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TWO MODES OF DEDUCTIVE INFERENCE

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1. This study is a sequel to the author's [1], which tried to exhibit a system of natural deduction as a mere typographical variant of an axiom system G. The aim was to provide a proof technique combining the formal advantages of deduction from axioms with the intuitive advantages of deduction from assumptions or premises.

To some readers, however, the central notion of a *context*—and the notation embodying it—were psychologically opaque or unmanageable in practice. Furthermore, the conventions bridging the axiomatic method and the method of natural deduction were jerry-built, piecemeal and by example. Thus, the paper failed to meet some reasonable standards of simplicity and directness. For these reasons, a fresh approach seems in order.

The burden of the sequel, therefore, is to rejustify the claim that the best features of the axiomatic method and of the method of natural deduction may be secured within the framework of the first alone; by formulating a new axiom system G′ that is tractable to generalized deductive routine, by showing that some straightforward conventions for rewriting its formulae yield a method of proof indistinguishable in practice from well known techniques of natural deduction, and by proving that the system is both complete and sound in the sense that all and only valid quantificational formulae are among its theorems.

G′ and its metalanguage are entirely new. The choice of primitives is in line with popular tastes, the notion of context is abandoned in favor of a more transparent descriptive device (2.1) and the major link between the axiomatic and natural methods of deduction is forged in one stroke by recursion (6.1). The presentation is so ordered as to facilitate comparison with that of the parental essay; nevertheless, it is entirely self-contained, thus sparing those with no interest in comparative anatomy the labor of repeated cross-reference.

2. The primitives of G′ are 0-place predicate letters (sentence letters) 'p', 'q', 'r', 's' and their subscripted variants, m-place predicate letters ($m \geq 1$) 'F^m', 'G^m', 'H^m' and their subscripted variants, variables 'w', 'x', 'y', 'z' and their subscripted variants, the negation sign '-', parentheses, the conditional sign '$\supset$' and the universal quantifier sign '$\forall$'.

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The formulae of G′ are all and only expressions identified by these rules: an n-place predicate letter ($n \geqslant 0$) followed by a string of n variables is a formula; if P is a formula then $-P$ is a formula; if P is a formula and Q is a formula then $(P \supset Q)$ is a formula; if X is a variable and P is a formula then $\forall XP$ is a formula.

Henceforth, 'P', 'Q', 'R' and 'S', plain or subscripted, are to be construed as metalinguistic variables ranging over formulae, 'X' and 'Y' as ranging over variables. Thus, $-(P \supset Q)$ is the negation of the conditional whose antecedent is P and whose consequent is Q, $\forall XP$ is the universal quantification of P with respect to X, and so on.

An occurrence of X in P is free in P if it lies within no part of P of the form $\forall XQ$, else it is bound in P. X itself is free or bound in P according as it has free or bound occurrences in P. The formula that is like P except for having free occurrences of Y at all places where P has free occurrences of X will be denoted by '$(P{:}\,Y/X)$'. Thus, if P lacks free occurrences of X, or if Y is X; $(P{:}\,Y/X)$ is simply P. The practice of writing, e.g., '$-(P{:}\,Y/X)$' for '$(-P{:}\,Y/X)$' or '$((P{:}\,Y/X) \supset (Q{:}\,Y/X))$' for '$((P \supset Q){:}\,Y/X)$' should be self-explanatory.

It will be necessary to speak repeatedly of strings of formulae. To this end, '0' is adopted as a metalinguistic constant purporting to name the empty string of formulae, and the Greek letters 'ϕ', 'ψ' and 'χ' will be used as metalinguistic variables ranging over strings of formulae. What is to count as a string of formulae is specified by these rules: 0 is a string of formulae; if ϕ is a string of formulae then ϕP is a string of formulae. Each string of formulae is said to be *initial in* itself; furthermore, if ϕ is initial in ψ, then ϕ is initial in ψP. In other words, ϕ is initial in ψ if and only if ϕ is ψ or is an initial segment of ψ.

