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# A CHARACTERIZATION OF $\mathbf{S}^m$ BY MEANS OF TOPOLOGICAL GEOMETRIES

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In a recent paper in this Journal [1], the author characterized $\mathbf{R}^m$ as a topological space using the concept of a topological geometry. The purpose of the present paper is to present a similar characterization for the $m$-sphere $\mathbf{S}^m$. The terminology and propositions referred to by number are those of [1].

*Theorem 1: Let X and G form an m-arrangement, $m \geq 2$, and suppose X is second countable. Then if $S = \{x_0, \ldots, x_m\}$ is a linearly independent subset of X and $T = \{p_0, \ldots, p_m\}$ is any maximal linearly independent subset of $\mathbf{R}^m$ with the usual Euclidean geometry $\overline{G}$, then there is a homeomorphism d which maps C(S) onto C(T) and $F^iC(S)$ onto $F^iC(T)$, $i = 0, \ldots, m$, such that $d(G_{C(S)}) = \overline{G}_{C(T)}$.*

*Proof:* Set $d(x_i) = p_i$, $i = 0, \ldots, m$. Let $S_1 = \bigcup_{i<j} \overline{x_i x_j}$. By **3.27**, $d|S$ can be extended to $d_1: S_1 \to K^1C(T)$, the *1*-skeleton of $C(T)$ such that $d_1$ is a homeomorphism onto which carries $\overline{x_i x_j}$ onto $\overline{p_i p_j}$. Set $S_2 = \bigcup_{i<j<k} C(\{x_i, x_j, x_k\})$ Define $d_2: S_2 \to K^2C(T)$, the *2*-skeleton of $C(T)$ as follows: If $C(\{x_i, x_j, x_k\}) \subseteq S_2$, $d_2 = d_1$ on $\mathrm{Bd}\,C(\{x_i, x_j, x_k\})$. Choose $z \in \mathrm{Int}\,C(\{x_i, x_j, x_k\})$. Then $\mathbf{f}_1(x_i, z)$

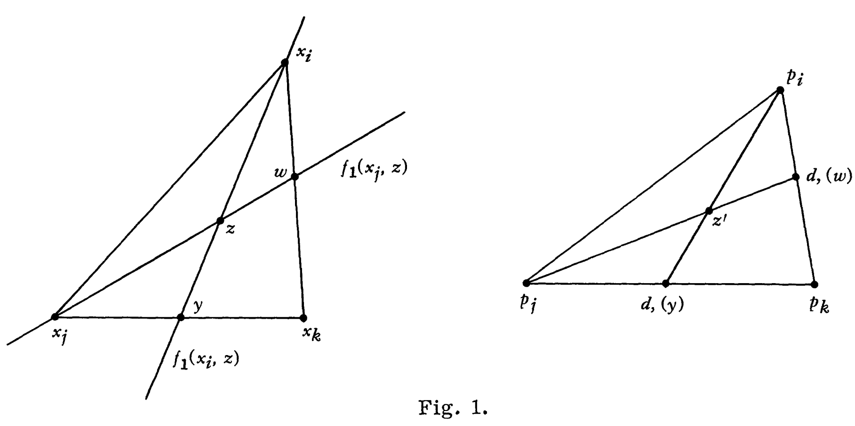


Fig. 1.

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$\cap\ \overline{x_j x_k} = \{y\}$, $\mathsf{f}_1(x_j, z)\ \cap\ \overline{x_i x_k} = \{w\}$, and $\overline{p_j d_1(w)} \cap \overline{p_i d_1(y)}$ contains a single point $z'$. Set $z' = d_2(z)$.

Set $S_3 = \bigcup_{i<j<k<q} C(\{x_i, x_j, x_k, x_q\})$. Define $d_3: S_3 \to K^3 C(T)$, the $3$-skeleton of $C(T)$, as follows: If $C(\{x_i, x_j, x_k, x_q\}) \subseteq S_3$, let $d_3 = d_2$ on $\operatorname{Bd} C(\{x_i, x_j, x_k, x_q\})$. Choose $z \in \operatorname{Int} C(\{x_i, x_j, x_k, x_q\})$. $\mathsf{f}_1(x_i, z)$ intersects $F^i C(\{x_i, x_j, x_k, x_q\})$ in a single point $y$; $\mathsf{f}_2(\{x_i, x_j, y\})\ \cap\ \mathsf{f}_2(\{x_i, x_k, y\}) = \overline{y x_i}$, and $\mathsf{f}_i(x_q, z) \cap F^q C(\{x_i, x_j, x_k, x_q\}) = \{w\}$. Then $\mathsf{f}_2(\{x_j, x_q, w\})\ \cap\ \mathsf{f}_2(\{x_i, x_j, y\}) \cap \mathsf{f}_2(\{x_i, x_k, y\}) = \{z\}$. Define $\{d_3(z)\} = \mathsf{f}_2(\{p_j, p_q, d_2(w)\}) \cap \mathsf{f}_2(\{p_i, p_j, d_2(y)\}) \cap \mathsf{f}_2(\{p_i, p_k, d_2(y)\})$. This process can be continued until we obtain $d_m = d: C(S) \to C(T)$.

