Applying this operation, we have always to remember that in protothetic there are no vacuous quantifiers, each variable can occur in the same quantifier only once, and any sequence of quantifiers is combined into one. The rule comprises no other points dealing with quantifiers. In chapter III a proof will be sketched that the remaining quantificational operations are provable in protothetic as theses.

The operation of detachment is formulated as follows: If in the system there are theses B and C having the forms:

$$B. \quad \phi \equiv \psi$$

$$C. \quad \phi$$

i.e. B is a thesis which has no main quantifier and equivalence is its main functor, and C is a thesis equiform with the first argument of B, then we can add to the system a new proved thesis A:

$$A. \quad \psi$$

which is to be equiform with the second argument of B.

It should therefore be noted, that in protothetic we do not have detachment under quantifiers.

Roughly speaking, in protothetic the restrictions concerning substitution are analogous to those which exist in the functional calculi. However, there is one important difference, viz. in Leśniewski's system only senseful expressions can be substituted for a variable occurring in the main quantifier of a thesis. This means that: 1) If there is need to substitute for such a variable a formula in which there occur the expressions belonging to unprimitive semantical categories, this can be done only if these semantical categories have been previously introduced into the system by way of suitable definitions. I.e., that in an application of substitution only such constants can be used as are either primitive or previously defined, and also that in case of a substitution in which a variable belonging to an unprimitive category is involved, an arbitrary constant of the same category must be previously defined in the system. 2) Since, the, so called, incomplete formulas do not belong to any semantical category and, therefore, have no sense in the system, a substitution of such formulas for the variable functors is not permitted. Instead of such forbidden operations in order to get in protothetic the same results which are performed in the other systems, suitable multi-link functors must always be used. The following example will explain the matter briefly. Suppose that having a thesis:

$$Z10 \quad [f] : f(f([u] . u)) . \equiv . [q] . f(q)$$ [26]

in which, evidently, "f" is a variable proposition-forming functor for one argument, we would like to obtain by substitution a thesis:

$$Z11 \quad [p] :: p \equiv : p \equiv . [u] . u \therefore \equiv . [q] . p \equiv q$$

We cannot make this by putting an incomplete expression "$p \equiv$" for the variable "f" because "$p \equiv$" has no semantical sense as a whole, but first of all we have to add to the system a definition:

$$Z12 \quad [r q] : p \equiv q . \equiv . \equiv_1 \langle p \rangle (q)$$

An expression "$\equiv_1 \langle p \rangle$" occurring in the definiendum of $Z12$ belongs to the same category as "f", and, therefore, we can substitute it for this variable in $Z10$. Thus in this way we obtain a thesis:

$$Z13 \quad [p] : \equiv_1 \langle p \rangle (\equiv_1 \langle p \rangle ([u] . u)) . \equiv . [q] . \equiv_1 \langle p \rangle (q)$$

Now, if there are at our disposal suitable theses from the propositional calculus, and an operation of distribution of quantifiers, then from these assumptions and from $Z13$ and $Z12$ we can easily obtain $Z11$, dismissing from $Z13$ the defined auxiliary term.

Concerning substitution the following final remark must be added. The rule does not allow one to change directly the variables which occur in an interior quantifier of a thesis. But, as will be shown in Chapter III, this can always be achieved by appropriate deductions providing that a new variable will be equiform with no variable occurring in the quantifiers of the thesis under consideration.

As was mentioned above, in order to obtain a complete system of protothetic from $\mathfrak{S}_1$, the latter system must be fortified so that we shall have a guarantee that in it for any semantical category, either primitive or introduced by definition of a fitting constant, an appropriate principle of bivalency is provable. Since in protothetic there can theoretically be an undefined number of various semantical categories, this cannot be accomplished by a construction of an axiom-system having a finite number of theses.[27] On the other hand, due to the fact that equivalence is the sole primitive term of $\mathfrak{S}_1$ the principle of bivalency for propositions formulated in this or that way must be included in any axiom-system of it. Otherwise, we are unable to obtain a full system of $\mathfrak{S}_1$, or even of the complete propositional calculus, unless we do like to change the rules of procedure of protothetic in an unnatural manner.[28] Hence, in order to obtain from $\mathfrak{S}_1$ a complete system of protothetic, Leśniewski at first strengthened $\mathfrak{S}_1$ by a new rule allowing one to add to the theory a thesis expressing bivalency for any given semantical category already senseful in the system, other than the category of propositions.