A function mapping the class of non-empty strings of formulae into the class of formulae is given by the following recursion:

2.1 $$f(0P) = P; \qquad f(\phi PQ) = f(\phi(P \supset Q)).$$

For example, $f(0PQR) = f(0P(Q \supset R)) = f(0(P \supset (Q \supset R))) = (P \supset (Q \supset R))$ and $f(0P(P \supset Q)Q) = f(0P((P \supset Q) \supset Q)) = f(0(P \supset ((P \supset Q) \supset Q))) = (P \supset ((P \supset Q) \supset Q))$. Clearly, the converse of f is not a function.

3. We are now in position to describe the axioms of G′ and to state the rules of inference.

3.1 *Axiom schema.* $f(\phi PP)$, *i.e.*, $f(\phi(P \supset P))$.

3.2 *From* $f(\phi\text{-}P)$ *one may infer* $f(\psi PQ)$, *i.e.*, $f(\psi(P \supset Q))$, *provided that* ϕ *is initial in* ψ.

3.3 *From* $f(\phi\text{-}PP)$, *i.e.*, $f(\phi(-P \supset P))$, *one may infer* $f(\psi P)$, *provided that* ϕ *is initial in* ψ.

3.4 *From* $f(\phi P)$ *and* $f(\psi PQ)$, *i.e.*, $f(\psi(P \supset Q))$, *one may infer* $f(\chi Q)$, *provided that both* ϕ *and* ψ *are initial in* χ.

3.5 *From* $f(\phi \forall XP)$ *one may infer* $f(\psi(P{:}\,Y/X))$, *provided that* ϕ *is initial in* ψ.

3.6 *From* $f(\phi(P{:}\,Y/X))$ *one may infer* $f(\psi \forall XP)$, *provided that* ϕ *is initial in* ψ *and* Y *is free neither in* $\forall XP$ *nor in any formula of* ϕ.

A proof in G′ is a sequence $P_1, P_2, \ldots, P_m$ in which, for each k, $1 \leq k \leq m$, P_k is an axiom or is inferred from one or more preceding formulae in accord with the stated rules of inference. A formula P is provable in (is a theorem of) G′ if and only if there exists a proof in G′ whose terminal formula is P.

Familiar truth functional connectives other than '-' and '⊃', as well as existential quantifiers, are introduced in the following list of definitions.

3.7 $(P \cdot Q)$ *for* $-(P \supset -Q)$.
3.8 $(P \vee Q)$ *for* $(-P \supset Q)$.
3.9 $(P \equiv Q)$ *for* $-((P \supset Q) \supset -(Q \supset P))$.
3.10 $\exists X$ *for* $-\forall X-$.

4. Supposing some formula of the form $f(0 P_1 P_2 \ldots P_n Q)$ to be a candidate for theoremhood, one may hope to elect it to that office by adopting the following strategy: first set down the axioms $f(0P_1P_1)$, $f(0P_1P_2P_2), \ldots$, $f(0P_1P_2 \ldots P_nP_n)$ and then apply the rules of inference.

For example, suppose that $f(0P(P \supset Q)Q) = (P \supset ((P \supset Q) \supset Q))$ is to be proven. The proof schema below shows how to proceed.

(1)	$f(0PP)$	Axiom
(2)	$f(0P(P \supset Q)(P \supset Q))$	Axiom
(3)	$f(0P(P \supset Q)Q)$	(1), (2), 3.4

The same procedure suffices to establish theoremhood of the following $f(0(P \supset (Q \supset R))(P \supset Q)PR) = ((P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R)))$:

(1)	$f(0(P \supset (Q \supset R))(P \supset (Q \supset R)))$	Axiom
(2)	$f(0(P \supset (Q \supset R))(P \supset Q)(P \supset Q))$	Axiom
(3)	$f(0(P \supset (Q \supset R))(P \supset Q)PP)$	Axiom
(4)	$f(0(P \supset (Q \supset R))(P \supset Q)PQ)$	(2), (3), 3.4
(5)	$f(0(P \supset (Q \supset R))(P \supset Q)P(Q \supset R))$	(1), (3), 3.4
(6)	$f(0(P \supset (Q \supset R))(P \supset Q)PR)$	(4), (5), 3.4