By the manner in which they were defined, each $d_i$, $i = 1, \ldots, m$, is 1-1, onto, has the property that $d_i(G_{S_i}) = G_{K^i C(T)}$ and is a homeomorphism. The proof of this latter fact is quite analogous to several of the proofs in chapter V of [1].

*Definition 1: Let X have geometry G. By a triangulation of X (with respect to G) we mean a collection K of simplices $\{C_\nu\}$, $\nu \in N$, of X such that i) $\bigcup_N C_\nu = X$; ii) if $C_\nu$ and $C'_\nu$ are arbitrary elements of K, then $C_\nu \cap C'_\nu$ is a simplex; and iii) if $C_\nu$, $C'_\nu \in K$, then $C_\nu \subseteq C'_\nu$ implies $C_\nu = C'_\nu$.*

*Definition 2: A space X with geometry G of length m-1 is called a spherical m-arrangement if:*

*1) Each 0-flat consists of precisely two points. If x and y are distinct points of the same 0-flat, we say they are antipodal.*

*2) G is semi-projective.*

*3) Every linearly independent subset of X has a convex hull.*

*4) If W is any convex subspace of X, then W with geometry $G_W$ is a $(\delta(W)+1)$-arrangement.*

*5) If f is a k-flat contained in a k+1-flat g, the f disconnects g into two convex components.*

Unless specifying otherwise, all further statements will refer to a space $X$ with geometry $G$ such that $X$ and $G$ form a spherical $m$-arrangement, $m \geq 1$.

*Lemma 1: X is connected.*

*Proof:* Suppose $X = A \cup B$, $A \cap B = \phi$, $A$, $B$ non-empty open subsets of $X$. Either $A$ or $B$ (or both) contains infinitely many points; assume $\operatorname{card} A \geq \aleph_0$. Choose $x \in B$ and $y \in A - \mathsf{f}_0(x)$. Then $\overline{xy}$ exists by definition 2, 3), is connected and contains both $x$ and $y$, hence $x$ and $y$ are in the same component of $X$, a contradiction.

*Lemma 2: If $\{x,y\}$ is linearly independent, then $\overline{xy} \subseteq \mathsf{f}_1(x,y)$.*

*Proof:* $\overline{xy} \cap \mathsf{f}_1(x,y)$ is a convex set (2.3) which contains $x$ and $y$, hence $\overline{xy} \cap \mathsf{f}_1(x,y) \supseteq \overline{xy}$, therefore $\overline{xy} \cap \mathsf{f}_1(x,y) = \overline{xy}$.

*Lemma 3: A subset W of X is convex iff i) W contains no antipodal points, and ii) $\{x,y\} \subseteq W$ implies $\overline{xy} \subseteq W$.*

*Proof:* The intersection of $W$ with any $0$-flat is connected if $W$ is convex, or if i) holds. Suppose $f$ is any $1$-flat and ii) holds. Then if $\{x, y\} \subseteq f \cap W$, $\overline{xy} \subseteq f \cap W$ (lemma 2), hence $x$ and $y$ are in the same component of $f \cap W$, therefore $f \cap W$ is connected. If $W$ is convex, then by definition 2, 4), $\overline{xy} \subseteq W$. This suffices by **2.1.2.**

*Lemma 4: G is a topological geometry.*

*Proof:* If $\{W_\lambda\}$, $\lambda \epsilon \Lambda$, is a family of convex sets and $\{x, y\} \subseteq W_\lambda$ for each $\lambda$, then $\overline{xy} \subseteq W_\lambda$ for each $\lambda$, hence since $\bigcap_\Lambda W_\lambda \subseteq W_\lambda$ for each $\lambda$, by lemma 3 $\bigcap_\Lambda W_\lambda$ is convex.

