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CONTEXT LOGIC I: FUNDAMENTAL CONCEPTS, NOTATIONS, AND DERIVED NOTIONS

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INTRODUCTION

The objective of this first paper* on the construction of a formal system for denoting and connoting context is to expand and augment the formal properties of a context operator introduced in [1], which models context transformations. Formally, such a representation allows one to keep track of "implications in context," "substitutions in context," and "contextual shifts." As was shown in [1], the context operator provides a strategy whereby we can develop the formal properties of intensionality and extensionality [2]. By extensionality we denote sets and connote the relativity of members of sets. By intensionality we denote structure of sets and connote elements of sets.

We will show in a subsequent paper that context, along with the intension-extension integration it generates, is the basis of a construction of a theory of sets that embraces the sense of Cantor's original definition of a "set" as a comprehension (concatenation) of definite distinct objects of our intuition into a whole. The inferential process of comprehending into a whole constructs a "set" and not merely determines the objects involved. We shall obtain a "setless" set theory, which means that the *comprehension within* a *context* is *primitive* and out of this will arise the concept of "set."

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A system incorporating context-dependency provides a theory in which we can simultaneously discuss the intension and extension of the set under consideration. Such a theory also permits the development and use of self-referencing statements without requiring an infinite regress. It also allows analysis of the concept of the set of all sets, and generates along the directions of Morse [3] a logico-set theory where the logic and set theory form a unified formal system.

Such an approach appears necessary for the design of machines capable of forming computational and organizational strategies for situations outside the scope of their program. Dreyfus in his [4] has critiqued the field of artificial intelligence, and he concludes that progress toward artificial intelligence must await computers of an entirely different nature than exist today, and that these machines must in some way embody contextual organization and behavior.

Our approach consists in developing a notation for the context logic, studying the interrelations of context operators, aiming toward the definition within this language of the concept of set along with the ancillary properties attributed to sets, generalizing the concept of a "null set" to that of a "relative null set," and giving good reasons for assuming that the Axiom of Choice is the intensional dual of the Axiom of Subsets.

BACKGROUND

(A) Many investigators have contributed to the formal development of a theory of context. Church [5] has developed the ideas of intension and extension along with C. I. Lewis [2], L. S. Stebbing [6], and J. Myhill [7]. Such workers have also made fundamental contributions to the logical theory of intension and extension first explicitly studied in modern logic by C. S. Peirce [18]. Our specific approach was inspired by Haskell B. Curry [8], Fitch [22], and Anderson and Belnap [16], whose important E system has embedded within it intimations of the context operator. The algebraic structure is modelled on the structures found in S. McLane [9], P. Freyd [10], B. Mitchell [11], and F. W. Lawvere [20]. The reader is referred to [21] in which appears an excellent survey of Categorical Algebra and its applications to the theory of Lawvere which attempts to show that set theory can be recovered from category theory. In this paper the notions of category theory will be explicated as they are introduced.

(B) Before presenting the formalism we will first review what has been considered to be the connotation of context and how contexts occur in inference.

Context is usually assumed to connote the interdependencies of "points of view," topical relevancy and continuity of thought. The "point of view" or "context of structuring" assigns a relevancy to the data. From this relevancy there emerges a continuity between data being structured and the topic which determines the process of the structuring. The "point of view" philosophy of context has been explicated by Mach [12] and James [13]. They consider that the integration of an element of data is brought about not by the phenomenal features exhibited by the element in question but by the view from which it is considered.

Relevancy does not occur simply through the simultaneity—in phenomenal time—of events under consideration. We find in Gurwitsch [14] a discussion of why the theme, which assigns a context, arises out of the thematic field but must not be confused with it. Said another way, context cannot simply be the background because what constitutes "background" and what is "object of interest" requires a decision on the part of the observer and thus points to the non-existence of context in the thematic field itself. Indeed, a context-selection had to be made by the observer. Rubin [15], who has studied the ground field problem, contends that the phenomenal difference between figure and ground is most closely related to the different roles the contour plays for them. For him, the context-selection is performed by the contour.

One can accept, without much difficulty, that context is not immediately given in experience but has something to do with the observer's organizational modes.

Context plays a role in logic [14] when during a chain of inference we proceed continuously from one proposition to another, each proposition in its turn becoming the theme. At any single stage of the inferential process we are aware of indices pointing back to the earlier stages. This sense of continuity not only unifies what has passed but also functions as a predictive mechanism for future stages. Context selects from the middle ground of immediate consequences one that is appropriate to the horizon for the modelling system [1].

The argument that context plays no role in defining logical relationships leads to the unfortunate conclusion that it would be incorrect to assign any meaning to the logical position of a proposition relative to other propositions. In fact, in the theories of formal implication we find logicians building systems of logic that attempt to capture the concept of relevance between antecedent and consequent of an implication. A necessary condition for an implication to hold is the requirement that the intensional meaning of the consequent is contained in the intensional meaning of the antecedent [16]. The idea is that one cannot derive one proposition from another without invoking relevance. A study of their formalisms reveals that these logicians are attempting to capture the role of context in deductive systems.