Should this strategy fail to yield a proof of $f(0P_1P_2 \ldots P_nQ)$, one may try a second: set down $f(0P_1P_1), f(0P_1P_2P_2), \ldots, f(0P_1P_2 \ldots P_nP_n)$ as before, then add the further axiom $f(0P_1P_2 \ldots P_n\text{-}Q\text{-}Q)$ and apply the rules of inference. The aim here is to get $f(0P_1P_2 \ldots P_n\text{-}QQ)$, from which the theorem will follow by 3.3.

Thus, $f(0PQP) = (P \supset (Q \supset P))$ is a theorem:

(1)	$f(0PP)$	Axiom
(2)	$f(0PQQ)$	Axiom
(3)	$f(0PQ\text{-}P\text{-}P)$	Axiom
(4)	$f(0PQ\text{-}P(P \supset -Q))$	(3), 3.2
(5)	$f(0PQ\text{-}P\text{-}Q)$	(1), (4), 3.4
(6)	$f(0PQ\text{-}P(Q \supset P))$	(5), 3.2
(7)	$f(0PQ\text{-}PP)$	(2), (6), 3.4
(8)	$f(0PQP)$	(7), 3.3

The same strategy establishes $f(0(-P \supset -Q)QP) = ((-P \supset -Q) \supset (Q \supset P))$:

(1)	$f(0(-P \supset -Q)(-P \supset -Q))$	Axiom
(2)	$f(0(-P \supset -Q)QQ)$	Axiom
(3)	$f(0(-P \supset -Q)Q\text{-}P\text{-}P)$	Axiom
(4)	$f(0(-P \supset -Q)Q\text{-}P\text{-}Q)$	(1), (3), 3.4
(5)	$f(0(-P \supset -Q)Q\text{-}P(Q \supset P))$	(4), 3.2
(6)	$f(0(-P \supset -Q)Q\text{-}PP)$	(2), (5), 3.4
(7)	$f(0(-P \supset -Q)QP)$	(6), 3.3

The reader may have noticed a certain redundancy in some of these examples; the third, for instance, could have been shortened to three lines. But here we are concerned less with elegance than with routine.

5. For G′, as for most other deductive systems, derived rules of inference are convenient sources of power—power being measured as units of work accomplished per unit of elapsed time. Some of the rules stated below are left unjustified; proof schemata appended to the remainder are to be imagined as prefaced by assumptions that the provisos on the rules are satisfied. Rules 5.1-5.12 have the common proviso that ϕ is initial in ψ.

5.1 *From $f(\phi P)$ one may infer $f(\psi P)$.*

(1)	$f(\phi P)$	Given
(2)	$f(\phi PP)$	Axiom
(3)	$f(\psi P)$	(1), (2), 3.4

5.2 *From $f(\phi\text{--}P)$ one may infer $f(\psi P)$.*

(1)	$f(\phi\text{--}P)$	Given
(2)	$f(\phi\text{-}PP)$	(1), 3.2
(3)	$f(\psi P)$	(2), 3.3

5.3 *From $f(\phi P)$ one may infer $f(\psi\text{-}PQ)$, i.e., $f(\psi(-P \supset Q))$.*

(1)	$f(\phi P)$	Given
(2)	$f(\phi\text{-}P\text{-}P)$	Axiom
(3)	$f(\phi\text{-}P(P \supset Q))$	(2), 3.2
(4)	$f(\psi\text{-}PQ)$	(1), (3), 3.4

5.4 *From $f(\phi P\text{-}P)$, i.e., $f(\phi(P \supset -P))$, one may infer $f(\psi\text{-}P)$.*

