AN EQUATIONAL AXIOMATIZATION OF ASSOCIATIVE NEWMAN ALGEBRAS

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An associative Newman algebra is a Newman algebra\(^1\) in which the binary multiplicative operation \(\times\) is associative for all elements belonging to the carrier set of the considered system. In [2], p. 265 and p. 271, Theorem 5 and Example E10, Newman has established that such an algebraic system is a proper extension of his complemented mixed algebra,\(^2\) and that it is a direct join of an associative Boolean ring with unity element and a Boolean lattice (i.e. a Boolean algebra). Moreover, he has shown there that this system can be constructed by an addition of a rather weak formula, viz. \(K1\) given in section 1 below, as a new postulate, to the axiom-system formulated in [2] of Newman algebra. On the other hand, it is almost self-evident that an associative Newman algebra is not necessarily a Boolean algebra.

In this note it will be shown that the addition of formula \(K1\) mentioned above, as a new postulate, to the set of axioms of system \(\mathfrak{B}\) discussed in [3] allows us to construct a very simple and compact equational axiom-system for associative Newman algebra.

1. We define a system under consideration as follows:

*Any algebraic system*\

\[ \mathfrak{D} = \langle B, =, +, \times, - \rangle \]

with *one binary relation* \(=\), *two binary operations* \(+\) and \(\times\), and *one unary operation* \(-\), *is an associative Newman algebra, if it satisfies the postulates*

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1. An acquaintance with the the papers [2] and [3] is presupposed. An enumeration of the formulas used in this note is a continuation of the enumeration which is given in [3]. As in that paper, the properties of "even" and "odd" elements will be not discussed in this note, and the axioms \(AI-A11\) given below will be used mostly tacitly in the deductions.

2. I.e., of Newman algebra, cf. [3].

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A1-A11, C1, C2, F1, F2 and F3 of System $\mathfrak{B}$ (defined in [3], section 1) of Newman algebra, and, additionally, an axiom

$$K1\quad [ab], a, b \in A. a + a = a \times \bar{a}. b + b = b \times \bar{b}. \therefore a \times (b \times b) = (a \times b) \times b$$

Concerning the form of $K1$, cf. [2], p. 285, Theorem 5, and $D2$ given in section 2.2 of [3]. The following algebraic table

$$
\begin{array}{c|cc}
+ & 0 & \eta \\
\hline
0 & 0 & \eta \\
\eta & \eta & 0
\end{array}
\begin{array}{c|cc}
\times & 0 & \eta \\
\hline
0 & 0 & 0 \\
\eta & \eta & \eta
\end{array}
\begin{array}{c|c}
x & \bar{x} \\
\hline
0 & 0
\end{array}
$$

which is constructed by Stone, cf. [4], p. 730, example *P6i, and [2], p. 268, and which is adjusted here to the primitive unary operation of complementation of system $\mathfrak{B}$ shows that this system is not necessarily a Boolean algebra. Namely this example satisfies all postulates of $\mathfrak{B}$, but falsifies

$$[a]: a \in B. \therefore a = a + a$$

for $a/\eta$: (i) $\eta = \eta$, and (ii) $\eta + \eta = 0$.

2 Let us assume the axioms of $\mathfrak{B}$. Since, clearly, system $\mathfrak{B}$ is a subsystem of $\mathfrak{B}$, we have at our disposal all formulas which are proved in sections 2.2 and 3.1 of [3]. Moreover, since it has been established, cf. [3], section 2.3, that system $\mathfrak{B}$ is inferentially equivalent or inferentially equivalent up to isomorphism to the original formalization of Newman algebra, we know that any formula which is proved in [2] is also provable analogously in the field of $\mathfrak{B}$. Hence, we can add the following formulas

$$F34\quad [abc]: a, b, c \in B. \therefore a + (b + c) = (a + b) + c \quad [Cf. P18 in [2], p. 260]$$
$$F35\quad [ab]: a, b \in B. a + a = a. \therefore (a \times b) + (a \times b) = a \times b \quad [Cf. P19 in [2], p. 261]$$
$$F36\quad [ab]: a, b \in B. a + a = 0. \therefore (a \times b) + (a \times b) = 0 \quad [Cf. P19 in [2], p. 261]$$
$$F37\quad [abc]: a, b, c \in B. a + a = a. b + b = b. c + c = c. \therefore a \times (b \times c)$$
$$= (a \times b) \times c \quad [Cf. P32 in [2], p. 263]$$

to the set of formulas which are already proven in sections 2.2 and 3.1 of [3].

Moreover, we have

$$H1\quad [abc]: a, b, c \in B. \therefore a \times (b + c) = (c \times a) + (b \times a)^3 \quad [C1; F26; F33]$$

Then:

$$K2\quad [ab]: a, b \in A. a + a = 0. b + b = 0. \therefore a \times b = (a \times b) \times b \quad [K1; F7; D2]$$

3. Formula $H1$ is accepted by Croisot, cf. [1], p. 27, as an axiom in his axiomatization of distributive lattice, with the constant element $I$.