John Dewey, [17] and the instrumentalists pointed out the importance of interacting with reality for the operational development of context. Dewey held that experiential acts develop contexts of knowing and these contexts arise out of the settings involved. If one sterilizes the environment in which the object under consideration is viewed, then artificial gaps are created between the medium and the object leading to a loss of natural continuity between the process of knowing and the known. As a result, a theory of context explicitly takes into account the process of generating context operators out of the experimental effort, i.e., the interaction between the knower and the known.

Various theories of implication have failed to formalize contextual operations because they could not capture the essential aspect of implication as an operation determined by the meanings of the propositions it connects. Many have theorized that implication is a necessary connection between meanings. Baylis [19] states that if A implies B then "the intensional meaning of B is identical with a part of the intensional meaning of A." The need to capture the condition of relevance resulted in the Theory of Entailment [16] by Alan Ross Anderson and Neul D. Belnap, Jr.

Further on in the paper we shall show that the requirement of Context independence of certain diagrams results in a major part of the Anderson-Belnap system E. This is not to be taken to mean that any formula in the $\mathcal{C}^*$ system is provable in the E system or that the $\mathcal{C}^*$ logic is a proper subset of E. Rather, there is a model of E in the $\mathcal{C}^*$ logic. The differences between the two systems are discussed in section 10 below.

FORMAL SYSTEM

1. *Primitive Objects.* The intuitive interpretation of the symbols may be listed as follows:

- ( ) Corresponds to a place holder with no specific restrictions.
- $\longrightarrow$ Stands for the concept of mapping or entailment ("arrow").
- $=$ Denotes the undefined equality between mappings.
- ϕ: Denotes that a symbol appearing on the left is the name of the "arrow" on the right, i.e.: $\phi: (\ ) \longrightarrow (\ )$ or $(\ ) \xrightarrow{\phi} (\ )$
- $*$ Capital letters with stars represent context-selection operator, i.e.: $\mathcal{A}^*$, $\mathcal{B}^*$, $\mathcal{C}^*$, $\mathcal{E}^*$, ... (A-star, B-star, C-star, etc.)
- $'$ The prime is applied to a ( ) giving a $(\ )'$ such that ( ) need not be equal to $(\ )'$.

2. *Well-Formed Diagrams* (W.F.D.). The mapping symbol $\longrightarrow$ may appear between (a) two place holders, (b) two other arrows, (c) a place holder and an arrow, (d) an arrow and a place holder, or (e) from place holder back to itself. A special form of (e) is termed $1_{(\ )}$ or the identity mapping.

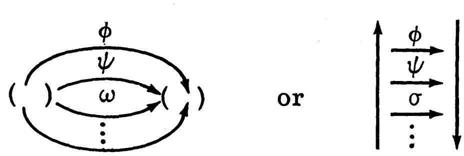
(a) $(\ ) \longrightarrow (\ )$ (b) $\downarrow \longrightarrow \uparrow$ (c) $(\ ) \longrightarrow \uparrow$

(d) $\downarrow \longrightarrow (\ )$ (e) $\circlearrowright(\ )$ or $1_{(\ )}$

We shall term the place holder ( ) to the left of an arrow the *origin* and the place holder ( ) to the right the *extremity*. (a) to (e) shall be termed "diagrams".

3. *Rules for Generating W.F.D's.*

(a) There may be many arrows with the same extremity and origin:



(b) There may be arrows running both ways between objects, e.g.:

$$\begin{tikzcd} (\ ) \arrow[r, shift left] & (\ ) \arrow[l, shift left] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} (\ ) \arrow[r, shift left] & {} \arrow[l, shift left] \arrow[d] \\ & {} \end{tikzcd}$$

(c) Any configuration of arrows obeying the above is a diagram, i.e., substituting into the place holders W.F.D's:

$$\begin{tikzcd} (\ ) \arrow[r] & (\ ) \arrow[d] \\ & (\ ) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} (\ ) \arrow[r] \arrow[dr] & (\ ) \arrow[d] \\ & (\ ) \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} (\ ) \arrow[r] \arrow[dr] & (\ ) \\ & (\ ) \arrow[u] \end{tikzcd}$$

4. *Composition of Maps.* At this point, we shall discuss the relation of strings of maps, i.e., relative to a given *extremity* and *origin,* (i.e., strings with same origin and extremity). The following shall be the compositional axioms. Given three arrows ϕ, σ, ω the following statements (A1) to (A4) are equivalent, where dot is an undefined composition operator.

(A1) $\phi \circ \sigma$ *and* $\sigma \circ \omega$ *is defined*
(A2) $(\phi \circ \sigma) \circ \omega$ *is defined*
(A3) $\phi \circ (\sigma \circ \omega)$ *is defined*
(A4) $(\phi \circ \sigma) \circ \omega$ *and* $\phi \circ (\sigma \circ \omega)$ *are defined and equal.*
(A5) *An identity arrow is a map* $1(\ )$ *such that, if* $1(\phi) \circ \phi$ *or* $\phi \circ 1(\phi)$ *is defined, it is equal to* ϕ.