But a strict formulation of such a rule appeared to be extremely complicated and difficult. For this reason, using a remark of Tarski, Leśniewski later changed it entirely.[29] Tarski had shown that in the field of the complete propositional calculus the principle of bivalency for any semantical category is inferentially equivalent with the law of extensionality for expressions of the same category. Therefore, having the complete propositional calculus included in $\mathfrak{S}_1$ Leśniewski was able to replace a rule concerning the principle of bivalency by a rule of extensionality for variable functors. This modification did not change the range of protothetic, but merely allowed one to formulate the rule in rather a simple way.[30]

In the rule of $\mathfrak{S}_5$ a point concerning the theses of extensionality for variable functors is formulated as follows. Having equivalence as the primitive term the law of extensionality for propositions can be expressed by a thesis:

$Z14$ $[pq] \therefore p \equiv q . \equiv : [f] : f(p) . \equiv . f(q)$[31]

As it was mentioned this thesis cannot be added to the system by rule, but has to be assumed or provable in $\mathfrak{S}_5$ without direct reference to the point ϵ. A structure of the theses which this point permits to add is similar. Let us suppose that the variables f and g are variable functors (ordinary or multilink[32]) belonging to the same semantical category α and that a formula "$f \overset{\alpha}{\equiv} g$" is a corresponding equivalence between these two variables. Then, a suitable law of extensionality can be expressed by the formula:

$$\text{T)} \quad [fg] \therefore f \overset{\alpha}{\equiv} g . \equiv : [\phi] : \phi\langle f\rangle . \equiv . \phi\langle g\rangle$$

where "ϕ" belongs to the category of proposition-forming functors for one argument which belongs to the same category as "f" or "g". Since, obviously, the formula "$f \overset{\alpha}{\equiv} g$" can be defined by the quantifiers and equivalence in the following way:

$$[a_1 a_2 \ldots a_n] : f(a_1 a_2 \ldots a_n) . \equiv . g(a_1 a_2 \ldots a_n).$$ [33]

for the ordinary variable functors and:

$$[a_1 a_2 \ldots a_n] : f \langle a_1 \rangle (a_2 \ldots a_n) . \equiv . g \langle a_1 \rangle (a_2 \ldots a_n)$$

a.s.o. for the multi-link functors, the formulas of extensionality which the rule allows one to add to the system have the forms always analogous to *T1*, viz.:

$$T2\ \ [fg] \therefore [a_1 a_2 \ldots a_n] : f(a_1\, a_2\, \ldots a_n) . \equiv . g(a_1\, a_2 \ldots a_n) : \equiv : [\phi] : \phi \langle f \rangle \,\dot{:}\, \equiv . \phi \langle g \rangle$$

or

$$T3\ \ [fg] \therefore [a_1\, a_2 \ldots a_n] : f \langle a_1 \rangle (a_2\, a \ldots a_n) . \equiv . g \langle a_1 \rangle (a_2 \ldots a_n) : \equiv : [\phi] : \phi \langle f \rangle . \equiv . \phi \langle g \rangle$$

a.s.o. for any possible similar combinations.

A formula of extensionality having one of these forms can be added to the system as its new thesis only under a condition, that any variable occurring in its main and interior quantifiers belongs to a semantical category which has already a sense in the system. Hence, if there is a need to add such a thesis and not all the semantical categories involved are yet introduced into the system, then the arbitrary constants belonging to these categories must be previously defined.

The rule says only what theses of extensionality may be added to the system, but gives no prescriptions concerning the use of them. The situation is entirely analogous to that which we discussed in a case of definitions. A well formed formula of extensionality added to the system automatically becomes a proved thesis which later can be transformed in any way permitted by the rule of procedure of $\mathfrak{S}_5$. Thus, normally, in protothetic each application of extensional reasoning requires the use of a special definition. E.g., suppose that we have at our disposal propositional calculus, *Z14* and theses of the form: [34]

a) $\alpha \equiv \beta$

b) $\phi(\alpha)$

and "$\phi(\alpha)$"is such that from *a)* and *b)* a formula "$\phi(\beta)$" cannot be proved without *Z14*. Then, in order to obtain it we have to make the following steps: [35]

c) $[f] : f(\alpha) . \equiv . f(\beta)$	[From *Z14* and *a*]
d) $[p] : \phi(p) . \equiv \mathfrak{A}(p)$	[A definition]
e) $\mathfrak{A}(\alpha)$	[*d;b*]
f) $\mathfrak{A}(\beta)$	[*c;e*]
g) $[p] : \mathfrak{A}(p) . \equiv . \phi(p)$	[From *d* and propositional calculus]
$\phi(\beta)$	[*g;f*]

