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TWO NOTES ON VECTOR SPACES WITH RECURSIVE OPERATIONS

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In [1] the author studied an $\aleph_0$-dimensional vector space $\overline{U}_F$ over a countable field F; it consists of an infinite recursive set ε_F of numbers (i.e., non-negative integers), an operation $+$ from $\varepsilon_F \times \varepsilon_F$ into ε_F and an operation $\cdot$ from $F \times \varepsilon_F$ into ε_F. If the field F is identified with a recursive set, both $+$ and $\cdot$ are partial recursive functions. Let β be a subset of ε_F. We call β a *repère*, if it is linearly independent; β is an *α-repère*, if it is included in a r.e. repère. A subspace V of $\overline{U}_F$ is an *α-space*, if it has at least one *α-basis*, i.e., at least one basis which is also an α-repère. We write c for the cardinality of the continuum. It can be shown [1, pp. 367, 385, 386 and 2, §2] that among the c subspaces of $\overline{U}_F$ there are c which are α-spaces and c which are not. The present paper* contains improvements of two results obtained in [1]. Henceforth the notations and terminology of [1] will be used.

1. HAMILTON'S THEOREM. Every two α-bases of an *isolic* α-space are recursively equivalent. This result [1, p. 375, Corollary 2] was strengthened by A. G. Hamilton [2] to:

every two α-bases of any α-space are recursively equivalent.

This means that $\dim_\alpha V$ can be defined for any α-space V. The following proof is shorter than Hamilton's; it is a modification of the proof of T1 in [1].

Proof. Let β and γ be α-bases of the α-space V, say $\beta \subset \overline{\beta}$, $\gamma \subset \overline{\gamma}$, where $\overline{\beta}$ and $\overline{\gamma}$ are r.e. repères. If V is finite-dimensional we are done, hence we suppose that $\dim V = \aleph_0$; thus $\beta, \overline{\beta}, \gamma$ and $\overline{\gamma}$ are infinite sets. We have $V = L(\beta) = L(\gamma)$, $V \leq L(\overline{\beta})$, $V \leq L(\overline{\gamma})$. Note that $L(\overline{\beta})$ need not equal $L(\overline{\gamma})$. There is no loss of generality in assuming that $\overline{\beta} \subset L(\overline{\gamma})$. For suppose this were not the case; take $\beta_0 = \overline{\beta} \cap L(\overline{\gamma})$; then $\beta \subset \beta_0$, where β_0 is a r.e. repère included in $L(\overline{\gamma})$. Assume therefore that $\overline{\beta} \subset L(\overline{\gamma})$. Put $\gamma^* = \overline{\gamma} \cap L(\overline{\beta})$, then

$$\beta \subset \overline{\beta} \subset L(\overline{\gamma}),\ \gamma \subset \gamma^* \subset \overline{\gamma},\ \gamma^* \subset L(\overline{\beta}),$$

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where $\overline{\beta}$, γ^* and $\overline{\gamma}$ are infinite r.e. repères. Let c_n be a one-to-one recursive function ranging over γ^*. Define the sequences $\{\overline{\beta}_n\}$, $\{\beta_n\}$ and the function b_n^* as in [1]; statements (i′), (ii′), (iii′) again hold for all n, and can be proved in the same way[1]. Let $p(x)$ be the function with domain γ^* which maps c_n onto b_n^*, for $n\epsilon\varepsilon$; put $\beta^* = \rho b_n^*$. Again, $p(x)$ is a partial recursive one-to-one function; it maps the r.e. set γ^* onto the r.e. subset β^* of the r.e. set $\overline{\beta}$ in such a way that

$$c_n \epsilon \gamma \longleftrightarrow p(c_n) \epsilon \beta, \text{ for } n \epsilon \varepsilon.$$