(1)	$f(\phi(P \supset -P))$	Given
(2)	$f(\phi\text{--}P\text{--}P)$	Axiom
(3)	$f(\phi\text{--}PP)$	(2), 5.2
(4)	$f(\phi\text{--}P\text{-}P)$	(1), (3), 3.4
(5)	$f(\psi\text{-}P)$	(4), 3.3

5.5 *From $f(\phi PQ)$, i.e., $f(\phi(P \supset Q))$, one may infer $f(\psi\text{-}Q\text{-}P)$, i.e., $f(\psi(-Q \supset -P))$.*

(1)	$f(\phi(P \supset Q))$	Given
(2)	$f(\phi\text{-}Q\text{-}Q)$	Axiom
(3)	$f(\phi\text{-}Q\text{--}P\text{--}P)$	Axiom
(4)	$f(\phi\text{-}Q\text{--}PP)$	(3), 5.2

(5)	$f(\phi\text{-}Q\text{--}PQ)$	(1), (4), 3.4
(6)	$f(\phi\text{-}Q\text{--}P(Q \supset \text{-}P))$	(2), 3.2
(7)	$f(\phi\text{-}Q\text{--}P\text{-}P)$	(5), (6), 3.4
(8)	$f(\psi\ \text{-}Q\text{-}P)$	(7), 3.3

5.6 *From* $f(\phi P\text{-}Q)$, *i.e.*, $f(\phi(P \supset \text{-}Q))$, *one may infer* $f(\psi Q\text{-}P)$, *i.e.*, $f(\psi(Q \supset \text{-}P))$.

5.7 *From* $f(\phi\text{-}PQ)$, *i.e.*, $f(\phi(\text{-}P \supset Q))$, *one may infer* $f(\psi\ \text{-}QP)$, *i.e.*, $f(\psi\ \text{-}Q \supset P))$.

5.8 *From* $f(\phi\text{-}P\text{-}Q)$, *i.e.*, $f(\phi(\text{-}P \supset \text{-}Q))$, *one may infer* $f(\psi QP)$, *i.e.*, $f(\psi(Q \supset P))$.

5.9 *From* $f(\phi(P \supset Q)P)$, *i.e.*, $f(\phi((P \supset Q) \supset P))$, *one may infer* $f(\psi P)$.

(1)	$f(\phi((P \supset Q) \supset P))$	Given
(2)	$f(\phi\text{-}P\ \text{-}P)$	Axiom
(3)	$f(\phi\text{-}P(P \supset Q))$	(2), 3.2
(4)	$f(\phi\text{-}PP)$	(1), (3), 3.4
(5)	$f(\psi P)$	(4), 3.3

5.10 *From* $f(\phi\text{-}(P \supset Q))$ *one may infer* $f(\psi P)$.

(1)	$f(\phi\text{-}(P \supset Q))$	Given
(2)	$f(\phi((P \supset Q) \supset P))$	(1), 3.2
(3)	$f(\psi P)$	(2), 5.9

5.11 *From* $f(\phi\text{-}(P \supset Q))$ *one may infer* $f(\psi\ \text{-}Q)$.

(1)	$f(\phi\text{-}(P \supset Q))$	Given
(2)	$f(\phi\text{--}Q\text{--}Q)$	Axiom
(3)	$f(\phi\text{--}Q(\text{-}Q \supset \text{-}P))$	(2), 3.2
(4)	$f(\phi\text{--}Q(P \supset Q))$	(3), 5.8
(5)	$f(\phi\text{--}Q((P \supset Q) \supset \text{-}Q))$	(1), 3.2
(6)	$f(\phi\text{--}Q\text{-}Q)$	(4), (5), 3.4
(7)	$f(\psi\ \text{-}Q)$	(6), 3.3

5.12 *From* $f(\phi(P{:}Y/X))$ *one may infer* $f(\psi \exists XP)$.