The rule of $\mathfrak{S}_5$, outlined above, may appear very heavy and inconvenient in its practical applications. However, its structure is determined by the conditions which according to Leśniewski's requirement any well formalized deductive theory belonging to his system must fulfill. And, if these conditions are taken into consideration, a formulation of the rule appears entirely natural. Moreover, as we shall see in Chapter II, in practice a direct application of the prescriptions required by the rule is necessary only for the first few steps, when we are beginning to make the deductions from this or that axiom-system of $\mathfrak{S}_5$. Later it is easy to prove several metarules of procedure which allow one to obtain the really significant theorems in a short and secure way. [35]

§ 3. Having properly constructed a system of the complete protothetic and established the principles of its rule of procedure, Leśniewski began to make investigations concerning possible simplifications of the axiom-system of $\mathfrak{S}_5$. These researches appeared to be difficult, often requiring long and complicated deductions. Outlining here a history of these endeavors [36] I shall give no hints how it is possible to prove that this or that thesis (or a set of several theses) is a single axiom (or an axiom-system) of protothetic, since a simple inspection of the proofs presented in the next chapter allows one to reconstruct the deductions without essential difficulties. Also it is entirely unnecessary to give here the proofs that any thesis, presented below, is a true formula of protothetic, because anyone can check it easily using the bi-valued logical matrix. The following methatheorem **L** remarked by Leśniewski plays an important role in investigations concerning the axiom-system of protothetic. [38]

METATHEOREM **L**: An axiom-system of protothetic having the rules of procedure inferentially equivalent with the rule of $\mathfrak{S}_5$ constitutes a complete system, if in its field the following three conditions are satisfied:

a) The system $\mathfrak{S}$ (i.e. the complete equivalential calculus of propositions) is provable.

b) There are provable the following four laws of the logical product for conjunction: [39]

$K1 \quad 1 \equiv 1 . 1$
$K2 \quad 0 \equiv 0 . 0$
$K3 \quad 0 \equiv 0 . 1$
$K4 \quad 0 \equiv 1 . 0$

c) The principle of bivalency for propositions is provable as a metarule saying that if a formula "$\phi(p)$" has a sense in the system and the formulas "$\phi(0)$" and "$\phi(1)$" are already proved, than a formula "$[p] . \phi(p)$" is also a thesis of this system. [40]

In fact, it is easy to be convinced that this metatheorem is correct. Having the rule of $\mathfrak{S}_5$, the mentioned conditions, and using a banal definition:

$Z15 \quad [pq] : p \equiv p . q . \equiv . p \supset q$

we can obtain at once the complete propositional calculus. Hence, we have obviously the logical laws of simplification and identity from which by substitution and detachment we get:

$Z16 \quad [f] \therefore f(0) . \supset : f(1) . \supset . f(0)$
$Z17 \quad [f] \therefore f(0) . \supset : f(1) . \supset . f(1)$

And in virtue of the condition c a thesis $Z1$ can be proved from $Z16$ and $Z17$ without any difficulty. Thus, the restricted system of protothetic is here provable and, since we have the rule of $\mathfrak{S}_5$, the complete system too. Therefore, in order to show that an investigated axiom-system of protothetic is complete it suffices only to prove that this system satisfies the conditions of Leśniewski's metatheorem. This allows one to omit many long and uninteresting deductions in such proofs.

The first axiom-system of $\mathfrak{S}_5$ contains, as we know the following theses:

$A1 \quad [pqr] \therefore p \equiv q . \equiv . q \equiv p : \equiv . r \equiv q$

$A2 \quad [pqr] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r$

$A3 \quad [gp] :\!\cdot\!: [f] :: g(pp) . \equiv \therefore [r] : f(rr) . \equiv . g(pp) : \equiv : [r] : f(rr) . \equiv . g(p \equiv . [q] . q , p) :: \equiv . [q] . g(qp)$

Written in the authentic symbolism of Leśniewski these axioms are expressed by 136 signs counting the variables, symbols of quantifiers and parentheses. In 1923 Leśniewski remarked that in this axiom-system $A1$ can be substituted by the thesis: [41]

$A1^* \quad [pqr] \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$

which did not change the length of the axiom-system, but showed the possibility of abbreviating the first single axiom of protothetic. This thesis, found in 1923 and named A_a contains 290 signs (in the authentic symbolism):