If $f$ is an $m$-$1$-flat, then $f$ is closed since $X$-$f$ is open. Suppose we have shown that all flats of dimension greater than $k$ are closed and suppose $f$ is a $k$-flat, $0 \leq k \leq m-1$. Let $g$ be any $k+1$-flat which contains $f$. Since $g$-$f$ is open in $g$, $f$ is closed in $g$, a closed set, hence $f$ is closed. $\phi$ is always closed.

*Lemma 5: If $m = 1$ and $X$ is second countable, then $X$ is homeomorphic to* $\mathbf{S}^1$.

*Proof:* Let $f = \{x_0, x_1\}$ be an arbitrary $0$-flat in $X$ and $A$ and $B$ be the open convex components of $X$-$f$. Since $A$ is closed in $X$-$f$, but not in $X$ (or lemma 1 would be contradicted), we may suppose $x_0 \epsilon \mathrm{Cl}A$. Suppose $x_1 \notin \mathrm{Cl}A$. Let $g = \{w, z\} \neq f$ be some $0$-flat in $X$ and $A'$ and $B'$ be the open convex components into which $g$ disconnects $X$; we may suppose $x_1 \epsilon A'$. Now $x_1 \epsilon \mathrm{Cl}B$, for if $x_1 \notin \mathrm{Cl}B$, then $\{x_1\}$ is both open and closed in $X$, contradicting lemma 1. But then $B \subseteq B \cup \{x_1\} \subseteq \mathrm{Cl}B$, therefore $B \cup \{x_1\}$ is connected, hence is convex. Thus, using lemma 4, we see that $A'$ splits into components $A' \cap (B \cup \{x_1\})$ and $A' \cap A$, hence $A'$ could not be convex. We have thus shown that $\mathrm{Cl}A = A \cup \{x_0, x_1\}$; a similar argument shows $\mathrm{Cl}B = B \cup \{x_0, x_1\}$. A simple argument shows that $\mathrm{Cl}A$ and $\mathrm{Cl}B$ are both irreducibly connected between $x_0$ and $x_1$. Applying theorem 11.17 of Wilder [2], chapter I, we see that $X$ is homeomorphic to $\mathbf{S}^1$.

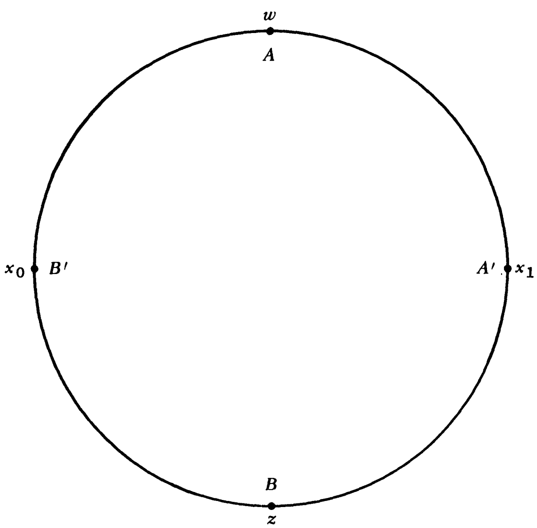

Fig. 2.

*Lemma 6: If $f$ is a $k$-flat, then $f$ with the subspace topology and geometry $G_f$ forms a spherical $k$-arrangement.*

*Proof:* The only part of definition 2 which is not clearly applicable is 3). We must show that if $S = \{x_0, \ldots, x_i\}$ is a linearly independent subset of $f$, then $C(S) \subseteq f$: If $i = 1$, then the lemma follows from lemma 2 since $\mathbf{f}_1(x_0, x_1) \subseteq f$. Suppose lemma 6 is true for $i - 1 \geq 1$. Then $C(S - \{x_0\}) \subseteq f$. But then by definition 2, 4) and lemma 2, $\bigcup_{x \in C(S - \{x_0\})} \overline{x_0 x} = C(S) \subseteq f$.

*Theorem 2: If X is second countable, then X is homeomorphic to* $\mathbf{S}^m$.

*Proof*: Lemma 5 proves this theorem for $m = 1$. Assume theorem 2 has been proved for all spherical $k$-arrangements, $1 \leq k \leq m - 1$, and suppose $X$ and $G$ form a spherical $m$-arrangement. Let $S = \{x_0, \ldots, x_m\}$ be a maximal linearly independent subset of $X$ and $\{y_0, \ldots, y_m\}$ be the set of points such that $y_i$ is antipodal to $x_i$, $i = 0, \ldots, m$. Set $S_i = S - \{x_i\}$. Each $\mathbf{f}_{m-1}(S_i)$ disconnects $X$ into convex open components $A_i$ and $B_i$; we say suppose that $x_i \in A_i$ for each $i$. We first prove

*Lemma 7*: $\mathbf{f}_{m-1}(S_i) = \operatorname{Fr} A_i = \operatorname{Fr} B_i$.