(1)	$f(\phi(P{:}Y/X))$	Given
(2)	$f(\phi\text{-}\exists XP\text{-}\exists XP)$	Axiom
(3)	$f(\phi\text{-}\exists XP\text{--}\forall X\text{-}P)$	(2), 3.10
(4)	$f(\phi\text{-}\exists XP\ \forall X\text{-}P)$	(3), 5.2
(5)	$f(\phi\text{-}\exists XP\text{-}(P{:}Y/X))$	(4), 3.5
(6)	$f(\phi(P{:}Y/X)\ \exists XP)$	(5), 5.8
(7)	$f(\psi \exists XP)$	(1), (6), 3.4

5.13 *From* $f(\phi\forall X(P \supset Q))$ *one may infer* $f(\psi \exists XPQ)$, *i.e.*, $f(\psi(\exists XP \supset Q))$, *provided that* ϕ *is initial in* ψ *and* X *is not free in* Q.

(1)	$f(\phi\forall X(P \supset Q))$	Given

Let Y be any variable that is not free in (1).

(2)	$f(\phi((P{:}Y/X) \supset Q))$	(1), 3.5

(3)	$f(\phi\text{-}Q\text{-}(P{:}Y/X))$	(2), 5.5
(4)	$f(\phi\text{-}Q\forall X\text{-}P)$	(3), 3.6
(5)	$f(\psi\text{-}\forall X\text{-}PQ)$	(4), 5.7
(6)	$f(\psi\,\exists XPQ)$	(5), 3.10

5.14 *From* $f(\phi\exists XP)$ *and* $f(\psi(P{:}Y/X)Q)$, *i.e.*, $f(\psi\,((P{:}Y/X)\supset Q))$, *one may infer* $f(\chi Q)$, *provided that both* ϕ *and* ψ *are initial in* χ *and* Y *is free neither in* $\exists XP$ *nor in any formula of* ψ *nor in* Q.

(1)	$f(\phi\exists XP)$	Given
(2)	$f(\psi\,(P{:}Y/X)Q)$	Given
(3)	$f(\psi\text{-}Q\text{-}(P{:}Y/X))$	(2), 5.5
(4)	$f(\psi\text{-}Q\forall X\text{-}P)$	(3), 3.6
(5)	$f(\psi\text{-}\forall X\text{-}PQ)$	(4), 5.7
(6)	$f(\psi\,\exists XPQ)$	(5), 3.10
(7)	$f(\chi Q)$	(1), (6), 3.4

5.15 *From* $f(\phi\forall XP)$ *one may infer* $f(\psi\text{-}\exists X\text{-}P)$, *from* $f(\phi\text{-}\exists X\text{-}P)$ *one may infer* $f(\psi\forall XP)$, *from* $f(\phi\text{-}\forall XP)$ *one may infer* $f(\psi\exists X\text{-}P)$, *from* $f(\phi\exists X\text{-}P)$ *one may infer* $f(\psi\text{-}\forall XP)$, *from* $f(\phi\forall X\text{-}P)$ *one may infer* $f(\psi\text{-}\exists XP)$ *and from* $f(\phi\text{-}\exists XP)$ *one may infer* $f(\psi\forall X\text{-}P)$, *provided that* ϕ *is initial in* ψ.

6. Perhaps enough has been said to show that G′ is like a system of natural deduction in being amenable to explicitly statable strategies for constructing proofs. Three simple conventions for rewriting formulae will work it into something resembling a system of natural deduction in still other respects.

As already noted, the converse of f is not a function: to distinct arguments of f there may correspond the same value. Thus, $f(0PQRS) = f(0PQ(R\supset S)) = f(0P(Q\supset(R\supset S))) = f(0(P\supset(Q\supset(R\supset S)))) = (P\supset(Q\supset(R\supset S)))$. But the functionality of f itself suggests our first convention: construe any argument of f as abbreviating the corresponding value, and ignore the notation '$f(\ )$' of functional application when reading axiom schemata or rules of inference. In effect, we adopt a recursion applicable to whole lines of proof:

6.1 $0P$ *for* P; ϕPQ *for* $\phi(P\supset Q)$.