4. The deductions presented below are also due to Newman, cf. [3], p. 265, Theorem 5, but they are given in a very compact way, or even verbally. In order to make this note more clear it was necessary to present these deductions in a formal way.
It is clear that in the field of Newman algebra regardless of its formalization $K1$ is inferentially equivalent to $K2$.

$K3 \ [abc] : a, b, c \in B . \ a + a = 0 . \ b + b = 0 . \ c + c = 0 . \ \supset . \ a \times (b \times c) = (a \times b) \times c$

$PR \ [abc] : Hp(4) . \ \supset .$

5. $((b \times a) + (b \times c)) + ((b \times a) + (b \times c)) = (b + b) \times (a + c)$ \ [1; C1; C2]

6. $a + c + (a + c) = (a + a) + (c + c) = 0 + 0 = 0 \ [1; F26; F34; 2; 4; F12]

7. $(a \times (b \times c)) + (a \times (b \times c)) = (a + a) \times (b \times c) = 0 \times (b \times c) = 0$

8. $((b \times a) + (b \times c)) = b \times (a + c) = (b \times (a + c)) \times (a + c)$ \ [1; C1; K2; 3; 6]

9. $0 = (b \times a) + (b \times c) + ((b \times a) + (b \times c))$

$L1 \ [abc] : a, b, c \in B . \ \supset . \ a \times (b \times c) = (a \times b) \times c$

$PR \ [abc] : Hp(1) . \ \supset .$

2. $d, e, f, g, m, n \in B .$

3. $d + d = d .$

4. $e + e = 0 .$

5. $a = d + e .$

6. $f + f = f .$

7. $g + g = 0 .$

8. $b = f + g .$

9. $m + m = m .$

10. $n + n = 0 .$

11. $c = m + n .$

12. $(d \times f) + (d \times f) = d \times f . \ [2; F35; 3; 6]$

13. $(e \times g) + (e \times g) = 0 . \ [2; F36; 4; 7]$

14. $(f \times m) + (f \times m) = f \times m . \ [2; F35; 6; 9]$

15. $(g \times n) + (g \times n) = 0 . \ [2; F36; 7; 10]$

16. $a \times (b \times c) = a \times ((f + g) \times (m + n))$

\[1; 2; 8; 11\]

\[5; F32; 6; 9; 7; 10\]

\[F32; 3; 14; 4; 15\]

\[F37; 3; 6; 9; K3; 4; 7; 10\]
Hence, it is shown that the formulas \( H1 \) and \( L1 \) are provable in the field of system \( \mathcal{D} \).

3. Now, let us assume, as the axioms, \( A1-A11, F1, F2, H1 \) and \( L1 \). Then:

\[ F3 \quad [ab]: a, b \in B \implies a = (b \times b) \times a \]

\[ [ab]: \text{Hp}(1) \implies a = a \times (b \times b) = (b \times (b \times b)) + ((b \times (b + b)) \times a) \]

\[ = (b \times ((b + b) \times a)) + ((b \times (b + b)) \times a) \]

\[ = ((b \times (b + b)) \times a) \times (b \times b) = (b \times b) \times a \]

\[ F26 \quad [ab]: a, b \in B \implies a + b = b + a \]

\[ F33 \quad [ab]: a, b \in B \implies a \times (b + c) = (a \times b) + (a \times c) \]

\[ \text{PR} \quad [ab]: \text{Hp}(1) \implies a \times (b + c) = (a \times b) + (a \times c) \]

Thus, in the field of the remaining axioms \( C1, C2 \) and \( F3 \) follow from \( F1, F2, H1 \) and \( L1 \).

4. The proofs given in the sections 2 and 3 above show clearly that in the axiom-system of \( \mathcal{D} \) the formulas \( H1 \) and \( L1 \) can be accepted, as the postulates, instead of \( C1, C2, F3 \) and \( K1 \). In [2], p. 271, Example 10, it is proved that \( K1 \) (and, therefore, \( L1 \)) is not the consequence of \( C1, C2, F1, F2 \) and \( F3 \). Matrices \( M1, M2, M3, M5 \) and \( M6 \), cf. section 4 in [3], each of which verifies \( K1 \) and \( L1 \) show that the formulas \( C1, C2, F1, F2 \) and \( F3 \) are mutually independent. Since \( M3 \) verifies \( F2 \) and \( H1 \), but falsifies \( F1, M5 \) verifies \( F1 \) and \( H1 \), but falsifies \( F2 \), and \( M1 \) verifies \( F1 \) and \( F2 \), but falsifies \( H1 \) for \( a/b, b/0, c/\gamma \): (i) \( \beta \times (0 + \gamma) = \beta \times \gamma = \gamma \) and (ii) \( (\gamma \times \beta) + (0 \times \beta) = \beta + 0 = \beta \), we know that the formulas \( F1, F2 \) and \( H1 \) are also mutually independent.

Thus, it is established that system \( \mathcal{D} \) of an associative Newman algebra can be based either on the set of mutually independent postulates \{\( C1; C2; F1; F2; F3; K1 \}\} or on the set of mutually independent postulates \{\( F1; F2; H1; L1 \}\).

**REFERENCES**


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