$A_a \quad [fp] :: f([pq] : p \equiv q . \equiv . q \equiv p , p) . \equiv \therefore f([hs] :\!\cdot\!: h([pqr] \therefore p \equiv r . \equiv . q \equiv p : \equiv . r \equiv q , s) . \equiv :: h([kt] :\!\cdot\!: k([pqr] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r , t) . \equiv :: k([gp] :\!\cdot\!: [f] :: g(pp) . \equiv \therefore [r] : f(rr) . \equiv . g(pp) : \equiv : [r] : f(rr) . \equiv . g(p \equiv . [q] . q , p) :: \equiv . [g] . g(qp) , t) . \equiv \therefore [pqr] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r , s) . \equiv \therefore [pqr] \therefore p \equiv r . \equiv . q \equiv p : \equiv . r \equiv q , p) . \equiv : [pq] : p \equiv q . \equiv . g \equiv p$

In the same year using $A1^*$ instead of $A1$ Leśniewski replaced A_a by a short single axiom (232 signs):

$A_b \quad [hs] :\!\cdot\!: h([pqr] \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r , s) . \equiv :: h([kt] :\!\cdot\!: k([pqr] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r , t) . \equiv :: k([gp] :\!\cdot\!: [f] :: g(pp) . \equiv \therefore [r] : f(rr) . \equiv . g(pp) : \equiv : [r] : f(rr) . \equiv . g(p \equiv . [q] . q , p) :: \equiv . [q] . g(qp) , t) . \equiv \therefore [pqr] \therefore p \equiv . q \equiv r : \equiv : p \equiv q . \equiv r , s) . \equiv \therefore [pqr] \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$

which in 1923 was replaced by the shorter (156 signs):

$A_c \quad [fpqr] :: f(p \equiv p , q) . \equiv \therefore f([g] \vdots\!\cdot\!\vdots\ g(pp) . \equiv \vdots\,\vdots\ g(r \equiv . p \equiv r , p) . \equiv :\!\cdot\!: [h] :\!\cdot\!: [k] :: [s] : k(ss) . \equiv . h(pp) : \equiv \therefore h(pp) . \equiv : [s] : k(ss) . \equiv . h(p \equiv . [t] . t , p) : \equiv . [t] . h(tp) , q) . \equiv \therefore p \equiv q . \equiv : r \equiv q . \equiv . p \equiv r$

In 1926 Wajsberg found the first two single axioms for the equivalential calculus of propositions [42] which enabled Leśniewski to find a shorter axiom (124 signs):

A_d $[fgpqrx] \vdots\vdots f([k]::[s]:k(ss).\equiv.h(pp):\equiv.\therefore h(pp).$
$\equiv:[s]:k(ss).\equiv.h(p\equiv:[t].t,p),q).\equiv:\cdot:f([t].h(tp),q)$
$::p\equiv.q\equiv r:\equiv.\therefore r\equiv x.\equiv x:\equiv.p\equiv q$

In the same year Wajsberg proved that on the base of the rule of $\mathfrak{S}_5$ from the protothetical thesis:

W^{**} $[pqrst]\therefore p\equiv q.\equiv:[g]:g(r\equiv s.\equiv t,q).\equiv.g(s\equiv t.\equiv r,$
$p)$

the system $\mathfrak{S}$ can be deduced, which enabled him to construct an axiom (120 signs):

A_e $[fhpqrst]:\cdot:f([g]\therefore h(pp).\equiv:g(qh(pp))\equiv.g(qh(p\equiv.[t]$
$.t,p)),r)\equiv::f([t].h(tp),r)\equiv.\therefore p\equiv q.\equiv:[g]:g(r\equiv s.$
$\equiv t,q).\equiv.g(s\equiv t.\equiv r,p)$

which in its turn was replaced by a shorter axiom of Leśniewski (116 signs):

A_f $[fhpqrs]\vdots\vdots f([t].h(tp)q).\equiv:\cdot:f([k]::h(pp).\equiv.\therefore$
$[s].k(ss).\equiv.\therefore[s]:k(ss).\equiv.h(p\equiv.[t].t,p)q).\equiv::$
$p\equiv q.\equiv r:\equiv s\therefore\equiv.\therefore s\equiv:p\equiv.q\equiv r$

Finally in 1926 Wajsberg found an axiom (106 signs):

A_g $[fp]\,\vdots\because[s].f(sp).\equiv\vdots\because[g)\vdots\cdot\vdots f(pp).\equiv\vdots\vdots[t].g(tt).\equiv$
$:\cdot:[qrt]:\cdot:g(t\equiv t.\equiv t,t).\equiv::f(p\equiv.[s].s,p)\equiv.\therefore p\equiv$
$.q\equiv r:\equiv:r\equiv.q\equiv p$

and Leśniewski a single axiom, counting only 82 signs:

A_h $[fpqrst]:\cdot:p\equiv q.\equiv::[g)::f(pf(p[u].u)).\equiv.\therefore[u].f$
$(qu).\equiv:g(r\equiv s.\equiv t,q).\equiv.g(s\equiv t.\equiv r,p)$

This axiom which Leśniewski used as a base for his formalization of the rule of procedure of $\mathfrak{S}_5$[43] could not be substituted by a shorter axiom during the next eleven years in spite of the endeavors made by Leśniewski and several of his students[44]. Even a result of Łukasiewicz who in 1933 proved that each of the following theses:

Ł1 $[pqr]\therefore p\equiv q.\equiv:r\equiv q.\equiv.p\equiv r$

Ł2 $[pqr]\therefore p\equiv q.\equiv:p\equiv r.\equiv.r\equiv q$

Ł3 $[pqr]\therefore p\equiv q.\equiv:r\equiv p.\equiv.q\equiv r$

is a single shortest axiom of the equivalential calculus of propositions did not help to solve this problem. It only enabled Leśniewski to replace A_n by an axiom:

A_i $[fpqrst]:\cdot:p\equiv q.\equiv::[g]::f(pf(p[u].u)).\equiv.\therefore[u].f$
$(gu).\equiv:g(r\equiv t.\equiv.s\equiv r,q).\equiv.g(s\equiv t,p)$

which has the same length as A_h (82 signs), but from which the necessary deductions can be obtained in a shorter way.

Only in 1937 was I able to advance this stubborn problem. Namely, analyzing the previous results and contents of the metatheorem **L** I remarked that the following set of assumptions S:

a) System $\mathfrak{S}$ (E.g. Łukasiewicz's axiom *Ł1*)

b) The protothetical theses:

$S1\ [fp]::f(p).\equiv.\cdot.\ f(p\equiv.[u].u).\equiv:[q]:f(p).\equiv.f(q)$

$S2\ [pq].\cdot.\ p\equiv q.\equiv:[f]:f(p).\equiv.f(q)$

$S3\ [pq]::p\equiv q.\equiv.\cdot.\ [f].\cdot.f(p).\equiv.f(q):\equiv.p\equiv q$

constitutes a complete axiom system of $\mathfrak{S}_5$ [46]. In fact these assumptions satisfy the metatheorem **L**. Namely: the point a evidently gives the condition $\mathfrak{a}$. Thesis S1 which is nothing else than the principle of bivalency for propositions expressed by the equivalence [47] fulfills, obviously, the condition $\mathfrak{c}$. And in virtue of the system $\mathfrak{S}$ and the theses $S2$ and $S3$ the condition $\mathfrak{b}$ can be obtained in a very easy way, if a thesis:

$Z15\ \ [pq]::[f].\cdot.f(p).\equiv.f(q):\equiv q.\cdot.\equiv.\ p.q$

is adopted as a definition of conjunction [48].

In the same year 1937 I proved that the set of assumptions S is not mutually independent, hence either the thesis $S2$ or the thesis $S3$ is a consequence of the remaining axioms. In other words I showed that in the condition of Leśniewski's metatheorem either the thesis $K4$ or the thesis $K3$ is superfluous, because in virtue of the remaining conditions $K4$ is a consequence of $\{K1; K2; K3\}$ and $K3$ of $\{K1; K2; K4\}$ providing that an adopted definition of conjunction is analogous to $Z15$.

Applying this result to a thesis which I knew previously, viz.:

$A_j\ \ [pq]\because p\equiv q.\equiv:\cdot:[fst]:\cdot:f(pf(p[u].u)).\equiv::[r].f(qr).\equiv.\cdot.\ s\equiv t.\equiv q:\equiv:t\equiv.s\equiv p$

I obtained at once a new single axiom of protothetic counting only 72 signs [49] and immediately after that Leśniewski improved this result, showing that a thesis

$A_f\ \ [pq]\because p\equiv q.\equiv:\cdot:[f]:\cdot:f(pf(p[u].u)).\equiv::[rs]::f(qr).\equiv.\cdot.\ s\equiv r.\equiv q:\equiv:r\equiv.s\equiv p$

counting 71 signs can also serve this purpose.