Theorem 1. *If* $1_{(\phi)}$ *and* $\bar{1}_{(\phi)}$ *are two distinct identity maps and* $1_{(\phi)} \circ \phi$ *and* $\bar{1}_{(\phi)} \circ \phi$ *are defined, then* $1_{(\phi)} = \bar{1}_{(\phi)}$.
Proof: Let $1_{(\phi)} \circ . \phi = \phi$ and $\bar{1}_{(\phi)} \circ \phi = \phi$. Therefore $1_{(\phi)} \circ (\bar{1}_{(\phi)} \circ \phi) = 1_{(\phi)} \circ \phi = \phi$ (A5). By the composition axioms above, therefore, $1_{(\phi)} \circ 1_{(\phi)}$ is defined. Therefore we can conclude $1(\phi) = 1(\phi) \circ \bar{1}(\phi) = \bar{1}(\phi)$.

Convention: We shall denote the left identity map of a given map ϕ as $D(\phi)$ and the right identity map as $R(\phi)$.

Theorem 2. $\phi \circ \sigma$ *is defined if and only if* $D(\phi) = R(\sigma)$.
Proof: (a) $\phi \circ \sigma$ defined implies $D(\phi) = R(\sigma)$. Since $\phi \circ \sigma$ is defined and $\phi = \phi \circ D(\phi)$, then $(\phi \circ D(\phi))$ is defined. Therefore $D(\phi) \circ \sigma$ is defined and $R(\sigma) \circ \sigma$ is defined by the composition axioms. Since $D(\phi)$ and $R(\sigma)$ are both identity maps, by theorem 1, we have $D(\phi) = R(\sigma)$.
(b) $D(\phi) = R(\sigma)$ implies $\phi \circ \sigma$ defined. If $D(\phi) = R(\sigma)$ = identity map, then $\phi \circ 1\phi$ and $1_\phi \circ \sigma$ are defined and thus $\phi \circ \sigma = (\phi \circ 1_{(\phi)}) \circ \sigma = \phi \circ (1_{(\sigma)} \circ \sigma)$ is defined.
Theorem 3. *If* $\phi \circ \sigma$ *is defined, then* $D(\phi \circ \sigma) = D(\sigma)$ *and* $R(\phi \circ \sigma) = R(\phi)$.

Proof: Since $\sigma \circ D(\sigma)$ is defined, therefore $(\phi \circ \sigma) \circ D(\sigma)$ is defined and $D(\phi \circ \sigma) = D(\sigma)$. $R(\phi \circ \sigma) = R(\sigma)$ can be proven similarly.

The motivation for our rules on composing maps is that we want to be able to say when we can "grow" strings of Arrows. Given $(\ ) \xrightarrow{\phi} (\ )$ and $(\ ) \xrightarrow{\psi} (\ )$, when can we "hook them up" to get $(\ ) \xrightarrow{\phi} (\ ) \xrightarrow{\psi} (\ )$. It happens simply when the "right end" of ϕ *is* the "left end" ψ i.e., $D(\psi) = R(\phi)$. We can now get long strings. $(\ ) \xrightarrow{\phi_1} (\ ) \xrightarrow{\phi_2} (\ ) \xrightarrow{\phi_3} (\ ) \ldots (\ ) \xrightarrow{\phi_n} (\ )$, etc. These may be instances when strings have the same initial point and final point. For example:

$$\begin{tikzcd} (\ ) \arrow[r, "\phi"] \arrow[dr, "\sigma"'] & (\ ) \arrow[d, "\psi"] \\ & (\ ) \end{tikzcd}$$

If we specify that the strings are equal, then we cay say: $(\ ) \xrightarrow{\phi} (\ ) \xrightarrow{\psi} (\ ) = (\ ) \xrightarrow{\sigma} (\ )$. For convenience we shall say in this case "the diagram formed by the two equivalent strings *Commutes*".

Thus $\left\{ \begin{tikzcd} (\ ) \arrow[r, "\phi"] \arrow[d, "\mu"'] & (\ ) \arrow[d, "\sigma"] \\ (\ ) \arrow[r, "\lambda"] & (\ ) \end{tikzcd} \text{ commutes} \right\}$ equals $\left\{ \sigma \circ \phi = \lambda \circ \mu . \right\}$

Note: It may be the case that a "closed" diagram does not commute.

5. *Context Operators.* $\mathcal{C}^*$, $\mathfrak{D}^*$, $\mathcal{E}^*$, ...

(A6) *A context operator $\mathcal{C}^*$ assigns an arrow to the origin-endpoint of an identity map.*

Given $\begin{tikzcd} (\ ) \arrow[loop above] \end{tikzcd}$, then $\begin{tikzcd} (\ ) \arrow[loop above] \arrow[d, "\mathcal{C}^*(\ )"] \\ (\ ) \end{tikzcd}$

where the extremity of $\mathcal{C}^*(\ )$ is either a place holder or an arrow, i.e., the origin of $\mathcal{C}^*(\ )$ is the origin of the identity map.

Definition. A map ϕ is defined to be a map *in the context* $\mathcal{C}^*$ if and only if the following diagram is commutative:

$$\begin{tikzcd} (\ ) \arrow[rr, "\phi"] \arrow[dr, "\mathcal{C}^*(D(\phi))"'] & & (\ ) \arrow[dl, "\mathcal{C}^*(R(\phi))"] \\ & (\ ) & \end{tikzcd}$$

i.e., $\mathcal{C}^*(D(\phi)) = \mathcal{C}^*(R(\phi)) \circ \phi$. The underlying thought preserved here is that a mapping in a context must have as one of its attributes the preservation of a shared meaning between its extremity and origin i.e., a relevancy between antecedent and consequent [16].