The last relation implies that $p(\gamma) = \beta^* \cap \beta$, hence $p(\gamma) \subset \beta$. Keeping in mind that $\gamma^* \subset L(\overline{\beta}) \leq L(\overline{\gamma})$, one realizes that the set $p(\gamma^*)$, i.e., β^* need not equal $\overline{\beta}$. We claim, however, that $p(\gamma) = \beta$. For suppose $p(\gamma) \subsetneqq \beta$, say $b \epsilon \beta - p(\gamma)$. Clearly,

$$\beta - p(\gamma) = \beta - (\beta^* \cap \beta) = \beta - \beta^* \subset \overline{\beta} - \beta^*,$$

hence $b \epsilon \overline{\beta} - \beta^*$; thus $b \epsilon \overline{\beta} - (b_0^*, \ldots, b_n^*)$, for $n \epsilon \varepsilon$. If b were equal to c_0, then "$1 \cdot b$" would be the expression of c_0 as a linear combination of elements in $\overline{\beta}$, hence $b_0^* = b = c_0$. Similarly we see (using $\overline{\beta}_n$ instead of $\overline{\beta}$) that $b = c_{n+1}$ implies $b_{n+1}^* = b = c_{n+1}$. Our hypothesis $b \notin \beta^*$ therefore implies $b \neq c_n$, for $n \epsilon \varepsilon$, hence $b \notin (c_0, \ldots, c_n)$, for $n \epsilon \varepsilon$. On the other hand, $b \epsilon \beta \subset V = L(\gamma) \leq L(\gamma^*)$; let k be the largest number n such that b, when expressed as a linear combination of elements in γ^*, has a non-zero coordinate w.r.t. c_n. We now have $b \epsilon L(c_0, \ldots, c_k)$, $b \epsilon \overline{\beta} - (b_0^*, \ldots, b_k^*)$ and $b \notin (c_0, \ldots, c_k)$. This implies the false statement that the set

$$\overline{\beta}_k = [\overline{\beta} - (b_0^*, \ldots, b_k^*)] \cup (c_0, \ldots, c_k)$$

is not a repère. Hence $p(\gamma) \subsetneqq \beta$ must be false. Thus $p(\gamma) = \beta$ and $\gamma \simeq \beta$.

2. R. E. SPACES. A space, i.e., a subspace of $\overline{U}_F$, is called *r.e.*, if it is r.e. when considered as a set, i.e., (every space being non-empty), if it is the range of a recursive function. According to [1, P3] a space is r.e. if and only if it has a r.e. basis. This suggests that among r.e. spaces those with a *recursive* basis might be of special interest. The following result shows that this is not the case: *every r.e. space has a recursive basis.* Before proving this proposition we shall introduce some notations and terminology and discuss three lemmas.

If f is a function from ε into ε, its value at n will be denoted by "$f(n)$" or "f_n". If α is a non-empty finite set, we write $\max \alpha$ for its maximum. Let $\sigma \subset \varepsilon_F$, $q \epsilon \sigma$, $p \epsilon \varepsilon_F$. Then σ_{-q} stands for $\sigma-(q)$, and $\sigma_{-q,p}$ for $\sigma_{-q} \cup (p)$.

DEFINITION. The repères β_1 and β_2 are *equivalent* [written: $\beta_1 \text{ eq } \beta_2$], if $L(\beta_1) = L(\beta_2)$.

DEFINITION. Let $\sigma \subset \varepsilon_F$, $q \epsilon \sigma$, $p \epsilon \varepsilon_F$, where σ is a repère. Then the element q of σ *may be replaced by p*, if $\sigma_{-q,p}$ is a basis of $L(\sigma)$.

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[1]We note the following misprints. In line 14 from the foot of p. 373, replace "$\overline{V}$" by "V" and in line 8 from the foot of p. 374, replace "β" by "$\overline{\beta}$".

Assume $\sigma \subset \varepsilon_F$, $q \epsilon \sigma$, $p \epsilon \varepsilon_F$, where σ is a repère. If $p = q$, we have $\sigma_{-q,p} = \sigma$, hence q may be replaced by itself. Now assume that q may be replaced by p, while $p \neq q$; then we have $p \notin \sigma$, for otherwise $\sigma_{-q,p}$ would equal the proper subset σ_{-q} of σ and not be a basis of $L(\sigma)$.