In 1938 using point ϵ of the rule, i.e. the point concerning the laws of the higher extensionalities, [50] I proved that in the axiom system S both theses $S1$ and $S2$ are superfluous. By this I showed that the condition b of Leśniewski's metatheorem can be dropped entirely, as the theses $K1$, $K2$, $K3$ and $K4$ are provable from the remaining conditions of **L**. This result not only elucidated a role which the rule of extensionality plays in an interior structure of protothetic, but enabled me to remark that a thesis (66 signs long):

$A_1\ \ [fpq]:\cdot:f(pf(p[u].u)).\equiv::[r]::f(p\equiv q.\equiv qr).\equiv.\cdot.\ p\equiv q.\equiv:r\equiv q.\equiv.p\equiv r$

can be adopted as a single axiom of protothetic. [51]

In the same year, starting from my aforesaid results and using my occasional remark that probably the condition $\mathfrak{a}$ could be weakened too, Leśniewski established a thesis (62 signs):

A_m $[p\,q] \vdots\vdots p \equiv q . \equiv :\cdot: [f] :\cdot: f(qf(q[u].u)) . \equiv :: [f] :: f(pr) \equiv .\therefore p \equiv : q \equiv . r \equiv p$

as a single axiom. [52]

Finally, in 1945 I proved that a thesis:

A_n $[p\,q] :: p \equiv q . \equiv .\therefore [f] .\therefore f(pf(p[u].u)) . \equiv : [r] : f(qr) . \equiv . q \equiv p$

is a single axiom of the system $\mathfrak{S}_5$ of protothetic. This thesis, whose length is expressed by 54 signs in the authentic symbolism of protothetic, seems to be also the shortest single axiom of the system $\mathfrak{S}_5$ although I have not been able to prove it yet.

In addition I remarked that each of the following theses (54 signs):

A_o $[p\,q] :: p \equiv q . \equiv .\therefore [f] .\therefore f(qf(q[u].u)) . \equiv : [r] : f(pr) . \equiv . q \equiv p$

A_p $[p\,q] :: p \equiv q . \equiv .\therefore [f] .\therefore f(pf(q[u].u)) . \equiv : [r] : f(qr) . \equiv . q \equiv p$

A_q $[p\,q] :: p \equiv q . \equiv .\therefore [f] .\therefore f(qf(p[u].u)) . \equiv : [r] : f(pr) . \equiv . q \equiv p$

can also serve as a single axiom of protothetic. [53]

These axioms differ on some respects from the axioms $A_j - A_m$. In order to establish that each of these theses can be a single axiom of $\mathfrak{S}_5$ it suffices to be acquainted with my aforenamed results and to apply this or that modification of the methods of deduction previously used for the same purpose in regard to the theses $A_a - A_i$. The proof of the sufficiency of A_n further requires the new deductions, rather long and elaborate, which will be presented in Chapter II.

Recollecting these problems in 1952, I found that the condition α of Leśniewski's metatheorem **L** can be replaced by a weaker assertion, viz., that the system $\mathfrak{S}$ can be substituted by a small fragment of it.

Hence summarizing my researches from this field I was able to formulate the following metatheorem **S**.

METATHEOREM **S**: An axiom-system of protothetic having the rules of procedure inferentially equivalent with the rule of $\mathfrak{S}_5$ constitutes a complete system, if in its field the following conditions are satisfied:

I. There are provable the following two theses:

F1. $[u] . u . \equiv . [u] . u$

F2. $[p\,q] .\therefore p \equiv : q \equiv p . \equiv q$

II. The principle of bivalency for propositions is provable as a metarule saying that if a formula "$\phi(p)$" has a sense in the system and the formulas "$\phi([u].u)$" and "$\phi([u].u . \equiv . [u].u)$" are already proved, then the formula " [p] . ϕ (p)" is also a thesis of this system.

A question remains open whether the thesis *F1* is indispensible or whether it can be proved from condition II and thesis *F2*.

NOTES

1) These results were obtained in the years 1937, 1938, 1945 and 1952. Some of them were included in my Polish paper "O aksjomatykach prototetyki" (On the axiom-systems of protothetic) which was to be published in the first volume of "*Collectanea logica*". During the siege of Warsaw in September, 1939 this almost ready volume perished when a printing office was burnt. *Cf.* an introduction to [23]. In 1953 I published a short resume in Polish concerning these researches, *cf.* [25].

2) A description of this symbolism is given in [6], *cf.* pp. 21-23. *Cf.* also [7].

3) Leśniewski considered that a rule of procedure occurring in his system must be formulated as fully as possible. Hence, an eventual rule concerning other quantifiers than the general ones would have to be formulated in such a way that it would give a possibility of an introducing into system not only the particular quantifiers but also any other kind of quantifiers possible in the bi-valued logic. E.g. a quantifier saying that a formula holds for at least two different instances of a variable bound by it, etc. On the other hand in Leśniewski's system a formalization of the rules of procedure is strictly constructive. This fact prevented a possibility to formulate such general rule concerning the quantifiers. And, therefore, not wishing to have in his system only a partially formulated rule he dropped the particular quantifier from his system. But, we can always use it "unofficially" as a pure abbreviation. *Cf.* [5], pp. 59-78, and [21]. Also *cf.* [4] .

