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GENTZEN-LIKE SYSTEMS FOR PARTIAL PROPOSITIONAL CALCULI: I

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1. *Introduction.* The existence of many decidable partial propositional calculi leads to a desire for the construction of Gentzen-like systems (with provable cut-elimination theorems) for such calculi. Two problems arise in the construction of such systems. First, one must to some extent formalize what one means by a Gentzen-like system. Second, one must find appropriate deduction rules for the systems so that the cut-elimination theorem can be proved. Here we consider the general definition and the two simpler one axiom sub-partial propositional calculi of the system of [1]. Part II of this paper will consider the other one axiom subcalculus.

2. *Classical Gentzen-like Systems.* The classical Gentzen-like systems, for example [2], might be characterized as follows.

(i) The objects manipulated are n-tuples of well-formed formulae.
(ii) Certain of these n-tuples with (easily) effectively recognizable forms are taken as axioms.
(iii) Deduction rules each take some n-tuple (or pair of n-tuples) and make (easily) effectively recognizable changes in the n-tuples. Most of these changes are permutations of the symbols of the n-tuples or the introduction of a new symbol or formula into the n-tuple.
(iv) A certain (easily) effectively recognizable subset of the n-tuples constructed from the axioms by the deduction rules is identifiable with the set of theorems of the classical theory for which the Gentzen-like system is being constructed.
(v) Often a theorem (cut-elimination) is provable on the basis of which one can effectively construct, given a n-tuple of the proper form for (iv), a finite set of possible proofs for it in such a way that one can actually decide if it is a theorem by determining if one of these possible proofs is actually a proof.

3. *Definitions.* Given a partial propositional calculus C with language L with the usual variables, constants, connectives and formation rules, let #

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be a symbol not occurring in **L**. Hence, # is not a well-formed formula. Then consider a system **G** as follows.

1. The objects θ of **G** are n-tuples composed of formulas from **L** together with #. We assume that the objects are Gödelized, so that we may refer to recursive sets of objects, etc.
2. A recursive subset α of the objects is given. Members of α are called axioms.
3. A finite set δ of recursive functions each from a recursive subset of θ (or $\theta \times \theta$) into θ, called the deduction rules, is given.
4. A recursive subset ρ of θ, called the possible theorems is given.

We construct a subset ω of θ, called the set of working proofs by the inductive definition: α is a subset of ω; if P and Q are in ω and $R = f(P)$ or $R = f(P,Q)$ for some f in δ, then R is in ω; ω is minimal, consistent with these conditions. Then let τ, the intersection of ω and ρ, be the set of theorems of **G**. Note that such systems include most classical (not necessarily Gentzen-like) systems of partial propositional calculus as special cases. What seems to distinguish classical Gentzen-like systems is the "easily" in (ii), (iii), and (iv) above and the possibility of proving (v).

4. *The First Axiom*. We consider the system [1] as having the three axioms

A1. $(((p \supset \mathrm{f}) \supset \mathrm{f}) \supset p)$
A2. $(p \supset (q \supset p))$
A3. $((s \supset (p \supset q)) \supset ((s \supset p) \supset (s \supset q)))$

and the deduction rules modus ponens (MP) and simultaneous substitution (SS). In the case of the subsystem with *A1* as its sole axiom and MP and SS as its rules, a formal conversion to a Gentzen-like system with cut-theorem works almost mechanically, since, for this system, MP is not an overly useful rule. In fact, we have the following proposition.

Proposition 1. *The set of consequences of A1 under the rules* MP *and* SS *consists of the consequences of A1 together with the single formula* $(\mathrm{f} \supset \mathrm{f})$ *under* SS *alone*.

Proof: Clearly each of the substitution instances of *A1* is a theorem. We adopt the terminology of [1], referring to the $A \supset B$ premiss as *major* and A premiss as *minor* in applications of MP. If one takes the instance of the axiom with $(\mathrm{f} \supset \mathrm{f})$ substituted for p as major premiss and the one with f substituted for p as minor premiss, $(\mathrm{f} \supset \mathrm{f})$ follows by MP.

To show that there are no other consequences, we do induction on the length of a proof, where by length l is meant the number of lines in the proof. If l is one, MP does not yet apply, so there are surely no theorems of forms other than the ones claimed. Let us assume that this is the case for all proofs of length $n - 1$. Now a proof of length n, if it is to prove a theorem not provable in $(n - 1)$-length proofs, must consist of a proof of length $n - 1$ plus a new line which follows by MP. Now, by the induction hypothesis each line before the last in this n-line proof is either a substitution instance of *A1* or is of the form (f $\supset$ f). Hence, in particular, no line is of the form f. Hence, (f $\supset$ f) is not the major premiss of the application of MP in question, since its antecedent is not available to serve as minor premiss. Nor can (f $\supset$ f) be the minor premiss of the application of MP in question, since, with major premiss a substitution instance of the axiom, its antecedent, which must be the same as the minor premiss, contains at least two implication signs. Hence, both the major and minor premisses are substitution instances of the axiom, say with Q and R substituted for p respectively. Comparison of the antecedent of the major premiss with the minor premiss shows that R must be f. When this substitution is made, the same comparison again shows that Q must be (f $\supset$ f). But the result of this application of MP is (f $\supset$ f) which is not a new theorem. Hence, all of the theorems provable in n-line proofs have the proper form and the theorem is proven. Q.E.D.