Thus, the first proof schema of section 4 may now be written as follows:

(1)	$0PP$	Axiom
(2)	$0P(P\supset Q)(P\supset Q)$	Axiom
(3)	$0P(P\supset Q)Q$	(1), (2), 3.4

Under this convention, each axiom is of the form ϕPP. The second convention now follows: replace the first occurrence of P in ϕPP by the numeral of the line in which ϕPP appears, and preserve the numerical representation of P in succeeding lines as long as convenience dictates. Accordingly, the proof schema above may be written in the following way, numerals being used autonymously:

(1)	$01P$	Axiom

(2)	$012(P \supset Q)$		Axiom
(3)	$012Q$		(1), (2), 3.4

In this form, proofs are faintly reminiscent of proofs by natural deduction. The resemblance is enhanced by use of a third convention: set off strings of numerals established by application of the second convention from the formulae succeeding them. Thus, rewriting the example above:

(1)	01	P	Axiom
(2)	012	$(P \supset Q)$	Axiom
(3)	012	Q	(1), (2), 3.4

With 6.1 at hand, a really striking likeness is obtained by systematically lifting conventions:

(4)	01	$((P \supset Q) \supset Q)$	(3), 6.1
(5)	0	$(P \supset ((P \supset Q) \supset Q))$	(4), 6.1
(6)		$(P \supset ((P \supset Q) \supset Q))$	(5), 6.1

Without instruction to the contrary, one viewing this layout might well suppose it to be a proof by natural deduction; that 'Axiom' is a misnomer for 'Premise'; that 6.1 is a conditionalization rule; that the strings of numerals are devices for indicating the scopes of premises; that but two of the six lines are valid. (The recurrence of '0', and its final elimination, might be puzzling.) The reader sees the twists in this assessment; nevertheless, it is not altogether bad, and the fact that it might have been made goes far to establish the claim of the introductory section. Our three conventions have yielded what may be called a system of quasi-natural deduction.

To forestall possible misunderstanding, we shall close this section with two versions of the same proof; one in regular axiomatic form, the other a quasi-natural deduction. Their lines are formulae of the object language, hence '0' and other metalinguistic signs do not occur. There is a single two-place predicate letter from which the numerical superscript is omitted.

(1)	$(\exists y \forall x Fxy \supset \exists y \forall x Fxy)$	Axiom
(2)	$(\exists y \forall x Fxy \supset (\forall x Fxy \supset \forall x Fxy))$	Axiom
(3)	$(\exists y \forall x Fxy \supset (\forall x Fxy \supset Fxy))$	(2), 3.5
(4)	$(\exists y \forall x Fxy \supset (\forall x Fxy \supset \exists y Fxy))$	(3), 5.12
(5)	$(\exists y \forall x Fxy \supset \exists y Fxy)$	(1), (4), 5.14
(6)	$(\exists y \forall x Fxy \supset \forall x \exists y Fxy)$	(5), 3.6

This is the full dress axiomatic version; the other follows.

(1)	1	$\exists y \forall x Fxy$	Axiom
(2)	12	$\forall x Fxy$	Axiom
(3)	12	Fxy	(2), 3.5
(4)	12	$\exists y Fxy$	(3), 5.12
(5)	1	$(\forall x Fxy \supset \exists y Fxy)$	(4), 6.1
(6)	1	$\exists y Fxy$	(1), (5), 5.14

(7)	1	$\forall x \exists y Fxy$	(6), 3.6
(8)		$(\exists y \forall x Fxy \supset \forall x \exists y Fxy)$	(7), 6.1

7. Church's system F^1 [2] is known to be complete in the sense that all valid quantificational formulae are among its theorems. Therefore, to show that G′ is complete, it is sufficient to show that all theorems of F^1 are theorems of G′.