Theorem 4. *Given ϕ and σ such that $\sigma \circ \phi$ is defined and ϕ and σ are in the context $\mathcal{C}^*$, then the composition $\sigma \circ \phi$ is in the context $\mathcal{C}^*$.*

Proof: By definition of "being in context $\mathcal{C}^*$" we have that the diagrams:

$$\begin{tikzcd} (\ ) \arrow[rr, "\phi"] \arrow[dr, "\mathcal{C}^*(D(\phi))"'] & & (\ ) \arrow[dl, "\mathcal{C}^*(R(\phi))"] \\ & (\ ) & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (\ ) \arrow[rr, "\sigma"] \arrow[dr, "\mathcal{C}^*(D(\sigma))"'] & & (\ ) \arrow[dl, "\mathcal{C}^*(R(\sigma))"] \\ & (\ ) & \end{tikzcd}$$

are commutative i.e., $\mathcal{C}^*(D(\phi)) = \mathcal{C}^*(R(\phi)) \circ \phi$ and $\mathcal{C}^*(D(\sigma)) = \mathcal{C}^*(R(\sigma)) \circ \sigma$. Since $\sigma \circ \phi$ is assumed defined and by prior theorems 2 and 3 $(\phi) = D(\sigma)$ and $D(\sigma \circ \phi) = D(\phi)$ and $R(\sigma \circ \phi) = R(\sigma)$, we have $\mathcal{C}^*(D(\sigma \circ \phi)) = \mathcal{C}^*(D(\phi)) = \mathcal{C}^*(R(\phi)) \circ \phi = \mathcal{C}^*(D(\sigma)) \circ \phi = \mathcal{C}^*(R(\sigma)) \circ \sigma \circ \phi = \mathcal{C}^*(R(\sigma)) \circ (\sigma \circ \phi) = \mathcal{C}^*(R(\sigma \circ \phi)) \circ (\sigma \circ \phi)$.

6. *Contextual Nesting*. Two context operators $\mathcal{C}^*$ and $\mathfrak{D}^*$ are defined to be "t ordered," written $\mathcal{C}^* <_t \mathfrak{D}^*$ if for any map ϕ such that:

$$\begin{tikzcd} (\ ) \arrow[rr, "\phi"] \arrow[dr, "\mathcal{C}^*(\mathrm{D}(\phi))"'] & & (\ ) \arrow[dl, "\mathcal{C}^*(\mathrm{R}(\phi))"] \\ & (\ ) & \end{tikzcd}$$

is a commutative diagram, then:

$$\begin{tikzcd} (\ ) \arrow[rr, "\phi"] \arrow[dr, "\mathfrak{D}^*(\mathrm{D}(\phi))"'] & & (\ ) \arrow[dl, "\mathfrak{D}^*(\mathrm{R}(\phi))"] \\ & (\ ) & \end{tikzcd}$$

is also commutative and t is such that $\mathfrak{D}^*(\mathrm{D}(\phi)) = t \circ \mathcal{C}^*(\mathrm{D}(\phi))$ and $\mathfrak{D}^*(\mathrm{R}(\phi)) = t \circ \mathcal{C}^*(\mathrm{R}(\phi))$. That is, for:

$$\begin{tikzcd} (\ ) \arrow[rr, "\phi"] \arrow[dr, "\mathcal{C}^*(\mathrm{D}(\phi))"] \arrow[ddr, "\mathfrak{D}^*(\mathrm{D}(\phi))"'] & & (\ ) \arrow[dl, "\mathcal{C}^*(\mathrm{R}(\phi))"'] \arrow[ddl, "\mathfrak{D}^*(\mathrm{R}(\phi))"] \\ & (\ ) \arrow[d, "t"] & \\ & (\ ) & \end{tikzcd}$$

each of the "side" diagrams is commutative.

7. *Generalized Modus Ponens*. Modus Ponens is the inference mode by which deductions are made in a logical schema. It lies at the very core of inferential thinking. Its standard form is: given A, $A \longrightarrow B$, then B (The Stoic form is "If the first then the second; the first, therefore the second). It has been used in logic since the time of Aristotle. Incorporating context dependency on inferences or mappings forces one to the conclusion that the accepted form of MP does not have the flexibility of conveying contextual information as exemplified by the following propositions:

1. "If it rains I shall carry an umbrella"
2. "It is raining"
3. by MP "I shall carry an umbrella."

It might be the case that instead of carrying an umbrella, I might decide to stay home, or even neglect the umbrella as an impediment. But, according to MP I literally must carry the umbrella in order to be "consistent".

A generalization of MP would have to be able to work when I am given the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr] \arrow[dddr] & B_1 \\ & B_2 \\ & \vdots \\ & B_n \end{tikzcd}$$

and A; it then selects out the pertinent next object.

Analysis of examples of inferences typified by the above leads one to conclude that in a context dependent inferential scheme Modus Ponens must be an operator based on the context operator, which incorporates in its definition the "sequential reasoning" which results in the "next step".