4) A meaning of multi-link functors will be explained later. In Leśniewski's system a form of parentheses is determined by a constant functor to which this kind of parentheses was prescribed the first time. Consequently, later the same form of parentheses must be used for any functor belonging to the same semantical category as the first one. *Cf.* [5], pp. 59-78.

5) *Cf.* also [24], p. 257.

6) *Cf.* [5], [6], [21] and [24], pp. 254-257.

7) We can base ontology and mereology on the simple classical propositional calculus, but in such case their rules of procedure must be strengthened by the addition of a rule concerning the operations of the quantifiers.

8) A description of this system is given in [20], pp. 55-79. The restricted system of protothetic cannot be confounded with the extended propositional calculus constructed by Łukasiewicz, *cf.* [12], [15]. In that system Łukasiewicz accepted much stronger rules of procedure than we have in protothetic. Besides, in the extended propositional calculus there are free variables.

9) Concerning this principle, *cf.* remarks of Łukasiewicz in [8] and [14]. *Cf.* also [26], pp. 22-27.

10) *Cf.* [31] and [20].

11) In Leśniewski's system this theory is adopted instead of a theory of logical types. It is applied also to the expressions belonging to protothetic. A short description of this theory is given in [20], pp. 45-47. *Cf.* a remark of Lejewski in [4].

12) A history of these modifications is given in [5], pp. 30-59.

13) *Cf.* [31] and [5], pp. 9-13. In [22] and [23] I presented the definitions of conjunction by the equivalence differing in their forms from the definitions of Tarski although they are based essentially on his idea. E.g:

$$[pq] \therefore p \,.\, q \,.\equiv : [f] : f(pq) \,.\equiv. f(qp \equiv p)$$

14) The computable systems of protothetic are discussed in [6], pp. 35-43. The proof of Leśniewski was never published, but it can be easily reconstructed. The proof of completeness of protothetic given in [20], pp. 90-97, is entirely different.

15) *Cf.* [27], pp. 61-62.

16) *Cf.* [5], pp. 30-35 and [6], pp. 5-6.

17) The quantifiers and parentheses do not have semantical meaning in Leśniewski's system. But in protothetic any senseful propositional formula with bound or free variables belongs to the category of propositions, e.g. *A3* or any senseful propositional part of it.

18) *Cf.* [5], pp. 15-30, and [6], pp. 17-21. *Cf.* also [10] and some remarks in [17].

19) A complete proof of it is given in [7].

20) An exceptionally precise formulation of them is given in [5], pp. 59-78.

21) The formal properties of protothetic are such that either an axiom-system of $\mathfrak{S}_5$ includes the law of extensionality concerning propositions or this law can be obtained from a suitable axiom-system by an application of a point ϵ of the discussed rule. Hence, this point allows only to add to the system the "higher" theses of extensionality. See in the Chapter II of this paper a proof of the matarule **S.**

22) A definition without a main quantifier is called an absolute protothetical definition, the definition with a main quantifier – a relative protothetical definition. The forms in some respect similar to the absolute protothetical definitions we have in [34], p. XII.

23) The forms of these parentheses are determined by a semantical category of a functor to which they belong. In case of a multi-link functor each pair of its parentheses has to have a form differing from the other pairs.

24) Ordinarily in this paper I use a dot as a symbol of conjunction, as in [34]. But a dot is very inconvenient when the parentheses are used. Although in *Z5* "$\wedge$" and "$\wedge_1$" belong to the different semantical categories we can use the same form of parentheses for both functors because the arguments of "$\wedge$" and "$\wedge_1$" belong to same the semantical category and a number of these arguments claused by parentheses indicates a category of a functor.

25) An idea of multi-link functors is akin to a theory of superposition of the functions of Schönfinkel. Leśniewski introduced these functors to his system independently from [19] of Schönfinkel. He was rather influenced by

some formulas in [34], e.g. **31.14*. *Cf.* [6], p. 44; [5]. p. 66 and [21], p. 159.

26) Concerning this thesis *cf.* [31], p. 17 and [6], pp. 30-55. In [15] Meredith showed that this formula can serve as a single axiom of the extended propositional calculus of Łukasiewicz. Meredith's proof cannot be carried out in protothetic since in this theory a rule of substitution adopted by Łukasiewicz is not valid.