The axioms of F^1 are given, in our notation, by the schemata following:

(i) $f(0PQP)$
(ii) $f(0(P \supset (Q \supset R))(P \supset Q)PR)$
(iii) $f(0(-P \supset -Q)QP)$
(iv) $f(0\forall X(P \supset Q)(P \supset \forall XQ))$, *if X is not free in P.*
(v) $f(0\forall XP(P{:}Y/X))$

There are two rules of inference:

(vi) *From $f(0P)$ and $f(0(P \supset Q))$ one may infer $f(0Q)$.*
(vii) *From $f(0P)$ one may infer $f(0\forall XP)$.*

A proof in F^1 is a sequence $P_1, P_2, \ldots, P_m$ which is such that, for each k, $1 \leq k \leq m$, P_k is an axiom or is inferred from one or more preceding formulae by (vi) or by (vii). A formula P is provable in (is a theorem of) F^1 if and only if there exists a proof in F^1 whose terminal formula is P.

Suppose of a proof in F^1 that all lines preceding some arbitrarily chosen line P_k are provable in G′. What is to be shown is that P_k is provable in G′. There are seven cases to consider. Cases (i)-(iii): P_k is either an axiom $f(0PQP)$, or $f(0(P \supset (Q \supset R))(P \supset Q)PR)$, or $f(0(-P \supset -Q)QP)$. Then P_k is provable in G′, as shown in section 4. Case (iv): P_k is an axiom $f(0\forall X(P \supset Q)(P \supset \forall XQ))$ having no free occurrence of X in P. Then P_k is provable in G′, as follows:

(1)	$f(0\forall X(P \supset Q)\forall X(P \supset Q))$	Axiom
(2)	$f(0\forall X(P \supset Q)P(Q{:}X/X))$	(1), 3.5
(3)	$f(0\forall X(P \supset Q)(P \supset \forall XQ))$	(2), 3.6

Case (v): P_k is an axiom $f(0\forall XP(P{:}Y/X))$. Then P_k is provable in G′, as follows:

(1)	$f(0\forall XP\forall XP)$	Axiom
(2)	$f(0\forall XP(P{:}Y/X))$	(1), 3.5

Case (vi): P_k is $f(0Q)$ and is inferred from preceding lines $f(0P)$ and $f(0(P \supset Q))$. By hypothesis, both $f(0P)$ and $f(0(P \supset Q))$ are provable in G′. Hence, there exists a proof in G′ (a proof of $f(0P)$ continued by a proof of $f(0(P \supset Q))$, say) in which both $f(0P)$ and $f(0(P \supset Q))$ are lines; whence, one may infer $f(0Q)$ as a further line by 3.4. That is, P_k is provable in G′. Case (vii): P_k is $f(0\forall XP)$ and is inferred from a preceding line $f(0P)$. By hypothesis, $f(0P)$ is provable in G′. Hence, there exists a proof in G′ whose terminal line is $f(0P)$, i.e., $f(0(P{:}X/X))$; whence, one may infer $f(0\forall XP)$ as a further line by 3.6. That is, P_k is provable in G′. Thus, in all cases, P_k is provable in G′. But P_k was any line of a proof in F^1. That all formulae provable in F^1 are provable in G′ follows by course of values induction.

What has been shown is that all theorems of F^1 are theorems of G′. But all valid quantificational formulae are theorems of F^1; hence, all valid qualificational formulae are theorems of G′. That is, G′ is complete.

8. In showing that all formulae provable in G′ are valid, free use will be made of well known laws of validity, implication and equivalence (e.g., see Quine, [3]). An auxiliary notation will be useful: if ϕ is empty, '$*\phi$' will refer to any valid formula devoid of free variables; if ϕ is non-empty, '$*\phi$' will refer to any conjunction of all and only formulae of ϕ. Thus, by laws alluded to above, $f(\phi P)$ is equivalent to $(*\phi \supset P)$ and if ϕ is initial in ψ then $*\psi$ implies $*\phi$.