We arrived at our definition of MP by attempting to identify elements of the context algebra with MP. First we noticed that asserting an A would correspond to our assertion that 1_A is in the Context $\mathcal{C}^*$, i.e., asserting the diagram:

$$\begin{tikzcd} 1_A \arrow[d, "\mathcal{C}^*(1_A)"] \\ (\ \ ) \end{tikzcd}$$

Next, asserting a particular $A \longrightarrow B$ would correspond to making a prediction on the above diagram exemplified by the following diagram:

$$\begin{tikzcd}[column sep=huge] 1_A \arrow[r, "\mathrm{Pd}(\mathcal{C}^*(1_A)) = \phi"] \arrow[d, "\mathcal{C}^*(1_A)"'] \arrow[dr, phantom, "\mathrm{Pd}"] & (\ \ ) \\ (\ \ ) & \end{tikzcd}$$

such that the domain of $\mathrm{Pd}(\mathcal{C}^*(1_A)) = \phi$ is 1_A.

Modus Ponens should be an operator whose value results, in context, from application to the above diagram, i.e. $\mathrm{MP}(\mathrm{Pd}\ \mathcal{C}^*(1_A)) \equiv \phi \equiv 1_B$ such that the following diagram would be commutative:

$$\begin{tikzcd} 1_A \arrow[rr, "\phi"] \arrow[dr, "\mathcal{C}^*(1_A)"'] & & 1_B \arrow[dl, "\mathcal{C}^*(1_B)"] \\ & (\ \ ) & \end{tikzcd}$$

The hierachy of operators would then be: $\mathcal{C}^*$, Pd, MP represented by the following diagram:

$$\begin{tikzcd}[column sep=huge] 1_A \arrow[r] \arrow[d, "\mathcal{C}^*(1_A)"'] \arrow[dr, phantom, "\mathrm{Pd}\ \ \mathrm{MP}"] & (\ \ ) \\ (\ \ ) & \end{tikzcd}$$

Notice that we have preserved the sequential ordering of the premises while adding the feature of context dependency.

Definition. Pd is an operator on $\mathcal{C}^*$ such that $\mathrm{Pd}(\mathcal{C}^*(1_A)) = \phi$ and such that $\mathrm{D}(\phi) = 1_A$.

Definition. Modus Ponens is an operator MP on a diagram:

$$\begin{tikzcd}[column sep=huge] 1_A \arrow[r, "\mathrm{Pd}(\mathcal{C}^*(1_A))"] \arrow[d, "\mathcal{C}^*(1_A)"'] \arrow[dr, phantom, "\mathrm{Pd}"] & (\ \ ) \\ (\ \ ) & \end{tikzcd}$$

whose value is $1_B \equiv \mathrm{R}(\mathrm{Pd}\,(\mathcal{C}^*(1_A))) \equiv \mathrm{R}\phi$ such that:

$$\begin{tikzcd} 1_A \arrow[rr, "\phi"] \arrow[dr, swap, "\mathcal{C}^*(1_A)"] & & 1_B \arrow[dl, "\mathcal{C}^*(1_B)"] \\ & (\;\;) & \end{tikzcd}$$

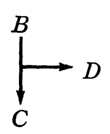
is commutative.

The definition of the operator MP reduces to the standard rule of inference when the context operator $\mathcal{Q}^*$ becomes a "zero" context operator $\mathcal{Q}^*$ such that $\mathcal{Q}^*(1_A)$ reduces to A, $\mathrm{Pd}(\mathcal{Q}^*(1_A))$ reduces to $A \longrightarrow B$, and $\mathrm{MP}(A, A \longrightarrow B) = B$. Thus we have the old definition of Modus Ponens embedded in this more generalized form.

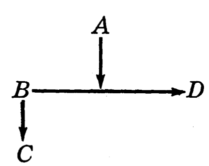
8. *Canonical Diagrams.* A natural question to ask is whether there are diagram configurations that are independent of the context(s) and particular maps employed. Such diagrams would lend themselves to being considered as "logical" statements and would be the basis for analysis of standard inference schemata. We shall demonstrate that there are indeed canonical diagrams and that they are the axioms for the Anderson-Belnap theory of entailment with the exception of one which we will discuss. The Anderson-Belnap system was generated to capture the concept of relevence between the antecedent and consequent of an inference. Since the system is purely deductive in terms of the Context operators we see that the only theorems that would be permissible are canonical diagrams and these are the general modes of reasoning used when we wish to consistently derive contextually independent theorems.

Preliminary machinery is necessary to define the procedure for obtaining canonical diagrams. We shall make use of the following bookkeeping devices.

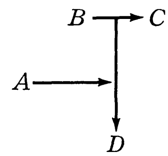
(a) There is a direct correspondence between writing arrow expressions linearly, e.g., $A \longrightarrow ((B \longrightarrow C) \longrightarrow D)$ and diagramatically. The rule is that we always work *out* from the innermost parenthesis as a unit. Thus $(B \longrightarrow C) \longrightarrow D$ corresponds to:



i.d., D is "pointed to be the arrow between B and C". For $A \longrightarrow ((B \longrightarrow C) \longrightarrow D)$ in the above example we would then have:



or any other equivalent configuration, i.e.:



Similarly we can reconstruct the linear form corresponding to such a diagram. It is evident that there is a unique way of transforming linear arrow forms to a diagram and the reverse.

