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RINGS OF TERM-RELATION NUMBERS AS NON-STANDARD MODELS

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1. *Purpose.* The concept of ultraproduct introduced by Łoś [3] has proved to be an important tool in model theory, as shown by A. Robinson's non-standard analysis, for example [4]. But in spite of the ultraproduct's usefulness, Vaught has pointed out its dramatic limitations and the essential need for other ways of developing models [5, p. 311]. With these points in mind, we shall show how systems of term-relation numbers can be used as model-theoretic devices. We hope this will have the heuristic value of suggesting other ways to construct finitary models that will contrast with the idea of infinite direct product basic to Łoś's theory. The systems mentioned in this paper have the following algebraic limitation (which may not be decisive): they are commutative rings with zero divisors, and since the least ideal that contains all those zero divisors is the whole ring, it is not possible to map these rings into a field through the usual method of forming the difference ring. This leaves unanswered the question of whether or not rings of term-relation numbers can be mapped nontrivially into an integral domain by some other method. Since we know so much more about integral domains than we do about ordered rings with zero divisors, the usefulness of the systems described here as generators of non-standard models of analysis remains undetermined. Nevertheless, these systems are non-standard models of certain areas of arithmetic and analysis.

2. *Rings of real term-relation numbers, order, finite infinitesimals.* A reading of [1] and [2] is necessary to follow the forthcoming discussion. In [2] the ring $T^{\infty'}$ is introduced; it is composed of term-relation numbers whose final components are drawn from the whole integral domain of ordinary integers. The paper also describes a generalization to rings of rational term-relation numbers. The corresponding definitions may now be broadened to introduce real and complex term-relation numbers. Details are left to the reader. The systems obtained are similar to those defined in [2], i.e., they are all commutative rings without identity and with proper zero divisors. Although most of the following considerations apply to both

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real and complex term-relation numbers, for simplicity, let us concentrate here on the former, and in particular on the ring T of all terms.

Definition 1. A binary relation $R(a,b)$ other than equality holds in the ring of term-relation numbers T if and only if, when a and b are given the same parenthesis structure with n final terms and n-1 final relations, R holds for at least n pairs of final components (terms or relations) such that the components of each pair occupy the same relative positions in a and b respectively. In symbols: if $a = t_1 r_1(t_2 r_2 t_3)$ and $b = u_1 v_1(u_2 v_2 u_3)$, then $R(a,b)$ holds in T iff at least three of the five expressions $R(t_1,u_1)$, $R(r_1,v_1), \ldots,$ $R(t_3,u_3)$ hold in the field of real numbers R_0 of terms or in the duplicate field of relations $\bar{R}_0$.

Definition 1 applies to the relations of strict simple ordering $>$ (greater than) and partial ordering $\geqslant$ (greater than or equal to) defined in R_0 (and $\bar{R}_0$). Therefore, $a > b$ holds in T if and only if Definition 1 is satisfied by the components of a and b. As a result of this definition, $>$ is not a strict simple ordering in T. For example, with $3\bar{2}1$ and $3\bar{1}2$ neither one is greater than the other: the ordering is not connected, although it is still transitive. Similarly $\geqslant$ becomes a strongly connected quasi-ordering, but it is neither symmetric, asymmetric, nor antisymmetric. For example, $3\bar{2}1$ and $3\bar{1}2$ are each greater than or equal to the other, but they are not equal to one another and—as we said—neither one is greater than the other (a different terminology for $\geqslant$ would be advisable, of course, one without any reference to logical disjunction).

The holding of an ordering relation in T may be defined in many other ways, and even total orderings may be introduced using a definition similar to Thieme's for the total ordering of complex numbers. But Definition 1 is justified because it now allows us to introduce finitary infinitesimals using Robinson's definition.

Definition 2. An element of T is an infinitesimal if and only if its absolute value is less than any positive real number.

According to this definition, and also according to the definition for the absolute value of a term-relation number introduced in [2], the collection of infinitesimals in T is the set of all term-relation numbers with only zero terms and arbitrary relations. In addition, we have two types of zero divisors in T: (i) term-relation numbers with some but not all terms zero, and (ii) infinitesimals. Infinitesimals form an ideal which is neither prime nor maximal. However, it is possible to imbed T in a commutative ring with identity in which the image of the class of nonzero divisors is a set of elements with inverse. To show this, let us call N the set of not zero divisors of T, and C the Cartesian product $T \times N$. We define the following equivalence relation: $(a,m) \equiv (b,n)$ iff $an = bm$. Let us call $\bar{C}$ the set of equivalence classes induced by $\equiv$, and let us define addition and multiplication in $\bar{C}$ as follows:

$$||(a,m)|| + ||(b,n)|| = ||an + bm,\ mn||$$
$$||(a,m)|| \cdot ||(b,n)|| = ||ab,\ mn||.$$

$\overline{C}$ is a commutative ring with identity $||(n,n)||$, and it contains a subring $\overline{C}_0$ of classes $||(an,n)||$ that is a homomorphic image of T. In this homomorphism, zero divisors map into zero divisors, and every element of the form $||(mn,m)||$ has $||(m,mn)||$ as an inverse.

3. *The Archimedean property, a comparison with hyper-real fields.* The Archimedean property for partially ordered rings may be defined as follows: if $a > 0$ and $b > 0$, then $na > b$ for some positive integer $n \geqslant 1$. Given the relationships between $>$ and $\geqslant$ in T, that definition may be extended to the quasi-ordered ring T, showing that rings of real term-relation numbers do possess the Archimedean property. Thus, although in hyper-real fields—or in general, in any elementary extension of a totally ordered field—the existence of infinitesimals is the necessary and sufficient condition for the field to be non-Archimedean, in the ring T the existence of infinitesimals is compatible with the Archimedean property. However, since zero divisors have no reciprocal in T, there are no infinites in T. If it were possible to map T in an order-preserving fashion in an integral domain, we could obtain finitary infinitely large elements—finite infinites, so to speak.

Finally, although T contains an isomorphic image of the field of real numbers R_0, it is not an elementary extension of R_0 in the sense of Tarski and Vaught. Hence, we should expect to have sentences hold in T which do not hold in R_0, and vice versa.

REFERENCES

[1] F. G. Asenjo, Relations irreducible to classes, *Notre Dame Journal of Formal Logic*, v. IV (1963), pp. 193-200.

[2] F. G. Asenjo, The arithmetic of the term-relation number theory, *Notre Dame Journal of Formal Logic*, v. VI (1965), pp. 223-228.

[3] J. Łoś, Quelques remarques, théorèmes, et problèmes sur les classes définissables d'algèbres, *Mathematical Interpretation of Formal Systems*, pp. 98-113, North Holland Publ. Co., Amsterdam, 1955.

[4] A. Robinson, *Introduction to Model Theory and to the Metamathematics of Algebra*, North Holland Publ. Co., Amsterdam, 1963.

[5] R. L. Vaught, Models of complete theories, *Bull, Amer. Math. Soc.*, v. 69 (1963), pp. 299-313.

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