*Proof*: If $m = 1$, then the lemma has already been proved during the proof of lemma 5. Suppose lemma 7 is true for $m - 1 \geq 1$. Let $w \in \mathbf{f}_{m-1}(S_i)$ and let $g$ be any $m-1$-flat distinct from $\mathbf{f}_{m-1}^{1}(S_i)$ which contains $w$. Then since $\mathbf{f}_{m-1}(S_i) \cap g$ disconnects $\mathbf{f}_{m-1}(S_i)$, each neighborhood $U$ of $w$ intersects both components, hence $w$ is both in $\operatorname{Fr} A_i$ and $\operatorname{Fr} B_i$, hence $\mathbf{f}_{m-1}(S_i) \subseteq \operatorname{Fr} A_i$ and $\mathbf{f}_{m-1}(S_i) \subseteq \operatorname{Fr} B_i$. However, since $X - \mathbf{f}_{m-1}(S_i) = A_i \cup B_i$ and $A_i$ and $B_i$ are both open, the inclusions also go the other way and $\mathbf{f}_{m-1}(S_i) = \operatorname{Fr} A_i = \operatorname{Fr} B_i$.

*Lemma 9:* $\bigcap_{i=0}^{m} \operatorname{Cl} A_i = C(S)$.

*Proof:* $\bigcap_{i=0}^{m} \operatorname{Cl} A_i = \bigcap_{i=0}^{m} (A_i \cup \mathbf{f}_{m-1}(S_i)) = \bigcup \{Y_0 \cap \ldots Y_m \mid Y_i = \mathbf{f}_{m-1}(S_i) \text{ or } Y_i = A_i\}$. Since $G$ is semi-projective, $\bigcap_{i=0}^{m} \mathbf{f}_{m-1}(S_i) = \phi$. Suppose $\{x, y\} \subseteq \bigcap_{i=0}^{m} \operatorname{Cl} A_i$ with $x$ and $y$ antipodal. We may suppose that $x \in A_{i_1} \cap A_{i_q} \cap \mathbf{f}_{m-1}(S_{j_1}) \cap \ldots \cap \mathbf{f}_{m-1}^{\prime}(S_{j_p})$ and $y \in A_{k_1} \cap \ldots \cap A_{k_s} \cap \mathbf{f}_{m-1}(S_{r_1}) \cap \ldots \cap \mathbf{f}_{m-1}(S_{r_t})$. Since all the $A_i$, $i = 0, \ldots, m$, are convex, no $A_i$ which contains $x$ can also contain $y$. Therefore in the sets above containing $x$ and $y$ all $\mathbf{f}_{m-1}(S_j)$, $j = 0, \ldots, 1$ are represented, and since $y$ is contained in every $m-1$-flat which contains $x$, it follows that $\{x, y\} \subseteq \bigcap_{j=0}^{m} \mathbf{f}_{m-1}(S_j)$, a contradiction to the fact that this intersection must be empty.

Suppose $\{x,y\} \subseteq \bigcap_{i=0}^{m} \operatorname{Cl} A_i$. $\mathbf{f}_1(x,y) \cap \bigcap_{i=0}^{m} A_i$ is convex (lemma 4 and 2.3), hence is connected, therefore $(\mathbf{f}_1(x,y) \cap \bigcap_{i=0}^{m} A_i) \cup \{x,y\} \subseteq \operatorname{Cl}(\mathbf{f}_1(x,y) \cap \bigcap_{i=0}^{m} A_i) = \mathbf{f}_1(x,y) \cap \bigcap_{i=0}^{m} \operatorname{Cl} A_i$ is connected. Since $x$ and $y$ cannot be antipodal $(\mathbf{f}_1(x,y) \cap \bigcap_{i=0}^{m} A_i) \cup \{x,y\}$ is convex and therefore contains $\overline{xy}$, hence $\overline{xy} \subseteq \bigcap_{i=0}^{m} \operatorname{Cl} A_i$. By lemma 3 then $\bigcap_{i=0}^{m} \operatorname{Cl} A_i$ is convex, therefore $C(S) \subseteq \bigcap_{i=0}^{m} \operatorname{Cl} A_i$.