(b) We next require a convention for substructuring a given diagram. Since there is a direct 1-1 correspondency between nested arrow sequences and diagrams, we need a convention for distinguishing the given elements in the diagrams from those elements that must be derived. We want to partition a given diagram (or corresponding arrow sequence) in terms of the antecedents L and the consequents R. This will be done with a two column list. On the left we place the antecedent and on the right the consequent, e.g., given $P_1 \longrightarrow Q_1$ we write:

L_1. P_1	R_1. Q_1.

If Q_1 is of the form $P_2 \longrightarrow Q_2$ we repeat the process, listing P_2 as L_2 and Q_2 as R_2. We continue this process until Q_n is no longer reducible. That is, with $R_{n-1} = P \longrightarrow Q_n$:

L_1. P_1	R_1. Q_1
.	.
.	.
.	.
L_n. P_n	R_n. Q_n

For example: $(D \longrightarrow B) \longrightarrow ((A \longrightarrow (B \longrightarrow C)) \longrightarrow (A \longrightarrow (D \longrightarrow C)))$:

L_1. $D \longrightarrow B$	R_1. $(A \longrightarrow (B \longrightarrow C)) \longrightarrow (A \longrightarrow (D \longrightarrow C))$
L_2. $A \longrightarrow (B \longrightarrow C)$	R_2. $A \longrightarrow (D \longrightarrow C)$
L_3. A	R_3. $D \longrightarrow C$
L_4. D	R_4. C

(c) The entries L_i are termed the "givens" or hypotheses of the diagram. They are of the form of a mapping or of an object. Since they are the hypotheses of the diagram we can designate them either by map names (if they are maps) or alternatively we can assign a context to them if they be objects. Thus if L_k is of the form $P_k \longrightarrow S_k$, we shall write it $P_k \xrightarrow{\phi_k} S_k$. If L_k is of the form P_k, then we assign it a context $\boldsymbol{C}_i{}^*$, e.g., P_k, $\boldsymbol{C}_i{}^*$. The first step in obtaining a canonical diagram is to show that there is a map in context $L_n \longrightarrow R_n$, making use of MP and the L_i assumed. If that is done we would have that map made up of combinations of the maps on the L_i. Then we can use a map introduction rule.

(d) *Rule.* In an n-lengthed column, for L_{n-k}, R_{n-k}, we can introduce a map $L_{n-k} \longrightarrow R_{n-k}$ if the quantity in L_{n-k} appears in R_{n-k}. If this is so then we know that the arrow sequence $L_{n-k} \longrightarrow R_{n-k}$ is independent of the symbol. If after arriving at L_2 using this procedure we find that the only variable left in R_2 is that found in L_2 then we call the resultant diagram Canonical, i.e., we could have assigned different Context(s) and nodal quantities and still have obtained the same diagram configuration. We will now see what diagrams are canonical.

Theorem 5. $\overset{\curvearrowright}{1_A}$ (*or* $1_A \xrightarrow{\text{id}} 1_A$) *is a canonical diagram.*

This means for any context C^* the following diagram is commutative.

$$\begin{tikzcd} 1_A \arrow[rr, "\text{id}"] \arrow[dr, "C^*(1_A)"'] & & 1_A \arrow[dl, "C^*(1_A)"] \\ & (\ ) & \end{tikzcd}$$

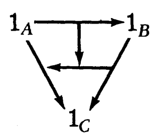
From the diagram this is obvious since $\text{id} \circ 1_A = 1_A$. Thus $C^*(\text{id} \circ 1_A) = C^*(1_A)$, and $C^*(1_A) = C^*(1_A) \circ \text{id}$.

Proof.

L_1. 1_A, C^*	R_1. 1_A, C^*

Since the only variables appearing on both sides are the same we have $1_A \longrightarrow 1_A$ is canonical.

Theorem 6. $(1_A \longrightarrow 1_B) \longrightarrow ((1_B \longrightarrow 1_C) \longrightarrow (1_A \longrightarrow 1_C))$, i.e.:



is a canonical diagram.

Proof: This diagram represents the process of transitivity of maps.

L_1. $1_A \longrightarrow 1_B$	R_1. $(1_B \longrightarrow 1_C) \longrightarrow (1_A \longrightarrow 1_C))$
L_2. $1_B \longrightarrow 1_C$	R_2. $1_A \longrightarrow 1_C$
L_3. 1_A	R_3. 1_C

Therefore we can say:

L_1. $1_A \xrightarrow{\phi} 1_B$

L_2. $1_B \xrightarrow{\sigma} 1_C$

L_3. 1_A, C^*

Given $(1_A, C^*)$ and $1_A \xrightarrow{\phi} 1_B$, by MP we have $(1_B, C^*)$, i.e., the following diagram is commutative:

$$\begin{tikzcd} 1_A \arrow[rr, "\phi"] \arrow[dr, "C^*(1_A)"'] & & 1_B \arrow[dl, "C^*(1_B)"] \\ & (\ ) & \end{tikzcd}$$