LEMMA L1. *Let $\sigma \subset \varepsilon_F$, $q \epsilon \sigma$, $p \epsilon \varepsilon_F$, where σ is a repère. Then the element q of σ may be replaced by p if and only if* (1) $p \epsilon L(\sigma)$, *and* (2) *when expressed as a linear combination of elements in σ, p has a non-zero coordinate with respect to q.*

LEMMA L2. *For every number n there exists an effective procedure which when applied to any given finite repère β of cardinality $\geq n$ yields a unique finite repère $\hat{\beta}$ such that $\hat{\beta}$ eq β and all elements of $\hat{\beta}$ are $\geq n$.*

LEMMA L3. *Let V be a finite-dimensional space over a finite field F. Then* $\mathrm{dim}\ V \geq n$ *implies* $\mathrm{max}\ V \geq n$.

Proofs of the Lemmas. L1 holds by elementary linear algebra. To establish L3 we assume $\mathrm{card}\ F = q$, $n \geq 1$, $\mathrm{dim}\ V \geq n$. Then $\mathrm{card}\ V = q^n \geq 2^n \geq n+1$, hence V cannot be a subset of $(0,\ldots,n-1)$ and $\mathrm{max}\ V \geq n$. Note that L3 also follows from L2. For, since by hypothesis, V has a finite basis of cardinality $\geq n$, it also has a finite basis all of whose elements are $\geq n$; again, $\mathrm{max}\ V \geq n$. It remains to prove L2. Let a finite repère β of cardinality $\geq n$ be given. If all elements of β are $\geq n$ (in particular, if β is empty or $n = 0$), we take $\hat{\beta} = \beta$. From now on we assume that $n \geq 1$ and that β contains at least one number $< n$. Let $\beta = (b_0, \ldots, b_t)$ with $\mathrm{card}\ \beta = t+1 \geq n \geq 1$; assume $b_0 < b_1 < \ldots < b_t$; thus $b_0 < n$. First consider the case that F is infinite. Let ϕ be the function from F into ε mentioned in [1, p. 363]. Put $r_n = \phi^{-1}(n)$, then $F = (r_0, r_1, \ldots)$, where $r_0 = 0_F$, $r_1 = 1_F$. Define for $0 \leq k \leq t$,

$$i_k = (\mu x)[r_x b_k \geq n], \quad \hat{b}_k = r_{i(k)} \cdot b_k .$$

Since $\hat{b}_k$ is a non-zero scalar multiple of b_k, the set $\hat{\beta} = (\hat{b}_0, \ldots, \hat{b}_t)$ satisfies the requirements. Now assume that F is finite. Consider the set $\tau = (b_0, b_0 + b_1, \ldots, b_0 + b_t)$. Since $0, b_1, \ldots, b_t$ are distinct, so are $b_0, b_0 + b_1, \ldots, b_0 + b_t$, hence $\mathrm{card}\ \tau = t+1 \geq n$. Also, $b_0 \neq 0$, for b_0 belongs to a repère. Let $1 \leq i \leq t$; then (b_0, b_i) is a repère, hence $b_0 + b_i \neq 0$. Since τ consists of at least n distinct non-zero numbers, τ contains at least one number $\geq n$. Put

$$i = (\mu x)[1 \leq x \leq t \,\&\, b_0 + b_x \geq n],$$
$$\hat{b}_0 = b_0 + b_i, \qquad \beta' = (\hat{b}_0, b_1, \ldots, b_t).$$

The element b_0 of β may be replaced by the element $\hat{b}_0 \geq n$, and β' eq β. Rearrange the sequence $\hat{b}_0, b_1, \ldots, b_t$ so that it becomes strictly increasing: $b_0' < b_1' < \ldots < b_t'$. If $b_0' \geq n$ we are done and put $\hat{\beta} = \beta'$. If $b_0' < n$ we define $\hat{b}_0'$ in terms of b_0' as we defined $\hat{b}_0$ in terms of b_0. Continuing this procedure we obtain (after at most $t+1$ replacements) a repère $\hat{\beta}$ which satisfies the requirements. Note that $\hat{\beta}$ is uniquely determined by β.

