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THREE-VALUED PROPOSITIONAL FRAGMENTS WITH CLASSICAL IMPLICATION

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In [1] V. Vučković discussed a generalized system of recursive arithmetic, for which also see [2], in which he found he could obtain the representing equations of a three-valued propositional logic containing classical implication, a weak negation and two systems of conjunction-alternation. He suggested a third system as the union of these two, retaining the weak negation, in fact the system A discussed in [3], but later realised that the model of such a union was unobtainable in the arithmetic. We show that any complete axioms for his matrices

C	0	1	2	N_1	N_2
$*0$	0	1	2	1	1
1	0	0	0	0	1
2	0	0	0	1	0

and an arbitrary three-valued function $\phi(x_1, \ldots, x_n)$ become two-valued or inconsistent when any unprovable formula is added to the axioms. (N_2 was not primitive in the original but defined as $KNN\alpha C\alpha N\alpha$.) Thus the system has more possibilities of extension, by new cases of ϕ, than was originally envisaged, but fewer in terms of already axiomatized ϕ.

In the statement of the axioms i,j take values 1 or 2. The rules are detachment and substitution.

1. $CCCpqrCCrpCsp$
2_j. CpN_1N_jp
$3_{i,j}$. $CN_ipN_1N_jp \qquad (i \neq j)$
4_i. CN_ipCpq
5_i. $CpCN_iqN_iCpq$
6. CCN_2ppCCN_1ppp
7. $Cx_1'Cx_2' \ldots . Cx_n'\phi(x_1, x_2, \ldots, x_n)' \qquad (n \geq 0)$

7 prescribes the writing of 3^n axioms in correspondence with the 3^n lines of the truth-table of ϕ. In each, α' is α or $N_1\alpha$ or $N_2\alpha$ according as α has the value $0,1,2$ in the corresponding line of the table.

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We prove two theses, utilizing the fact that 1 is complete for classical C.

8. $CCCprCCqrrCCpsCCqsCCrss$ [C

$8\ p/N_2p,\ q/N_1p,\ r/p, s/q = C6 - 9$

9. $CCN_2pqCCN_1pqCCpqq$

Lemma. *If $x_1, \ldots, x_m$ are all the variables in α, then all formulas $Cx_1' \ldots Cx_m'\alpha'$ are provable.*

The proof is by induction on the structure of α. Inferences holding in virtue of implication are referred to as C.

Case 1 (basis). α is a variable. Then α is one of $x_1, \ldots, x_n$ and the lemma holds by C.

For the remaining cases we make the induction hypothesis that the lemma holds for $\beta, \gamma, \alpha_1, \ldots, \alpha_n$. To show that the lemma holds in Cases 2 and 3 we need only remark that $C\beta'\alpha'$ is a substitution in an axiom, Cpp, or $CpCqp$, whence the result follows from the induction hypothesis and C.

Case 2. $\alpha = N_j\beta$

2.1 $\beta = 0$. Then $\beta' = \beta, \alpha' = N_1\alpha = N_1N_j\beta$. (Use 2_j).

2.2 $\beta > 0$. Then $\beta' = N_i\beta$; $\alpha' = \alpha = N_i\beta$ or $\alpha' = N_1\alpha = N_1N_j\beta$ according as $i = j$ or not. (Use C or $3_{i,j}$).

Case 3. $\alpha = C\beta\gamma$

3.1 $\gamma = 0$. Then $\gamma' = \gamma, \alpha' = \alpha = C\beta\gamma$. (Use C).

3.2 $\beta > 0$. $\beta' = N_i\beta$, $\alpha' = \alpha = C\beta\gamma$. (Use 4_i)

3.3 $\beta = 0, \gamma > 0$. Then $\beta' = \beta$, $\gamma' = N_i\gamma$, $\alpha' = N_i\alpha = N_iC\beta\gamma$. (Use 5_i).

Case 4. $\alpha = \phi(\alpha_1, \ldots, \alpha_n)$

Substitution in 7 gives $C\alpha_1' \ldots C\alpha_n'\phi(\alpha_1, \ldots, \alpha_n)'$ whence the lemma follows by C and the induction hypothesis.

The lemma is proved.

Theorem I. *If α takes the value 0 for all valuations of its variables, then α is provable.*

Proof. Representing the formulas of the lemma by $Cx_1'CX_{n-1}'\alpha'$ provable under the hypothesis of the theorem are

$$Cx_1CX_{n-1}'\alpha,\ CN_1x_1CX_{n-1}'\alpha,\ CN_2x_1CX_{n-1}'\alpha,$$

which by 9 and C give $CX_{n-1}'\alpha$. Eliminating all antecedents in this way we obtain α.

Theorem II. *If any unprovable formula in C, N_i, and already axiomatized ϕ is added to the axioms, the system becomes either two-valued or inconsistent.*

Proof. We may assume that any such formula α has at least three variables and that in any valuation which rejects it there are variables valued 0,1,2; since for all α, there is a formula β, viz. $C\pi_0CC\pi_1\pi_1CC\pi_2\pi_2\alpha$ in which π_0, π_1, π_2 are not in α, such that β is inferentially equivalent to α, and for every valuation of α there is a valuation of β with $\pi_0/0$, $\pi_1/1$, $\pi_2/2$ Let α, then, be $\Psi(p_1, \ldots, p_l, q_2, \ldots, q_m, r_1, \ldots, r_n)$ rejected for the valuations $p_i/0$, $q_j/1$, $r_k/2$. Putting p for all variables valued 2 in this valuation, N_1Cpp for those valued 1, Cpp for those valued 0, we obtain a thesis

(1) $\Psi^*(p)$

and so, by C,

(2) $CN_2p\Psi^*(p)$.

By the lemma we have

(3) $CN_2pN_1\Psi^*(p)$ or $CN_2pN_2\Psi^*(p)$

hence from (2), (3), 4_i and C,

(4) CN_2pq.

Detaching (4) q/p from 6 gives

(5) CCN_1ppp

which with 1 and 4_1 bases two-valued C,N_1. N_2 is the constant false functor. As for the ϕ-axioms, if there are any with an N_2-consequent but without an N_2-antecedent these evidently give inconsistency via (4). If all without N_2-antecedents lack an N_2-consequent they give a complete two-valued definition of ϕ. Those with N_2-antecedents but without an N_2-consequent are trivial consequences of (4) and C.

Conclusion. If the range of i,j in the axioms is allowed to be $1,\ldots, m - 1$, and 6 is extended to $CCN_{m-1}ppCCN_{m-2}pp\ldots CCN_1ppp$, the system is complete for tautologies in m-values and has m-1 distinct weak negations, such that $N_i\,\alpha = 0$ when $\alpha = i$ and otherwise $N_i\,\alpha = 1$. But when $m > 3$ we lose at once the degree of completeness. In four values we can add the unprovable $CN_3pN_1N_2q$ without becoming m-n valued or inconsistent, unless new constants have been introduced by the ϕ-axioms. For this formula is rejected just if p is valued 3 and q is valued 2.

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