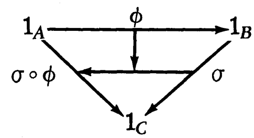
By MP, from $1_B \xrightarrow{\phi} 1_C$ and $(1_B, C^*)$ we get $(1_C, C^*)$, i.e.:

$$\begin{tikzcd} 1_B \arrow[rr, "\sigma"] \arrow[dr, "\mathcal{C}^*(1_B)"'] & & 1_C \arrow[dl, "\mathcal{C}^*(1_C)"] \\ & (\ ) & \end{tikzcd}$$

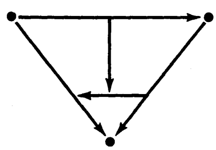
is commutative. Thus by the composition rule, the face, $(\sigma\circ\phi, \mathcal{C}^*.)$, i.e.:

$$\begin{tikzcd} 1_A \arrow[rr, "\sigma\circ\phi"] \arrow[dr, "\mathcal{C}^*(1_A)"'] & & 1_C \arrow[dl, "\mathcal{C}^*(1_C)"] \\ & (\ ) & \end{tikzcd}$$

is commutative. Therefore, R_2. $1_A \xrightarrow{\sigma\circ\phi} 1_C$, i.e.:

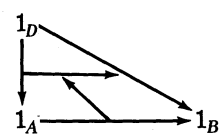


Thus, since L_2. $1_B \xrightarrow{\phi} 1_C$ | R_2. $1_A \xrightarrow{\sigma\circ\phi} 1_C$, we get R_1. $(1_B \longrightarrow 1_C) \xrightarrow{-\circ\phi} (1_A \longrightarrow 1_C)$. As L_1. $1_A \xrightarrow{\phi} 1_B$ | R_1. $(1_B \longrightarrow 1_C) \xrightarrow{-\circ\phi} (1_A \longrightarrow 1_C)$, we deduce that $L_1 \longrightarrow R_1$ is canonical. That is:



is a canonical diagram, i.e., transitivity is independent of context.

Theorem 7. $(1_A \longrightarrow 1_B) \longrightarrow ((1_D \longrightarrow 1_A) \longrightarrow (1_D \longrightarrow 1_B))$, i.e.:



is canonical.

This diagram represents left distribution.

Proof. The parsed diagram results in:

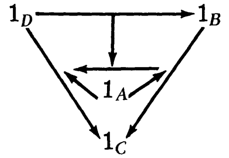
L_1. $1_A \longrightarrow 1_B$	R_1. $(1_D \longrightarrow 1_A) \longrightarrow (1_D \longrightarrow 1_B)$
L_2. $1_D \longrightarrow 1_A$	R_2. $1_D \longrightarrow 1_B$
L_3. 1_D	R_3. 1_B

Thus we have:

$1_A \xrightarrow{\phi} 1_B$
$1_D \xrightarrow{\sigma} 1_A$
$1_D, \mathcal{C}^*$

By MP on $(1_D, \mathcal{C}^*)$ and $1_D \xrightarrow{\sigma} 1_A$ we get $(1_A, \mathcal{C}^*)$. By MP on $(1_A, \mathcal{C}^*)$ and $(1_A \xrightarrow{\phi} 1_B)$ we get $(1_B, \mathcal{C}^*)$. By composition law $\phi\circ\sigma$ is in context $\mathcal{C}^*$, i.e., $1_D \xrightarrow{\phi\circ\sigma} 1_B$ is in $\mathcal{C}^*$. Hence, $(1_D \longrightarrow 1_A) \xrightarrow{\phi\circ -} (1_D \longrightarrow 1_B)$. Hence, $(1_A \longrightarrow 1_B) \xrightarrow{-\circ -} (1_D \longrightarrow 1_A) \longrightarrow (1_D \longrightarrow 1_B))$. The diagram is canonical.

Theorem 8. $(1_D \longrightarrow 1_B) \longrightarrow ((1_A \longrightarrow (1_B \longrightarrow 1_C)) \longrightarrow (1_A \longrightarrow (1_D \longrightarrow 1_C)))$, i.e.:

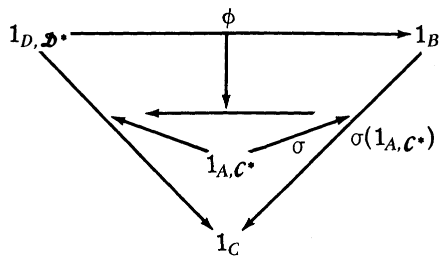


is a canonical diagram.