PROPOSITION A. *Every r.e. space has a recursive basis.*

Proof. Let $\overline{V}$ be a r.e. space. Then $\overline{V}$ has a r.e. basis, say $\overline{\beta}$. If $\overline{V}$ is finite-dimensional, $\overline{\beta}$ is a finite, hence recursive set. We therefore assume that $\dim\ \overline{V} = \aleph_0$; then $\overline{\beta}$ is an infinite r.e. set. Let b_n be a one-to-one recursive function ranging over $\overline{\beta}$. If F is infinite, the function $\overline{c}_n$ defined as in the proof of [1, P8] ranges over a recursive basis of $\overline{V}$. From now on we suppose that F is finite. Define $L_k = L(b_0, \ldots, b_k)$ and

$$\begin{aligned} M_0 &= L_0, & m_0 &= \max\ M_0, \\ M_1 &= L_{m(0)+2}, & m_1 &= \max\ M_1, \\ M_2 &= L_{m(0)+m(1)+4}, & m_2 &= \max\ M_2, \\ &\vdots & &\vdots \end{aligned}$$

Using L3 we see that $\dim\ L_k = k+1$ implies

$$m_{k+1} = \max\ L_{m(0)+\ldots+m(k)+2(k+1)} > m_k,$$

while $m_0 > 0$, since $(0) \subsetneq L_0$. Thus m_k is a strictly increasing recursive function all of whose values are positive. Clearly, $M_0 \leq M_1 \leq \ldots$ and the function m_k being strictly increasing, $M_0 < M_1 < \ldots$. It follows from $0 < m_k < m_{k+1}$ that (m_k, m_{k+1}) is a 2-element repère in M_1, where M_1 has dimension $m_0 + 3 \geq 4$; this repère can therefore be extended to an (m_0+3)-element basis of M_1 of the form

$$\beta_1 = (m_0, \underbrace{\qquad\qquad}_{m_0+1 \text{ nos } < m_1}, m_1).$$

Using L2 we see that the (m_0+1)-element repère $\beta_1 - (m_0, m_1)$ in M_1 is equivalent to a repère in M_1 all of whose elements are $\geq m_0 + 1$, but still $\leq m_1$ (since $m_1 = \max\ M_1$). Thus M_1 has a basis of the type

$$(*) \qquad (m_0, \underbrace{\qquad\qquad\qquad\qquad}_{m_0+1 \text{ nos between } m_0 \text{ and } m_1}, m_1).$$

The basis of M_1 which is not only of type (*), but also has the lowest Gödel number under

$$G(a_0, \ldots, a_k) = \prod_{i=0}^{k} p_i^{a_i}, \qquad a_0 < a_1 < \ldots < a_k,$$

is called the *minimal* basis of M_1; it can be effectively computed from the basis $(b_0, \ldots, b_{m(0)+2})$ of M_1; let its enumeration according to size be

$$m_0 = c_0, c_1, \ldots, c_{m(0)+2} = m_1.$$

The (m_0+3)-element repère $(c_0, \ldots, c_{m(0)+2})$ in M_1 is also a repère in M_2. Since M_2 has dimension $m_0 + m_1 + 5$, it can be effectively extended to a basis of M_2 of the form

$$(**) \qquad (c_0, \ldots, c_{m(0)+2}, \underbrace{\qquad\qquad\qquad\qquad}_{m_1+1 \text{ nos between } m_1 \text{ and } m_2}, m_2),$$

in fact, to the minimal such basis of M_2. Let its enumeration according to size be

$$c_0, \ldots, c_{m(0)+2}, c_{m(0)+3}, \ldots, c_{m(0)+m(1)+4} = m_2.$$

Continuing this procedure, we construct a strictly increasing recursive function c_n such that the set consisting of

$$c_0, \ldots, c_{m(0)+\ldots+m(k)+2k} = m_k,$$

is a basis of M_k. Thus c_n ranges over a recursive basis of the space

$$\bigcup_{k=0}^{\infty} M_k = L(b_0, b_1, \ldots) = L(\overline{\beta}) = \overline{V}.$$

This completes the proof.