On the basis of this theorem, we could just take the collection of 1-tuples of the forms $(((P \supset \text{f}) \supset \text{f}) \supset P)$ and (f $\supset$ f) as our axioms α, an empty set δ of deduction rules and ρ as all 1-tuples of well-formed formulae to get the appropriate set of theorems. However, for the sake of a more Gentzen-like system in which changes are made in small easily recognizable stages, and also to let the reader see, in an easy case, what is going on, we consider the following system, **G1**.

1. The objects in θ are 4-tuples composed of well-formed formulae and #.
2. The axiom schema are the 4-tuples of the form $(P,\#,\#,\#)$ and $(\#,\#,\#,(\text{f} \supset \text{f}))$ where P is any well-formed formula.
3. The rules in δ are the following:

   $(P,\#,\#,\#) \rightarrow (P,\text{f},\#,\#)$
   $(P,Q,\#,\#) \rightarrow (P,Q,\text{f},\#)$
   $(P,Q,R,\#) \rightarrow (P,Q,R,P)$
   $(P,Q,R,S) \rightarrow (\#,(P \supset Q),R,S)$
   $(\#,P,Q,R) \rightarrow (\#,\#,(P \supset Q),R)$
   $(\#,\#,P,Q) \rightarrow (\#,\#,\#,(P \supset Q))$

   where $P,Q,R,$ and S are variables over well-formed formulae and $X \rightarrow Y$ has the same meaning as $f(X) = Y$.
4. The set ρ of possible theorems is the set of all 4-tuples of the form $(\#,\#,\#,Q)$ where Q is any well-formed formula.

Note that, in effect, in this case, all that the deduction rules do is act as formation rules, since, as indicated above, by adding $(\#,\#,\#,(\text{f} \supset \text{f}))$, we have made any application of the rule MP superfluous.

5. *The Second Axiom*. We now consider the more interesting case of *A2* under MP. Following the example just done, a straightforward "translation" might be as follows. The objects in θ will be 3-tuples. The single axiom schema will be $(P,Q,\#)$ for P and Q any well-formed formulae. The deduction rules, with the same conventions as before will be

(a) $(P,Q,\#) \rightarrow (\#,P,(Q \supset P))$
(b) $(\#,P,Q) \rightarrow (\#,\#,(P \supset Q))$
(c) $[(\#,P,(Q \supset R))$ and $(\#,\#,P)] \rightarrow (\#,Q,R)$

the rule (c) representing MP (at least for cases in which the conclusion contains an implication sign, that is, all the cases in which we are interested). The set ρ will consist of all 3-tuples of the form $(\#,\#,Q)$. First let us be sure that our translation is a good one.

Proposition 2. *The classical theory and its translation have the same set of theorems in the sense that if P is a theorem of the classical theory, then $(\#,\#,P)$ is a theorem of the Gentzen-like theory, and conversely.*

Proof: In one direction, it is sufficient to show that all instances of the classical axiom *A2* are theorems of the new system and that for any two theorems of the new system, their consequence by MP is also a theorem of the new system. But, starting with the axiom $(P,Q,\#)$ and applying the rules (a) and (b) in turn, we get the theorem $(\#,\#,(P \supset (Q \supset P)))$. Now suppose that in the new system, we have proved $(\#,\#,(P \supset Q))$ and $(\#,\#,P)$. We want a proof of $(\#,\#,Q)$. Since (b) is the only rule that gives conclusions of the form $(\#,\#,(P \supset Q))$, this last formula must result from a previously proved formula $(\#,P,Q)$. Writing Q in the form $(R \supset S)$ and applying rule (c), we get $(\#,R,S)$. Now applying (b) we get $(\#,\#,Q)$ as desired. (Since, for the Q which we want to prove, all are theorems of the complete classical system, and since no such theorem lacks an implication sign, we are justified in writing Q as $(R \supset S)$.)

Similarly, in the other direction, any proof not involving rule (c) gives a substitution instance of the translated form of the axiom $(\#,\#,(P \supset (Q \supset P)))$, and any use of (c) (followed by uses of (b)) can be replaced by a single use of MP. Hence, the two systems have the same theorems. Q.E.D.