A straightforward argument and the induction hypothesis of theorem 2 show that $C(S_i) = \bigcup \{Y_0 \cap \ldots \cap Y_m \mid Y_j = \mathbf{f}_{m-1}(S_j) \text{ or } Y_j = A_j, j \neq i;\ Y_i = \mathbf{f}_{m-1}(S_i)\}$, $i = 0, \ldots, m$, and for $i \neq k$, $C(S_i \cap S_k) = \cup \{Y_0 \cap \ldots \cap Y_m \mid Y_j = \mathbf{f}_{m-1}(S_j) \text{ or } Y_j = A_j,\ j \neq i, k;\ Y_i = \mathbf{f}_{m-1}(S_i) \text{ and } Y_k = \mathbf{f}_{m-1}(S_k)\}$.

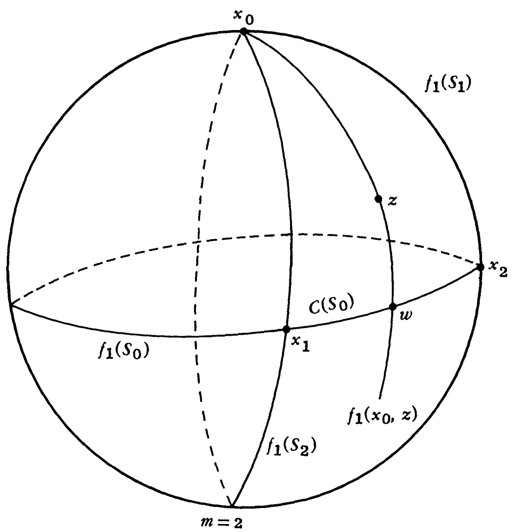

Fig. 3.

Suppose $z \epsilon (\bigcap_{i=0}^{m} \text{Cl} A_i) - C(S_0)$. Then if $z \epsilon \bigcap_{i=0}^{m} A_i$, $\mathbf{f}_1(x_0, z)$ must intersect $C(S_0)$ in an interior point $w$; for if not, then it must intersect some $\mathbf{f}_{m-1}(S_j)$, $j \neq 0$, in a point other than $x_0$ or $y_0$, which would imply $\mathbf{f}_1(x_0, z) \subseteq \mathbf{f}_{m-1}(S_j)$, or $z \notin \bigcap_{i=0}^{m} A_i$. In this case then $z \epsilon \overline{x_0 w}$. If $z \epsilon Y_0 \cap \ldots \cap Y_m$ where $Y_i = \mathbf{f}_{m-1}(S_i)$, then there is $w \epsilon C(S_i \cap S_0)$ such that $z \epsilon \overline{x_0 w}$. Thus by definition 2, 4) and **3.6**, $C(S) = \bigcap_{i=0}^{m} \text{Cl} A_i$.

A similar argument can be used to show that if $D \cup E = \{0, \ldots, m\}$ and $D \cap E = \phi$, then $\bigcap_{i \epsilon D} \text{Cl} A_i \bigcap_{i \epsilon j} \bigcap_E \text{Cl} B_j = C(\{x_i\}_{i \epsilon E} \bigcup \{y_j\}_{j \epsilon D})$.

It is easy to see that this procedure gives a triangulation of $X$, and if applied to $\mathbf{S}^m$, it will also give a triangulation of $\mathbf{S}^m$ (with respect to the usual "spherical" geometry on $\mathbf{S}^m$). It should be noted that considering $\mathbf{S}^m = \{(w_1, \ldots, w_{m+1}) \epsilon \mathbf{R}^{m+1} \mid w_1^2 + \ldots + w_{m+1}^2 = 1\}$, then the $i$-flats of $\mathbf{S}^m$ are the intersections of $\mathbf{S}^m$ with the $i+1$-dimensional vector subspaces of $\mathbf{R}^{m+1}$; the geometry on $\mathbf{S}^m$ thus obtained is semi-projective because the lattice of vector subspaces of $\mathbf{R}^{m+1}$ is modular. The triangulation of $\mathbf{S}^m$ contains exactly as many $m$-simplices related in precisely the same manner as in $X$. Using theorem 1, we can find a homeomorphism between $X$ and $\mathbf{S}^m$ by defining the homeomorphism one simplex at a time. Using the techniques of theorem 1 we can insure the necessary matching on the boundaries of the simplices in the triangulation.

## REFERENCES


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