If α is a subset of ε_F we denote the Turing degree of α by $\Delta(\alpha)$. Let $V = [\alpha, +, \cdot]$ be a space, i.e., a subspace of $\overline{U}_F = [\varepsilon_F, +, \cdot]$. Then the *Turing degree of* V [written: Δ_V] is defined as $\Delta(\alpha)$. In particular, V is called *decidable*, if $\Delta_V = 0$, i.e., if both V and $\overline{U}_F - V$ (considered as sets, i.e., as α and $\varepsilon_F - \alpha$) are r.e. With every set β we can associate a space V such that $\Delta_V = \Delta(\beta)$, namely the α-space $V = L[e(\beta)]$; this is discussed in [1, p. 368]. Consider the case that $\overline{V} = L[e(\overline{\sigma})]$, where $\overline{\sigma}$ is a r.e., but not recursive set. Then $\overline{V}$ is a r.e., but not decidable space; nevertheless, $\overline{V}$ has a recursive basis according to Proposition A. It is therefore of some interest that we can associate with every space V a unique basis π such that $\Delta_V = \Delta(\pi)$, the so-called *perfect* basis of V. Consequently, a space is decidable if and only if its perfect basis is recursive. We shall now discuss these matters in more detail.

If σ is a set and n a number we shall write $\sigma[n]$ for the set $\{y \in \sigma | y \leq n\}$.

DEFINITION. A repère β is *perfect*, if

$$x \in L(\beta) \longleftrightarrow x \in L(\beta[x]), \text{ for } x \in \varepsilon_F.$$

DEFINITION. A *perfect* basis of a space V is a basis of V which is also a perfect repère.

As an example we mention the fact that the canonical basis η of $\overline{U}_F$ [see 1, p. 365] is also the perfect basis of $\overline{U}_F$; this is true for every choice of the countable field F.

REMARK. Let $p_0, \ldots, p_r$ and $p_0, p_1, \ldots$ be strictly increasing sequences and let P denote the class of all perfect repères. Then

$$(p_0, \ldots, p_r) \in P \longleftrightarrow (\forall n \leq r)\,[(p_0, \ldots, p_n) \in P],$$
$$(p_0, p_1, \ldots) \in P \longleftrightarrow (\forall n)\,[(p_0, \ldots, p_n) \in P].$$

PROPOSITION B. *Every space V has exactly one perfect basis* π. *Moreover*, $\Delta_V = \Delta(\pi)$.

Proof. Let V be any space. If $V = (0)$, it only has the empty set as basis and this basis is perfect. Now assume $V \neq (0)$. Define

$$p_0 = (\mu x)[0 < x \;\&\; x \in V],$$
$$p_{n+1} = (\mu x)[p_n < x \;\&\; x \in V \;\&\; x \notin L(p_0, \ldots, p_n)],$$
$$\pi = \begin{cases} (p_0, \ldots, p_{k-1}), & \text{if } \dim V = k \geq 1, \\ (p_0, p_1, \ldots), & \text{if } \dim V = \aleph_0. \end{cases}$$

It is readily proved that π is a perfect basis of V and the only such basis. It follows from the definition of $p_0, \ldots, p_{k-1}$ or $p_0, p_1, \ldots$ in terms of V that π is Turing reducible to V. It remains to be proved that V is Turing reducible to π. Suppose that τ is a finite repère. Then we have for $x \epsilon \varepsilon_F$,

$$(*) \qquad x \epsilon L(\tau) \leftrightarrow \begin{cases} x \epsilon \tau \\ \text{or} \\ \tau \cup (x) \text{ is not a repère.} \end{cases}$$

Given a finite set σ we can effectively test whether σ is a repère [1, P2]. Thus it follows from (*) that given a number x and a finite repère τ, we can effectively test $x \epsilon L(\tau)$. We now conclude from

$$x \epsilon V \leftrightarrow x \epsilon L(\pi[x]), \text{ for } x \epsilon \varepsilon_F,$$

that V is Turing reducible to π.

REMARK. Let $\alpha \subset \varepsilon_F$. In discussing the decision problem of α we have only considered elements of ε_F. This is justified, since ε_F is a recursive set.

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