One would suppose that here we cannot simply eliminate the rule corresponding to MP and get the same set of theorems. But, let us add two further deduction rules:

(d) $(\#,P,Q) \rightarrow (R,P,Q)$
(e) $(R,P,Q) \rightarrow (\#,R,(P \supset Q))$

Proposition 3. *The new rules add no new theorems to the system.*

Proof: Clearly, neither rule alone adds any new theorems, since each involves a formula of the form (P,Q,R) which is the antecedent of no other rule. Together, they take us from a proof of $(\#,P,Q)$ to one of $(\#,R,(P \supset Q))$. Let us show how to do this without using the new rules. By assumption, we have a proof of $(\#,P,Q)$. By applying rule (a) to the axiom $((P \supset Q),R,\#)$ we get $(\#,(P \supset Q), (R \supset (P \supset Q)))$. By applying rule (b) to $(\#,P,Q)$ we get $(\#,\#,(P \supset Q))$. By applying rule (c) to this last pair, we get $(\#,R,(P \supset Q))$ as desired. Q.E.D.

But, with the presence of these extra two rules, the rule (c) becomes superfluous.

Proposition 4. (*Cut-elimination*) *In the system extended by the addition of rules* (d) *and* (e), *rule* (c) *may be eliminated without loss of theorems.*

Proof: Given proofs of $(\#,P,(Q \supset R))$ and $(\#,\#,P)$, we want a proof of $(\#,Q,R)$ not using rule (c). Further, since proofs are of finite length, it suffices to show that we can eliminate the first use of rule (c) in a proof. Hence, in particular, we may assume that the premiss $(\#, P, (Q \supset R))$ is not the result of an application of rule (c). Hence, as its form shows, it must result from an application of rule (a) or rule (e). If it results from rule (a), then R must be the same as P, so that the conclusion that we want is, in fact, $(\#, Q, P)$. But we have a proof of $(\#, \#, P)$ in which, by hypothesis, no application of rule (c) has been made. Hence, by its form, $(\#, \#, P)$ must have arisen by an application of rule (b) from $(\#, S, T)$ where P is $(S \supset T)$. Hence, take the provable formula $(\#, S, T)$ and apply rule (d) to get (Q, S, T) and then apply rule (e) to get $(\#, Q, (S \supset T))$, that is, $(\#, Q, P)$ as desired. The other possibility is that $(\#, P, (Q \supset R))$ arises by rule (e). We desire $(\#, Q, R)$. But since rule (e) was the rule applied, the statement proved in the previous line was (P, Q, R). But such statements arise only by rule (d). Hence going back one line farther, we have $(\#, Q, R)$ already as desired. Hence rule (c) is eliminable. Q.E.D.

We can now justify the name cut-elimination and show that a result of the type of (v) in section 2 above holds.

Proposition 5. *In the system* **G2** *with axiom schema* $(P, Q, \#)$ *and the deduction rules* (a), (b), (d), *and* (e), *given a proposed theorem* $(\#, \#, P)$, *one can effectively decide whether it has a proof in the system.*

Proof: The equivalence to the classical system is proved in Propositions 2, 3, and 4 above. We prove the decidability by induction on the length of a 3-tuple which we define as the sum of the lengths of the well-formed formulae appearing in the 3-tuple. By the equivalence to the classical system and because the classical system is a subset of the complete consistent classical system, no provable formula is of the form $(\#, \#, P)$ where P is a variable or the constant f standing alone. Hence, every provable formula contains an implication sign and the theorem is vacuously true for 3-tuples of lengths one and two. Now suppose that it is true for all 3-tuples with lengths n-1 or less. Since our conclusion is of the form $(\#, \#, R)$, and we know that R is of the form $(S \supset T)$, the step before the conclusion must have been $(\#, S, T)$ by rule (c). There are two possibilities. Either this statement came about by rule (a) or by rule (e). If it came about by rule (a), then T must be of the form $(Q \supset S)$ for some Q. If this is so, then $(\#, S, T)$ follows from the axiom $(S, Q, \#)$ by rule (a) and we are done. If T is not of this form, then the conclusion did not come by rule (a) and the remaining possibility is that it came by rule (e). Hence, T must be of the form $(Q \supset P)$ for some Q and some P. But since rule (e) was applied, the preceding step was (S, Q, P). But such a formula can arise from rule (d) only. Hence its antecedent is $(\#, Q, P)$. But since S has length at least one, this 3-tuple has length less than n and we can decide whether it has a proof in the system by the induction hypothesis. Q.E.D.

Corollary. *Given a theorem in the system* **G2** *for which one has decided that there is a proof, one may effectively construct such a proof.*

Proof: Just apply the method given in the proof of Proposition 5 for tracing proofs back a step at a time. The method stops eventually, since all proofs are of finite length and the proposition in question has a proof. Q.E.D.

Part II of this paper (to appear) will consider the third axiom, $A3$. The presence of three variables in the classical statement of the axiom greatly complicates the Gentzenization of the system.

REFERENCES

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