Suppose of a proof in G′ that all lines preceding some arbitrarily chosen line P_k are valid. What is to be shown is that P_k is valid. There are six cases to consider. Case (i): P_k is an axiom $f(\phi PP)$, i.e., $f(\phi(P \supset P))$. Because $(P \supset P)$ is valid, it is implied by any formula; in particular, by $*\phi$. Hence, $(*\phi \supset (P \supset P))$ is valid; hence, so is its equivalent $f(\phi(P \supset P))$. That is, P_k is valid. Case (ii): P_k is $f(\psi(P \supset Q))$ and is inferred from a preceding line $f(\phi\text{-}P)$. By the proviso on 3.2, ϕ is initial in ψ, so $*\psi$ implies $*\phi$. By hypothesis, $f(\phi\text{-}P)$ is valid; hence, so is its equivalent $(*\phi \supset \text{-}P)$. That is, $*\phi$ implies $\text{-}P$, But $\text{-}P$ implies $(P \supset Q)$; so, by transitivity of implication, $*\psi$ implies $(P \supset Q)$. That is, $(*\psi \supset (P \supset Q))$ is valid; hence, so is its equivalent $f(\psi(P \supset Q))$. That is, P_k is valid. Case (iii): P_k is $f(\psi P)$ and is inferred from a preceding line $f(\phi(\text{-}P \supset P))$. The argument establishing validity of P_k is analogous to that of Case (ii) and is omitted. Case (iv): P_k is $f(\chi Q)$ and is inferred from preceding lines $f(\phi P)$ and $f(\psi(P \supset Q))$. By the proviso on 3.4, both ϕ and ψ are initial in χ. Hence, $*\psi$ implies $*\phi$, also $*\psi$, also $(*\phi.*\psi)$. By hypothesis, $f(\phi P)$ and $f(\psi(P \supset Q))$ are both valid; hence their respective equivalents $(*\phi \supset P)$ and $(*\psi \supset (P \supset Q))$ are both valid. That is, $*\phi$ implies P and $*\psi$ implies $(P \supset Q)$. Hence, $(*\phi.*\psi)$ implies $(P.(P \supset Q))$. But $(P.(P \supset Q))$ implies Q; hence, $(*\phi.*\psi)$ implies Q; hence, $*\chi$ implies Q. That is, $(*\chi \supset Q)$ is valid; hence, so is its equivalent $f(\chi Q)$. That is, P_k is valid. Case (v): P_k is $f(\psi(P{:}Y/X))$ and is inferred from a preceding line $f(\phi \forall XP)$. By the proviso on 3.5, ϕ is initial in ψ, so $*\psi$ implies $*\phi$. By hypothesis, $f(\phi \forall XP)$ is valid; hence, so is its equivalent $(*\phi \supset \forall XP)$. That is, $*\phi$ implies $\forall XP$. But $\forall XP$ implies $(P{:}Y/X)$; hence, $*\phi$ implies $(P{:}Y/X)$; hence, $*\psi$ implies $(P{:}Y/X)$. That is, $(*\psi \supset (P{:}Y/X))$ is valid; hence, so is its equivalent $f(\psi(P{:}Y/X))$. That is, P_k is valid. Case (vi): P_k is $f(\psi \forall XP)$ and is inferred from a preceding line $f(\phi(P{:}Y/X))$. By the proviso on 3.6, ϕ is initial in ψ, so $*\psi$ implies $*\phi$. Furthermore, by the same proviso, Y has no free occurrence in $\forall XP$ or in the formulae of ϕ. By hypothesis, $f(\phi(P{:}Y/X))$ is valid; hence, so is its equivalent $(*\phi \supset (P{:}Y/X))$; hence, so is $\forall Y(*\phi \supset (P{:}Y/X))$; hence, so is $(*\phi \supset \forall Y(P{:}Y/X))$; hence, rewriting bound variables, so is its equivalent $(*\phi \supset \forall XP)$. That is, $*\phi$ implies $\forall XP$; hence, $*\psi$ implies $\forall XP$. That is, $(*\psi \supset \forall XP)$ is valid; hence, so is its equivalent $f(\psi \forall XP)$. That is, P_k is valid. Thus, in all cases, P_k is valid. But P_k was any line of a proof in G′. That all formulae provable in G′ are valid follows by course of values induction. That is, G′ is sound.

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