Proof. Parsing:

L_1.	$1_D \longrightarrow 1_B$	R_1.	$(1_A \longrightarrow (1_B \longrightarrow 1_C)) \longrightarrow (1_A \longrightarrow (1_D \longrightarrow 1_C))$
L_2.	$1_A \longrightarrow (1_B \longrightarrow 1_C)$	R_2.	$1_A \longrightarrow (1_D \longrightarrow 1_C)$
L_3.	1_A	R_3.	$1_D \longrightarrow 1_C$
L_4.	1_D	R_4.	1_C

From the L_i column we now have the original diagram in the following form:



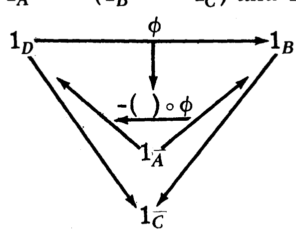
Notice that the value $\sigma(1_A, \mathcal{C}^*)$ is a map between 1_B and 1_C which is always dependent on what σ maps. As we have $1_{D,\mathfrak{D}^*} \xrightarrow{\phi} 1_B$ and $\sigma(1_A, \mathcal{C}^*)$ by the composition law and MP we have:

$$1_D \xrightarrow{\sigma(1_{A,\mathcal{C}^*}) \circ \phi} 1_C$$

is in the context $\mathfrak{D}^*$, thus we have R_3. We obtain R_2 as we have $(1_A, \mathcal{C}^*)$ and $\sigma(1_A, \mathcal{C}^*) \circ \phi$ mapping 1_D to 1_C, i.e.:

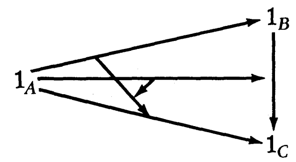
$$1_A \xrightarrow{\sigma(\ ) \circ \phi} (1_D \longrightarrow 1_C)$$

We obtain R_1 because of $1_A \xrightarrow{\sigma} (1_B \longrightarrow 1_C)$ and $1_A \xrightarrow{\sigma(\ ) \circ \phi} (1_D \longrightarrow 1_C)$, i.e.:



Since we now are left with ϕ and $-(\ ) \circ \phi$ we verify that the original diagram is canonical.

Theorem 9. $(1_A \longrightarrow (1_B \longrightarrow 1_C)) \longrightarrow ((1_A \longrightarrow 1_B) \longrightarrow (1_A \longrightarrow 1_C))$, *the distributary diagram,* i.e.:



is canonical.

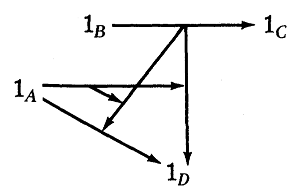
The proof of theorem 9 proceeds in the same manner as the preceding proofs. What it says is that we can distribute the antecedent through to the constituent members of a mapping.

9. *Relationship to* E. Given the fact that there exist diagrams, what relevance do they have in a formal theory of context? It turns out that for most part they are those axioms that Anderson and Belnap have for their theory of entailment E with some important modifications. The entailment theory was developed as a solution to one of the more intractable problems in contemporary philosophical analysis. The concept of formal deducibility was needed which would avoid paradoxes and preserve the idea of relevance between the antecedent and consequent of a mapping. The axioms which Anderson and Belnap chose as best capturing their intuitive notion of entailment are listed below.

E1 $A \longrightarrow A$
E2 $(A \longrightarrow B) \longrightarrow ((B \longrightarrow C) \longrightarrow (A \longrightarrow C))$
E3 $(A \longrightarrow ((B \longrightarrow C) \longrightarrow D)) \longrightarrow ((B \longrightarrow C) \longrightarrow (A \longrightarrow D))$
E4 $(A \longrightarrow (B \longrightarrow C)) \longrightarrow ((A \longrightarrow B) \longrightarrow (A \longrightarrow C))$
Rule (α). *From A and $(A \longrightarrow B)$ to infer B.*

Their interpretations of the arrow is that $A \longrightarrow B$ shall be true if and only if B "depends on the logical content of" A.

10. *Differences between Context Logic and* E. The axioms E1, E2 and E4 correspond to canonical diagrams within the context logic and the rule α is a special case of MP. The only real difference lies in E3 which is provable in the Anderson Belnap system but not in ours. For, looking at the diagram of this axiom we get:



and the corresponding parsing:

L_1. $1_A \longrightarrow ((1_B \longrightarrow 1_C) \longrightarrow 1_D)$	R_1. $(1_B \longrightarrow 1_C) \longrightarrow (1_A \longrightarrow 1_D)$
L_2. $1_B \longrightarrow 1_C$	R_2. $1_A \longrightarrow 1_D$
L_3. 1_A	R_3. 1_D

It is easily seen that given 1_A in $\mathcal{C}^*$ we can in no way show 1_D is in $\mathcal{C}^*$.

Only the map between $1_B \longrightarrow 1_C$ and 1_D is in C^*. Thus axiom E3 is not provable in our system. The reason why Anderson-Belnap system can include this axiom is because the E system is insensitive to the order of the indices in the subordinate proofs. Thus the major difference between the context logic and the E system results from our assigning particular interpretations to the indices and the fact that we find their order to be of importance.

We can summarize by saying that our mode of generating *canonical diagrams* corresponds to a restricted form of the Anderson-Belnap E system where the indices in the intelium subproof correspond in the context logic to maps and context operators related via the commutative diagrams. Thus, entailments in the E sense are those maps that appear only in canonical diagrams except for E_3.

Addendum: Dr. Stephen Waxman of Yeshiva University has brought to my attention a very interesting article by David H. Krantz, A Theory of Context Effects Based on Cross-Context Matching, *Journal of Mathematical Psychology*, vol. 5, No. 1, February 1968. This paper takes the notion of context as primitive and manifests the extensional properties of it in the language of semi-groups. Having done this, he develops a compelling system to study color adaptions and contrasts.

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