27) Similarly as in the functional calculus of omega degree.

28) This fact is connected with the properties of the possible definitions of conjunction by the equivalence.

29) *Cf.* [5], pp. 41-44.

30) *Cf.* in [5] T.E.IL, pp. 74-75.

31) In *Z14*, obviously, there occur the expressions belonging to three different semantical categories. Namely, "*f*" belongs to a category of proposition-forming functors for one propositional argument.

32) Discussing this point Słupecki omitted the case of multi-link functors, *cf.* [20], p. 81; p. 100. *Cf.* also [4].

33) After Słupecki, [20], p. 81, a functor of a type "$\overset{\alpha}{\equiv}$" can be called an interfunctorial equivalence.

34) In order to simplify the situation I omitted in the formulas *a*) and *b*) the possible quantifiers. If they are present we have to define in a point *d*) a multi-link functor.

35) Several such metarules will be established in Chapter II.

36) In [6], p. 23-35, Leśniewski gave a history of the successive simplifications of the axiom system of $\mathfrak{S}_5$ up to 1936. Here I repeat the essential points concerning this subject from that paper as only a few copies of that publication are preserved. The results obtained after 1936 are not discussed in [6].

37) *Cf.* e.g. [20], pp. 79-84, where there is given a method of verification of protothetical functors by the bi-valued matrix.

38) In Leśniewski's writings this metarule is not stated explicitly but he used it constantly. E.g. *cf.* [6], p. 34.

39) Here "0" and "1" are not defined constants, but the abbreviations of "$[u] \,.\, u$" and "$[u] \,.\, u \,.\!\equiv\!.\, [u] \,.\, u$" respectively.

40) Obviously, in "ϕ" there can occur the other variables.

41) *Cf.* [6], p. 24. Only in 1933 Łukasiewicz has proved that *A1** is a single axiom of the equivalential propositional calculus. *Cf.* [10] and [11].

42) *Cf.* [6] p. 27 and [33].

43) *Cf.* [5], pp. 59-78, and [6], pp. 14-16.

44) In connection with this problem Leśniewski preparing [6] in 1937 made the following remark (p.23): Das von mir oben in dem Resüme des §*11* meiner Mitteilung angegebene einzige Axiom der nach Direktiven des Systems $\mathfrak{S}_5$ gebauten Protothetik stellte sich leider bisher "siegreich" seit schon elf Jahren von anderen Forschern und von mir unternommenen Versugen entgegen, es um wenigstens ein einziges Wort zu verkürzen. Die Sachen haben sich hier jedoch so gestaltet, dass ich beinahe bis zum heutigen Tag wiederholt von immer neuen diese Aufgabe betreffenden Lösungsideen hörte. Alles dies bewegt mich, dass ich ohne zu warten, bis die mit der Entstehung des eben er-

wähnten Axioms und mit seinen kleined weiteren Umgestaltungen verbundenen Probleme eine angemessene Besprechung an einer passenden Stelle im Rahmen der Forsetzung meiner Mitteilung finden, hier im Interesse des Lesers, welcher sich mit selbständigen Forschungen in diesem Gebiet beschäftigen möchte, eine ganz allgemeine Übersicht der wichtigsten theoretischen Positionen angebe, die zum Entwickelungsbild des einzigen Axioms der Protothetik beitargen und verschiedene auf diesem Boden engwandte nutzliche konstructive Kunstgriffe illustrieren.

45) *Cf.* [10], [23], p. 10 and [6], p. 31.

46) In [20], p. 99 and [2] the authors do not mention my name discussing this axiom-system. *Cf.* [27], p. 60, note 17, and [25].

47) Thesis *S1* was established by Leśniewski.

48) Obviously, *Z15* is inferentially equivalent with *Z3*.

49) In an authentic symbolism of Leśniewski.

50) In 1928 applying a rule of extensionality I abbreviate an axiom-system of Leśniewski's ontology, *cf.* [21]. But a proof given there has nothing in common with the deductions used in protothetic.

51) The thesis A_l is not organic, when the axioms A_h – A_k are. Concerning a notion "organic formula", *Cf.* [27], p. 60.

52) It was the last investigation made by Leśniewski before his fatal illness. From his posthumus note I prepared a paper concerning this result with the intention to publish it in *Collectanea Logica.* Due to World War II it was not accomplished. *Cf.* [23].,an introduction.

53) It was not mentioned in [25].

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Remark: [32] is an English translation of [13]; and English translation of [22] is a part of [23]; [29] and [30] are a French version and [31] is an English translation of [28].

To be continued.

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